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CONSTRUCTING NOWHERE DIFFERENTIABLE FUNCTIONS FROM CONVEX FUNCTIONS

Abstract

We find an easy way to construct a continuous nowhere differentiable function from any nondecreasing convex function mapping the unit interval onto itself. We give a number of examples of nowhere differentiable functions constructed this way.

Examples of continuous nowhere differentiable real valued functions have been of interest in real analysis since the nineteenth century. In most such examples this function is expressed as the sum of a series of differentiable functions. (See, for example, [1], [3], [4], and [5].) In this note we show how a large class of nowhere differentiable functions can be so constructed where each summand is a convex function on certain intervals. We conclude with several concrete examples of nowhere differentiable functions we construct in this way. (For material on nowhere differentiable functions, consult the references in [1] and [3].)

We say that the real function f is <u>convex</u> on the interval J if whenever $a < b$ in J and $0 < t < 1$, we have $f(ta + (1-t)b) \geq tf(a) + (1-t)f(b)$; i.e., the graph of f on $[a, b]$ is never below the line joining the points $(a, f(a))$ and $(b, f(b))$. It follows that necessarily

$$\frac{f(ta + (1-t)b) - f(a)}{ta + (1-t)b - a} \geq \frac{f(b) - f(a)}{b - a}.$$

We say that f is <u>concave</u> on J if $-f$ is convex on J.

We offer the following assertion.

Theorem 1. *Let (a_n) be a sequence of nonnegative real numbers such that $\Sigma_n\, a_n < \infty$. Let (b_n) be a strictly increasing sequence of integers such that b_n*

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divides b_{n+1} for each n, and the sequence $(a_n b_n)$ does not converge to 0. For each index $j \geq 1$, let f_j be a continuous function mapping the real line onto the interval $[0,1]$ such that $f_j = 0$ at each even integer and $f_j = 1$ at each odd integer. For each integer k and each index j, let f_j be convex on the interval $(2k, 2k+2)$. Then the continuous function $\sum_{j=1}^{\infty} a_j f_j(b_j x)$ has a finite left or right derivative at no point.

PROOF. Assume that F has a finite right derivative $F'(x)$ at the point x. Because the sequence $(a_n b_n)$ does not converge to 0, we have $\limsup a_n b_n > 0$. Select $\epsilon > 0$ so that $11\epsilon < \sup a_n b_n$. Let p be a positive number such that if $0 < y - x < p$, then $|(F(y) - F(x))(y-x)^{-1} - F'(x)| < \epsilon$. Let N be an index such that $a_N b_N > 11\epsilon$ and consecutive zeros of $f_N(b_N x)$ differ by less than $\frac{p}{2}$; in other words $b_N^{-1} < \frac{p}{4}$. Let x_1 and x_3 be consecutive zeros of $f_N(b_N x)$ such that $x < x_1 < x_3$, $x_1 - x \leq x_3 - x_1 < \frac{p}{2}$ and let x_2 be the midpoint of the interval (x_1, x_3) Then $x_1 - x < x_2 - x < x_3 - x < p$ and moreover

$$x_1 - x \leq 2(x_2 - x_1), x_2 - x \leq 3(x_2 - x_1) \tag{1}$$

Let r_1, r_2, and r_3 be the real numbers for which

$$\begin{aligned}(F(x_2) - F(x_1))(x_2 - x_1)^{-1} &= F'(x) + r_3,\\ (F(x_2) - F(x))(x_2 - x)^{-1} &= F'(x) + r_2,\\ (F(x_1) - F(x))(x_1 - x)^{-1} &= F'(x) + r_1.\end{aligned}$$

Then $|r_2| < \epsilon$ and $|r_1| < \epsilon$. We have

$$\begin{aligned}(F'(x) + r_2)(x_2 - x) - &(F'(x) + r_1)(x_1 - x)\\ &= (F(x_2) - F(x)) - (F(x_1) - (F(x))\\ &= F(x_2) - F(x_1) = (F'(x) + r_3)(x_2 - x_1)\\ &= (F'(x) + r_3)(x_2 - x) - (F'(x) + r_3)(x_1 - x)\end{aligned}$$

and hence $r_2(x_2 - x) - r_1(x_1 - x) = r_3(x_2 - x_1)$. But from (1) we deduce $(x_2 - x)(x_2 - x_1)^{-1} \leq 3$ and $(x_1 - x)(x_2 - x_1)^{-1} \leq 2$. So

$$|r_3| \leq |r_2|(x_2 - x)(x_2 - x_1)^{-1} + |r_1|(x_1 - x)(x_2 - x_1)^{-1} \leq 3\epsilon + 2\epsilon$$

and it follows that

$$|(F(x_2) - F(x_1))(x_2 - x_1)^{-1} - F'(x)| \leq 5\epsilon. \tag{2}$$

The same argument with x_3 in place of x_2 (starting with (1)) shows that

$$|(F(x_3) - F(x_1))(x_3 - x_1)^{-1} - F'(x)| \leq 5\epsilon. \tag{3}$$

From (2) and (3) we obtain

$$|(F(x_2) - F(x_1))(x_2 - x_1)^{-1} - (F(x_3) - F(x_1))(x_3 - x_1)^{-1}| \le 10\epsilon. \tag{4}$$

Fix $j < N$. Because b_j divides b_N, necessarily x_1 and x_3 lie between consecutive zeroes of $f_j(b_j x)$. Thus $f_j(b_j x)$ is convex on the interval (x_1, x_3) and hence

$$\frac{a_j\big(f_j(b_j x_2) - f_j(b_j x_1)\big)}{x_2 - x_1} \frac{-a_j\big(f_j(b_j x_3) - f_j(b_j x_1)\big)}{x_3 - x_1} \ge 0 \ \ (j < N). \tag{5}$$

Now fix $j > N$. The points x_1 and x_3 are zeros of $f_j(b_j x)$ because b_N divides b_j. Moreover $0 = f_j(b_j x_1) = f_j(b_j x_3) \le f_j(b_j x_2)$ and hence

$$\frac{a_j\big(f_j(b_j x_2) - f_j(b_j x_1)\big)}{x_2 - x_1} - \frac{a_j\big(f_j(b_j x_3) - f_j(b_j x_1)\big)}{x_3 - x_1} \ge 0 \ \ (j > N). \tag{6}$$

By the choice of the index N, $\dfrac{a_N\big(f_N(b_N x_2) - f_N(b_N x_1)\big)}{x_2 - x_1} = a_N b_N > 11\epsilon$ and $f_N(b_N x_3) = f_N(b_N x_1) = 0$. So,

$$\frac{a_N\big(f_N(b_N x_2) - f_N(b_N x_1)\big)}{x_2 - x_1} - \frac{a_N\big(f_N(b_N x_3) - f_N(b_N x_1)\big)}{x_3 - x_1} > 11\epsilon. \tag{7}$$

We sum (5), (6) and (7) to obtain

$$(F(x_2) - F(x_1))(x_2 - x_1)^{-1} - (F(x_3) - F(x_1))(x_3 - x_1)^{-1} > 11\epsilon. \tag{8}$$

Finally (8) is inconsistent with (4), and it follows that F has no finite right derivatives at any point. Because F is an even function, F has no left derivative at any point either. □

We have a similar result for concave functions.

Corollary 1. *Let the sequences (a_n) and (b_n) and the functions f_j be as in Theorem 1. Let $g_j = 1 - f_j$ for each index j. Then the continuous function $G(x) = \sum_{j=1}^{\infty} a_j g_j(b_j x)$ has a finite left or right derivative at no point.*

PROOF. Observe that $G(x) = \sum_{j=1}^{\infty} a_j - F(x)$ and use Theorem 1. □

Note that for any index j and any integer k, the function g_j is concave on the interval $(2k, 2k + 2)$ in Corollary 1.

From any convex nondecreasing function f mapping $[0, 1]$ onto $[0, 1]$ we can construct a nowhere differentiable function as follows. Extend f to $[0, 2]$

be reflecting the graph of f in the line $x = 1$; that is, $f(x) = f(2-x)$ for $1 < x \leq 2$. Extend f to the real line by making it periodic with period 2. Then $F(x) = \sum_{j=1}^{\infty} 2^{-j} f(2^j x)$ suffices.

We conclude with some concrete examples of nowhere differentiable functions disclosed by our work. In what follows $m-1$ is a nonnegative real number and $b-2$ is a nonnegative integer. We denote j factorial by $j!$ and e^y by $\exp(y)$. For any real number x, let $K_0(x)$ denote the distance from x to the nearest even integer, and let $K_1(x) = 1 - K_0(x)$.

Example 1. $\sum_{j=1}^{\infty}(K_1(b^j x))^m/(b^j)$, $\sum_{j=1}^{\infty}(K_1(j!x))^m/(j!)$.

Example 2. $\sum_{j=1}^{\infty}(K_0(b^j x))^{1/m}/(b^j)$, $\sum_{j=1}^{\infty}(K_0(j!x))^{1/m}/(j!)$.

Example 3. $\sum_{j=1}^{\infty}\exp(K_1(b^j x))/(b^j)$, $\sum_{j=1}^{\infty}\exp(K_1(j!x))/(j!)$.
Put $f_j(x) = [-1+\exp(K_1(x))]/(e-1)$.

Example 4. $\sum_{j=1}^{\infty}(\tan(K_1(b^j x)))/(b_j)$, $\sum_{j=1}^{\infty}(\tan(K_1(j!x)))/(j!)$
Put $f_j(x) = (\tan(K_1(x)))/(\tan 1)$.

Example 5. $\sum_{j=1}^{\infty}(\sin(K_0(b^j x)))/(b^j)$, $\sum_{j=1}^{\infty}(\sin(K_0(j!x)))/(j!)$
Put $f_j(x) = (\sin(K_0(x)))/(\sin 1)$.

Example 6. $\sum_{j=1}^{\infty}(\arctan(K_0(b^j x)))/(b^j)$, $\sum_{j=1}^{\infty}(\arctan(K_0(j!x)))/(j!)$

Example 7. $\sum_{j=1}^{\infty}(\arcsin(K_1(b^j x)))/(b^j)$, $\sum_{j=1}^{\infty}(\arcsin(K_1(j!x)))/(j!)$

Example 8. $\sum_{j=1}^{\infty}(K_0(r_j x))/2^j)$, $\sum_{j=1}^{\infty}(K_0(s_j x))/j!)$,
where $r_j = 2^j$ and $s_j = j!$ if j is a prime integer and $r_j = s_j = 0$ otherwise.

Example 9. $\sum_{j=1}^{\infty}(K_0(2^j x))/((2+5j^{-1})^j)$, $\sum_{j=1}^{\infty}(K_1(2^j x))^j/2^j)$
Use $((2+5j^{-1})^j) = 2^j((1+(5/2)j^{-1})^j)$.

Example 10. $\sum_{j=1}^{\infty}(K_0(2^j x)^{1/3})/(2^j) + \sum_{j=1}^{\infty}(K_0(2^j x)^{1/5})/(2^j)$
Put $f_j(x) = (K_0(x)^{1/3} + K_0(x)^{1/5}))/2$.

References

[1] A. Baouche, S. Dubue, *A. Unified Approach for Nondifferentiable Functions,* J. Math. Anal. Appl. **182** (1994), 134–142.

[2] E. Hewitt, K. Stromberg, *Real and Abstract Analysis,* Springer-Verlag, New York 1965.

[3] N. Kono, *On Generalized Takagi Functions,* Acta. Math. Hung., **49** (1987), 315–324.

[4] T. Takagi, *A Simple Example of a Continuous Function Without Derivative,* Phys. Math. Soc. Japan, **1** (1903), 176–177.

[5] B. L. Van der Waerden, *Ein Einfaches Beispiel Einer Nichtdifferenzierbaren Stetigen Funktion* , Math. Zeitschrift. **32** (1930), 474–475.