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CONTROLLED CONVERGENCE THEOREM FOR STRONG VARIATIONAL BANACH-VALUED MULTIPLE INTEGRALS

Abstract

In this paper, a controlled convergence theorem is proved for n-dimensional strong variational Banach-valued integrals, also referred herein as Banach-valued Multiple Integrals. The methods used in the proof for one dimensional case given in [15], in which linearization was used, cannot be applied for the higher dimensional case. Instead, we follow the ideas in [17, Chapter 5, Section 21; 4; 18].

The Henstock integral for Banach-valued functions has been discussed in [1-3, 6-10, 12, 14-15, 22-29] and the Denjoy-Dunford, Denjoy-Pettis and Denjoy-Bochner integrals have been discussed in [11, 13, 21, 30, 31]. In some cases, the Henstock's Lemma may not hold for such integrals. We adopt here the stronger version (see Definition 1.2) which is in the form of Henstock's Lemma and referred to as strong variational (HL) Banach-valued integrals. A controlled convergence theorem for these integrals in the one dimensional case was established in [15]. We shall prove a similar convergence theorem for the n-dimensional case.

1 HL and ML Integrals and $AC_\delta^{**}(X)$

In this paper, we shall consider Banach-valued functions f defined on the n-dimensional euclidean space $\mathbb{R}^n$. An *interval* $E = [\alpha, \beta]$ in $\mathbb{R}^n$, where $\alpha =$

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$(\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, is the set of points $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, 2, \ldots, n\}$. We also write $E = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots \times [\alpha_n, \beta_n]$. Any point $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_i = \alpha_i$ or β_i for some i is called a *vertex* of E. The *volume* of $E = [\alpha, \beta]$ is given by $|E| = \prod_{i=1}^{n} (\beta_i - \alpha_i)$. If $A \subset \mathbb{R}^n$ and Lebesgue measurable, then $|A|$ denotes the Lebesgue measure of A.

Definition 1.1. Let E be an interval in $\mathbb{R}^n$. A *division* π of E is a finite collection of interval-point pairs (I, x), x the associated point of interval I, such that the intervals are non-overlapping and their union is E. Two intervals I_1 and I_2 are *non-overlapping* if $|I_1 \cap I_2| = 0$. If their union is a proper subset of E, then π is said to be a partial division of E.

Let δ be a positive function defined on interval E. A division π is said to be *δ-fine* if for each interval-point pair (I, x), we have $I \subset S(x, \delta(x))$ where x is a vertex of I and $S(x, \delta(x))$ is an open sphere centered at x with radius $\delta(x)$. Suppose x is not a vertex and x may not be in I, then π is said to be McShane δ-fine.

Definition 1.2. Let $(B, \| \, \|_B)$ denote a Banach space with norm $\| \, \|_B$. A B-valued function f defined on an interval E is said to be *strongly variational* or *HL integrable* on E with primitive F (which is an additive B-valued interval function), if for every $\epsilon > 0$ there exists a positive function δ on E such that for any δ-fine division $\pi = \{(I, x)\}$ of E, we have $\sum_{\pi} \|f(x)|I| - F(I)\|_B < \epsilon$.

In the following, we may use $\| \, \|$ instead of $\| \, \|_B$ if there is no confusion between $\| \, \|_B$ and the usual norm $\| \, \|$ in $\mathbb{R}^n$, and $\mathbb{R}^+ = (0, \infty)$.

Definition 1.3. A B-valued function f defined on E is *absolutely HL integrable* if both f and $\|f(\cdot)\|_B$ are HL integrable on E.

Remark 1.4. Note that $\|f(\cdot)\|_B$ is real-valued. Thus, $\|f(\cdot)\|_B$ is HL integrable on E if and only if $\|f(\cdot)\|_B$ is Henstock-Kurzweil integrable on E.

Definition 1.5. A B-valued function f defined on an interval E is said to be *ML integrable* on E with primitive F which is an additive interval function, if for every $\epsilon > 0$ there exists a positive function δ on E such that for any δ-fine McShane division $\pi = \{(I, x)\}$ of E; i.e., $I \subset S(x, \delta(x))$ in which x does not necessarily belong to I, $\sum_{\pi} \|f(x)|I| - F(I)\|_B < \epsilon$.

It is known that if $(B, \| \, \|_B)$ is $(\mathbb{R}, | \, |)$, the ML integral is equivalent to the usual McShane integral (Lebesgue integral).

Definition 1.6. A function F is *absolutely continuous* on an interval E if for every $\epsilon > 0$ there exists $\eta > 0$ such that $\sum_{\pi} \|F(I)\| < \epsilon$ for every partial division $\pi = \{(I, x)\}$ of E with $\sum_{\pi} |I| < \eta$.

Definition 1.7. Let E be an interval in $\mathbb{R}^n$ and $X \subset E$. A B-valued function F defined on E is said to be $AC_\delta^{**}(X)$ if for every $\epsilon > 0$, there exists a positive function δ defined on E and $\eta > 0$ such that for any two δ-fine partial divisions $\pi_1 = \{(I_n, x_n)\}_{n=1}^p$ and $\pi_2 = \{(J_m, y_m)\}_{m=1}^q$ of E, with $\pi_2 \subset \pi_1$ (any interval from π_2 lies in some interval in π_1) and $x_n, y_m \in X$ ($n = 1, 2, \ldots p; m = 1, 2, \ldots, q$), we have $\sum_{\pi_1 \setminus \pi_2} |I| < \eta$ implies $\sum_{\pi_1 \setminus \pi_2} \|F(I)\| < \epsilon$ where $\pi_1 \setminus \pi_2 = \{(I_i \setminus \cup_j J_j, x_i) | J_j \subset I_i\}$. If $I = I_i \setminus \cup_j I_j$, then $F(I) = F(I_i \setminus \cup_j J_j) = F(I_i) - \sum_j F(J_j)$ and $|I| = |I_i \setminus \cup_j J_j| = |I_i| - \sum_j |J_j|$. Note that π_2 above may be void.

Furthermore, F is ACG_δ^{**} if $E = \bigcup_{i=1}^{\infty} X_i$ such that F is $AC_\delta^{**}(X_i)$ for each i.

We should highlight that $\sum_{\pi_1 \setminus \pi_2} \|F(I)\| = \sum_{\pi_1 \setminus \pi_2} \sum_i \|F(I_i) - \sum_j F(J_j)\| < \epsilon$ can not be replaced by $\sum_{\pi_1 \setminus \pi_2} \sum \|F(I'_k)\| < \epsilon$ where I'_k are intervals and $\cup I'_k = I_i \setminus \cup_j J_j$. This can be observed in the last part of the proof of Theorem 4.1. For real-valued cases, these two inequalities are equivalent.

Definition 1.8. Let E be an interval in $\mathbb{R}^n$ and $X \subset E$. A sequence $\{F_k\}$ of B-valued interval functions defined on E is said to be $UAC_\delta^{**}(X)$ if for every $\epsilon > 0$, there exists a positive function δ defined on E and $\eta > 0$, both independent of k, such that for any two δ-fine partial divisions $\pi_1 = \{(I_n, x_n)\}$ and $\pi_2 = \{(J_m, y_m)\}$ of E, with $\pi_2 \subset \pi_1$ and tags x_n, y_m in X for all n, m, we have $\sum_{\pi_1 \setminus \pi_2} |I| < \eta$ implies $\sum_{\pi_1 \setminus \pi_2} \|F_k(I)\| < \epsilon$ where $\pi_1 \setminus \pi_2 = F(I_i \setminus \cup_j J_j)$ and $|I_i \setminus \cup_j J_j|$ are defined as in Definition 1.7.

Furthermore, $\{F_k\}$ is $UACG_\delta^{**}$ if $E = \cup_{i=1}^{\infty} X_i$ such that $\{F_k\}$ is $UAC_\delta^{**}(X_i)$ for each i.

In the next sections, we always assume that f is a B-valued function where B is a Banach space with norm $\| \ \|_B$.

2 Measurability of HL Integrable Functions

In this section, we shall mention the strong measurability of HL integral functions, which is known, see [3].

Definition 2.1. A function $f : E \to B$ is called *simple* if there exist $b_1, \ldots, b_n \in B$ and $A_1, \ldots, A_n$ Lebesgue measurable subsets of E such that $f = \sum_{i=1}^n b_i \chi_{A_i}$ where $\chi_{A_i}(x) = 1$ if $x \in A_i$ and 0 otherwise.

A function $f : E \to B$ is *strongly measurable* if there exists a sequence of simple functions f_n such that $\|f_n(x) - f(x)\| \to 0$ as $n \to \infty$ for almost all $x \in E$.

Theorem 2.2. *Pettis' Measurability Theorem [5]. Let* (Ω, Σ, μ) *be a measurable space and* $(B, \| \ \|_B)$ *a Banach space. A function* $f : \Omega \to B$ *is strongly measurable if and only if*

(i) f *is essentially separably valued; i.e., there exists* $A \in \Sigma$ *with* $\mu A = 0$ *and such that* $f(\Omega \setminus A)$ *is a (norm) separable subset of* B*; and*

(ii) f *is weakly measurable; i.e., for each* $x^* \in B^*$ *(the space of all continuous linear functionals defined on* B*),* $x^* f : \Omega \to \mathbb{R}$ *is* μ*-measurable, where* $(x^* f)(w) = x^*(f(w)), w \in \Omega$.

Theorem 2.3. *[3]. If* f *is* HL *integrable on* E*, then* f *is strongly measurable on* E.

3 Properties of ML Integrals

In this section, we shall state properties of ML integrals. The proofs are standard, see [16, 17, 18, 20].

Theorem 3.1. *If* f *is* ML *integrable on* E*, then it is* HL *integrable on* E.

Theorem 3.2. *If* f *is* ML *integrable on* E*, then its primitive* F *is absolutely continuous on* E.

Theorem 3.3. *If* f *is* ML *integrable on* E*, then* $\|f(\cdot)\|$ *is McShane integrable on* E.

Theorem 3.4. *If* f *is strongly measurable on* E*, then* $\|f^N(\cdot)\|$ *is McShane integrable on* E*, where*

$$f^N(x) = \begin{cases} f(x), & \text{if } \|f(x)\| \leq N \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.5. *(i) If* f *is* HL *integrable and bounded on* E*, then* f *is absolutely* HL *integrable on* E*. (ii) If* f *is absolutely* HL *integrable on* E*, then* $\|f(\cdot)\|$ *is McShane integrable on* E.

It is proved in [6, 24, 29] that f is ML integrable on E if and only if f is Bochner integrable there. It is also known [5] that a strongly measurable function f is Bochner integrable if and only if $\|f\|$ is Lebesgue (McShane) integrable. Hence, we have

Theorem 3.6. *[6, 24, 29] If* f *is* ML *integrable on* E*, then* $\|f(\cdot)\|$ *is McShane integrable. Conversely, if* f *is strongly measurable and* $\|f(\cdot)\|$ *is McShane integrable on* E*, then* f *is* ML *integrable.*

As a result of Theorems 2.4, 3.5 and 3.6, we have the following.

Theorem 3.7. *If f is absolutely HL integrable on E, then it is ML integrable on E.*

4 Controlled Convergence Theorem

Before we go to our main result, Theorem 4.8, we shall mention some properties of HL integrable functions and their primitives. The ideas of the proofs in this section follow [17, Chapter 5, Section 21; 4; 18].

Theorem 4.1. *If f is HL integrable on E, then its primitive F is ACG_δ^{**} on E.*

Proof. Let $X = \{x \in E : \|f(x)\| \le N\}$ and

$$f_X(x) = \begin{cases} f(x), & \text{if } x \in X \\ 0, & \text{otherwise.} \end{cases}$$

Since f is HL integrable on E, it follows from Theorems 2.3, 3.4 and 3.6 that f_X is ML integrable on E. In other words for every $\epsilon > 0$, there exists $\delta : E \to \mathbb{R}^+$ such that for any δ-fine division $\pi = \{(I, x)\}$ of E, $\sum_\pi \|F(I) - f(x)|I|\| < \epsilon$ and also, for every McShane δ-fine division $\pi = \{(I, x)\}$ of E in which x does not necessarily belong to I, $\sum_\pi \|F_X(I) - f_X(x)|I|\| < \epsilon$ where F and F_X are the primitives of f and f_X, respectively.

Now, suppose we take any δ-fine partial division $\pi = \{(I, x)\}$ with $x \in X$. Then we have

$$\begin{aligned}\sum_\pi \|F(I) - F_X(I)\| &\le \sum_\pi \|F(I) - f(x)|I|\| + \sum_\pi \|F_X(I) - f_X(x)|I|\| \\ &< \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

Next, let $\pi_1 = \{(I_n, x_n)\}_{n=1}^p$ and $\pi_2 = \{(J_m, y_m)\}_{m=1}^q$ be δ-fine partial divisions with tags in X and $\pi_2 \subset \pi_1$ such that $\sum_{\pi_1 \setminus \pi_2} |I| < \eta$ where $\pi_1 \setminus \pi_2 = \{(I_i \setminus \cup_j J_j, x_i) | J_j \subset I_i\}$, $F(I_i \setminus \cup_j J_j) = F(I_i) - \sum_j F(J_j)$ and $|I_i \setminus \cup_j J_j| = |I_i| - \sum_j |J_j|$. Observe that $\pi_1 \setminus \pi_2$ is a McShane δ-fine partical division. We then obtain

$$\begin{aligned}\sum_{\pi_1 \setminus \pi_2} \|F(I)\| \le &\sum_{\pi_1} \|F(I) - F_X(I)\| + \sum_{\pi_2} \|F(J) - F_X(J)\| \\ &+ \sum_{\pi_1 \setminus \pi_2} \|F_X(I) - f_X(x)|I|\| + \|\sum_{\pi_1 \setminus \pi_2} f_X(x)|I|\| \\ < &2\epsilon + 2\epsilon + \epsilon + N\eta = 5\epsilon + N\eta.\end{aligned}$$

Consequently, for every $\epsilon > 0$, there exists $\eta > 0$ (take $\eta < \dfrac{\epsilon}{N}$), such that the condition in the definition of $AC_{\delta}^{**}(X)$ holds. Hence, F is ACG_{δ}^{**} on E. $\square$

Lemma 4.2. *Let $f_n (n = 1, 2, \dots)$ be HL integrable on E with primitives F_n. Suppose $f_n(x) \to f(x)$ a.e. in E as $n \to \infty$ and $F_n(I)$ converges to a limit function $F(I)$ for each $I \subset E$. Then, f is HL integrable on E with primitive F if and only if for every $\epsilon > 0$, there exists a function $M(x)$ taking positive integer values such that for infinitely many positive integers $m(x) \geq M(x)$, and there exists a function $\delta : E \to \mathbb{R}^+$ such that for any δ-fine division $\pi = \{(I, x)\}$ of E, we have $\sum_{\pi} \|F_{m(x)}(I) - F(I)\| < \epsilon$.*

Remark 4.3. The phrase, "for infinitely many positive integers $m(x) \geq M(x)$, there is a function $\delta : E \to \mathbb{R}^+$", means for each x, there exists a sequence $\{n_k(x)\}$ of distinct positive integers with $n_k(x) \geq M(x)$ for all k and for each $n_k(x)$, there exists function $\delta : E \to \mathbb{R}^+$.

Proof of Lemma 4.2 We shall first prove the sufficiency part. Here, we may assume $f_n(x) \to f(x)$ everywhere as $n \to \infty$ so that given any $\epsilon > 0$ and $x \in E$, there exists a positive integer $M(x)$ such that $\|f_{m(x)}(x) - f(x)\| < \epsilon$ whenever $m(x) \geq M(x)$. Since each f_n and f are HL integrable on E, there exist positive functions δ_n and δ_0 defined on E such that for any δ_n-fine division $\pi = \{(I, x)\}$ of E, $\sum_{\pi} \|F_n(I) - f_n(x)|I|\| < \epsilon 2^{-n}$ and for any δ_0-fine division $\pi = \{(I, x)\}$ of E, $\sum_{\pi} \|F(I) - f(x)|I|\| < \epsilon$. Now, for every $m(x) \geq M(x)$, put $\delta(x) = min\{\delta_{m(x)}(x), \delta_0(x)\}$. Then, for any δ-fine division $\pi = \{(I, x)\}$ of E, we obtain

$$\begin{aligned}\sum_{\pi} \|F_{m(x)}(I) - F(I)\| &\leq \sum_{\pi} \|F_{m(x)}(I) - f_{m(x)}(x)|I|\| \\ &\quad + \sum_{\pi} \|f_{m(x)}(x)|I| - f(x)|I|\| + \sum_{\pi} \|f(x)|I| - F(I)\| \\ &< \sum_{n=1}^{\infty} \epsilon 2^{-n} + \epsilon|I| + \epsilon = 2\epsilon + \epsilon|I|.\end{aligned}$$

For the necessity, using the same notations as above and in the theorems, for each x, we choose a positive integer $m(x) \geq M(x)$ such that $\|f_{m(x)}(x) - f(x)\| < \epsilon$. Modify $\delta(x)$ such that $\delta(x) \leq \delta_{m(x)}(x)$ for $x \in [a, b]$. Then, for any

δ-fine division $\pi = \{(I, x)\}$ of E, we get

$$\begin{aligned}\sum_{\pi} \|f(x)|I| - F(I)\| \leq& \sum_{\pi} \|f(x)|I| - f_{m(x)}(x)|I|\| \\ &+ \sum_{\pi} \|f_{m(x)}(x)|I| - F_{m(x)}(I)\| + \sum_{\pi} \|F_{m(x)}(I) - F(I)\| \\ <& \epsilon|I| + \sum_{n=1}^{\infty} \epsilon 2^{-n} + \epsilon = 2\epsilon + \epsilon|I|.\end{aligned}$$

Thus, f is HL integrable on E with primitive F. □

Remark 4.4. The above lemma is known as the Basic Convergence Theorem. In this lemma, it is necessary that the condition holds for all $m(x) \geq M(x)$. However, it is sufficient to have only infinitely many $m(x) \geq M(x)$ for f to be HL integrable with primitive F.

Lemma 4.5. *Let f be HL integrable on E with primitive F. If F is $AC_{\delta}^{**}(X)$ with X closed in E, then f_X is absolutely HL integrable on E where*

$$f_X(x) = \begin{cases} f(x), & \text{if } x \in X \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Suppose f is HL integrable on E with primitive F. Then, for every $\epsilon > 0$, there exists $\delta_1 : E \to \mathbb{R}^+$ such that for any δ_1-fine division $\pi = \{(I, x)\}$ of E, $\sum_{\pi} \|f(x)|I| - F(I)\| < \epsilon$. Let $A_{\delta} = \sup \sum_{\pi} \|F(I)\|$ where the supremum is taken over all δ-fine divisions $\pi = \{(I, x)\}$ of E with $x \in X$ and let $A = \inf_{\delta} A_{\delta}$ where the infimum is taken over all $\delta : E \to \mathbb{R}^+$. By the definition of $AC_{\delta}^{**}(X)$, A exists and is finite. Note that

$$\begin{aligned}\left|\sum_{\pi} \|f_X(x)\||I| - A\right| =& \left|\sum_{\substack{\pi \\ (x \in X)}} \|f(x)\||I| - A\right| \\ =& \left|\sum_{\substack{\pi \\ (x \in X)}} \|f(x)\||I| - \sum_{\substack{\pi \\ (x \in X)}} \|F(I)\|\right| + \left|\sum_{\substack{\pi \\ (x \in X)}} \|F(I)\| - A\right| \\ \leq& \sum_{\substack{\pi \\ (x \in X)}} \|f(x)|I| - F(I)\| + \left|\sum_{\substack{\pi \\ (x \in X)}} \|F(I)\| - A\right|.\end{aligned}$$

Hence, $\|f_X(\cdot)\|$ is Henstock integrable on E and so it is McShane integrable on E. On the other hand, f is strongly measurable, by Theorem 2.3. By given, X

is closed. Then X is Lebesgue measurable. Hence, f_X is strongly measurable. By Theorem 3.6, f_X is ML integrable, so it is HL integrable. Consequently, it is absolutely HL integrable. □

Lemma 4.6. *Let $f_n (n = 1, 2, \dots)$ be HL integrable on E with primitives F_n. If $\{F_n\}$ is $UAC_\delta^{**}(X)$, with X closed in E, then for every $\epsilon > 0$ there exists $\delta : E \to \mathbb{R}^+$, independent of n, such that for every δ-fine partial division $\pi = \{(I, x)\}$ with $x \in X$, we have $\sum_\pi \|F_{n,X}(I) - F_n(I)\| < \epsilon$ where*

$$f_{n,X}(x) = \begin{cases} f_n(x), & \text{if } x \in X \\ 0, & \text{otherwise} \end{cases}$$

and $F_{n,X}$ denotes the primitive of $f_{n,X}$.

Proof. Note that $F_{n,X}$ exists due to Lemma 4.5. Since $\{F_n\}$ is $UAC_\delta^{**}(X)$, then for every $\epsilon > 0$ there exist $\eta > 0$ and $\delta : E \to \mathbb{R}^+$, both independent of n, such that for any two δ-fine partial divisions $\pi_1 = \{(I_n, x_n)\}$ and $\pi_2 = \{(J_m, y_m)\}$ with $\pi_2 \subset \pi_1$ and tags in X; we have $\sum_{\pi_1 \setminus \pi_2} |I| < \eta$ implies $\sum_{\pi_1 \setminus \pi_2} \|F_n(I)\| < \epsilon$ where $\pi_1 \setminus \pi_2 = \{(I_i \setminus \cup_j J_j, x_i) | J_j \subset I_i\}$, $F(I_i \setminus \cup_j J_j) = F(I_i) - \sum_j F(J_j)$ and $|I_i \setminus \cup_j J_j| = |I_i| - \sum_j |J_j|$. Now, choose an open set $G \supset X$ such that $|G - X| < \eta$. Further, assume that $S(x, \delta(x)) \subset G$ when $x \in X$ and $S(x, \delta(x)) \subset E \setminus X$ when $x \notin X$. Since both f_n and $f_{n,X}$ are HL integrable on E, there exists $\delta_n : E \to \mathbb{R}^+$ with $\delta_n(x) \leq \delta(x)$ for all x such that for any δ_n-fine division $\pi = \{(I, x)\}$ of E, we have $\sum_\pi \|F_n(I) - f_n(x)|I|\| < \epsilon$ and $\sum_\pi \|F_{n,X}(I) - f_{n,X}(x)|I|\| < \epsilon$.

Consider any δ-fine partial division $\pi = \{(I, x)\}$ with $x \in X$. Construct a δ_n-fine division for each I in π and denote the new division by $\pi_1 = \{(J, y)\}$. Split π_1 into π_2 and π_3 so that π_2 contains intervals with tags in X and π_3 otherwise. Note that $J \cap X = \phi$ and $F_{n,X}(J) = 0$ when J belongs to π_3. Also, $\sum_{\pi_3} |J| = \sum_{\pi_1 \setminus \pi_2} |J| < \eta$ and we use the definition of $UAC_\delta^{**}(X)$. Then, we obtain

$$\begin{aligned}\sum_{\pi}\|F_{n,X}(I)-F_n(I)\| &= \sum_{\pi_1}\|F_{n,X}(\cup J)-F_n(\cup J)\| \\ &= \sum_{\pi_2}\|F_{n,X}(\cup J)-F_n(\cup J)\| \\ &\quad + \sum_{\pi_3}\|F_{n,X}(\cup J)-F_n(\cup J)\| \\ &\le \sum_{\pi_2}\|F_{n,X}(J)-f_n(y)|J|\| \\ &\quad + \sum_{\pi_2}\|f_n(y)|J|-F_n(J)\| + \sum_{\pi_3}\|F_{n,X}(\cup J)\| \\ &\quad + \sum_{\pi_3}\|F_n(\cup J)\| < 2\epsilon + 0 + \epsilon \ = 2\epsilon. \qquad \square\end{aligned}$$

Lemma 4.7. *Let X be closed in E. For each n, let*

$$f_{n,X}(x) = \begin{cases} f_n(x), & \text{if } x \in X \\ 0, & \text{otherwise}\end{cases}$$

and let

$$f_X(x) = \begin{cases} f(x), & \text{if } x \in X \\ 0, & \text{otherwise.}\end{cases}$$

Suppose

(i) *$f_{n,X}(x) \to f_X(x)$ a.e. in E as $n \to \infty$ where each $f_{n,X}$ is HL integrable on E; and*

(ii) *the primitives $F_{n,X}$ of $f_{n,X}$ are $UAC^{**}_{\delta}(X)$.*

Then, f_X is HL integrable on E with primitive F_X and $F_{n,X}(E) \to F_X(E)$ as $n \to \infty$.

Furthermore, for every $\epsilon > 0$ there is an integer N such that for every $n \ge N$, there exists $\delta : E \to \mathbb{R}^+$ such that for any δ-fine division $\pi = \{(I, x)\}$ of E we have $\sum_{\pi}\|F_{n,X}(I) - F_X(I)\| < \epsilon$.

Proof. In view of (ii), we have, for every $\epsilon > 0$ there exist $\delta : E \to \mathbb{R}^+$ and $\eta > 0$, both independent of n, such that for any δ-fine partial division $\pi = \{(I, x)\}$ of E with tags in X, $\sum_{\pi}|I| < \eta$ implies $\sum_{\pi}\|F_{n,X}(I)\| < \epsilon$. Since (i) holds, by Egoroff's theorem (Recall that $f_{n,X}$ are strongly measurable.)

there exists an open set G with $|G| < \eta$ such that $\|f_{n,X}(x) - f_{m,X}(x)\| < \epsilon$ for $n, m \geq N$ and $x \in E \setminus G$. We may assume that if $x \in G$, then for any δ-fine division $\pi = \{(I, x)\}$ we have $I \subset G$.

Since $f_{n,X}$ is HL integrable on E, there exists $\delta_n : E \to \mathbb{R}^+$ such that $\delta_n(x) \leq \delta(x)$ for each x and for any δ_n-fine partial division $\pi = \{(I, x)\}$ of E, we have $\sum_\pi \|f_{n,X}(x)|I| - F_{n,X}(I)\| < \epsilon 2^{-n}$. Now, for any δ-fine division π of E and any n, $m \geq N$, take a δ_n-fine and δ_m-fine division π_1 finer than π and let $\pi_1 = \pi_2 \cup \pi_3$ where π_2 consists of interval-point pairs (I, x) with $x \in E \setminus G$ and π_3 otherwise. Then for any $n, m \geq N$, we have

$$\begin{aligned}\sum_\pi \|F_{n,X}(I) - F_{m,X}(I)\| &\leq \sum_{\pi_1} \|F_{n,X}(I) - F_{m,X}(I)\| \\ &\leq \sum_{\pi_2} \|F_{n,X}(I) - f_n(x)|I|\| + \sum_{\pi_2} \|f_n(x) - f_m(x)\||I| \\ &\quad + \sum_{\pi_2} \|f_m(x)|I| - F_{m,X}(I)\| \\ &\quad + \sum_{\pi_3} \|F_{n,X}(I)\| + \sum_{\pi_3} \|F_{m,X}(I)\| < 4\epsilon + \epsilon|E|.\end{aligned}$$

Thus, $F_{n,X}(I)$ converges in $(B, \| \, \|)$ for any interval I in E, say, $F_{n,X}(I) \to F_X(I)$ as $n \to \infty$ for any interval I in E and the required inequality holds.

It remains to show that f_X is HL integrable on E with primitive F_X. Applying the above inequality, we can find a subsequence $F_{n(j),X}$ of $F_{n,X}$ such that for any δ-fine partial division π of E, we have $\sum_\pi \|F_{n(j),X}(I) - F_X(I)\| < \epsilon 2^{-j}$ for $j = 1, 2, \ldots$. Then, put $M(x) = n(1)$ for each x and let $m(x)$ take values in $\{n(j); j \geq 1\}$. Consequently, for any δ-fine division $\pi = \{(I, x)\}$ of E,

$$\sum_\pi \|F_{m(x),X}(I) - F_X(I)\| \leq \sum_{j=1}^{\infty} \epsilon 2^{-j} = \epsilon.$$

Hence, the conditions in Lemma 4.2 are satisfied. □

Theorem 4.8 (Controlled Convergence Theorem)**.** *Suppose the following conditions are satisfied:*

(i) $f_n(x) \to f(x)$ *a.e. in* E *as* $n \to \infty$ *where each* f_n *is* HL *integrable on* E*;*

(ii) the primitives F_n *of* f_n *are* $UACG_\delta^{**}$*; and*

(iii) the primitives F_n *converge uniformly on* E*.*

Then, f *is also* HL *integrable on* E *and* $\int_E f_n \to \int_E f$ *as* $n \to \infty$*.*

PROOF. Since $\{F_n\}$ is $UACG_\delta^{**}$, we can choose $X_1 \subset X_2 \subset \dots$ such that $\cup_{i=1}^\infty X_i = E$ and $\{F_n\}$ is $UAC_\delta^{**}(X_i)$ for each i. Choose $Y_i \subset X_i$ such that Y_i is closed and $|X_i \setminus Y_i| < 2^{-i}$. Then $|E \setminus \bigcup_{i=1}^\infty Y_i| = |\bigcap_{i=1}^\infty (E \setminus Y_i)| \leq |E \setminus X_i| + |X_i \setminus Y_i|$ for each i. That is, $E = S \cup (\cup_{i=1}^\infty Y_i)$, where S is of measure zero. In view of $UAC_\delta^{**}(X_i)$ note that the following strong Lusin condition holds for F_n and F. That is, for every set $Z \subset X_i$ of measure zero and for every $\epsilon > 0$, there exists $\gamma_i : E \to \mathbb{R}^+$, independent of n, such that for any γ_i-fine partial division $\pi = \{(I,x)\}$ with $x \in Z$, we have $\sum_\pi \|F_n(I)\| < \epsilon 2^{-i}$ for each n and $\sum_\pi \|F(I)\| < \epsilon 2^{-i}$. In particular, we take $Z = S \cap X_i$ for $i = 1, 2, \dots$.

Suppose we let $X = Y_i$ and take the same $\gamma_i : E \to \mathbb{R}^+$ as above, independent of n. By Lemma 4.6, if $\pi = \{(I,x)\}$ is any γ_i-fine partial division with $x \in X$, then $\sum_\pi \|F_{n,X}(I) - F_n(I)\| < \epsilon 2^{-i}$ for each n and hence, $\sum_\pi \|F_X(I) - F(I)\| < \epsilon 2^{-i}$. Since $\{F_n\}$ is $UAC_\delta^{**}(X_i)$, we can also conclude that $\{F_{n,X}\}$ is $UAC_\delta^{**}(X)$ where $X = Y_i$. Then, by Lemma 4.5, Lemma 4.7 for each i, j, there exists an integer $n = n(i,j)$ and $\delta_n : E \to \mathbb{R}^+$ such that for any δ_n-fine division $\pi = \{(I,x)\}$ of E, we have $\sum_\pi \|F_{n,X}(I) - F_X(I)\| < \epsilon 2^{-i-j}$.

We may assume that for each i, $\{F_{n(i,j)}\}$ is a subsequence of $\{F_{n(i-1,j)}\}$. Now consider $f_{n(j)} = f_{n(j,j)}$ in place of the original sequence $\{f_n\}$. It is necessary to consider i and j, where i runs over all Y_i and j runs over all $F_{n(j)}$.

Now put $M(x) = n(i)$ when $x \in Y_i - (Y_1 \cup \dots \cup Y_{i-1})$ where $i = 1, 2, \dots$ and $Y_0 = \emptyset$. Let $m(x)$ take values in $\{n(j); j \geq i\}$ when $m(x) \geq M(x) = n(i)$. Next, put $\delta(x) = min\{\delta_{n(j)}(x), \gamma_i(x)\}$ when $x \in Y_i - (Y_1 \cup \dots \cup Y_{i-1})$ and $m(x) = n(j)$; and $\delta(x) = \gamma_i(x)$ when $x \in S \cap \{X_i - (X_1 \cup \dots \cup X_{i-1})\}$ where $X_0 = \emptyset$. For any δ-fine division $\pi = \{(I,x)\}$ of E, let

$$\pi_1 = \{(I,x) \in \pi | x \in \bigcup_{i=1}^\infty Y_i\} \text{ and } \pi_2 = \{(I,x) \in \pi | x \in S\}.$$

If we let $m = m(x)$ and $X = Y_i$ when $x \in Y_i - (Y_1 \cup \dots \cup Y_{i-1})$, then we have

$$\begin{aligned}\sum_\pi \|F_m(I) - F(I)\| \leq &\sum_{\pi_1} \|F_m(I) - F_{m,X}(I)\| + \sum_{\pi_1} \|F_{m,X}(I) - F_X(I)\| \\ &+ \sum_{\pi_1} \|F_X(I) - F(I)\| + \sum_{\pi_2} \|F_m(I)\| + \sum_{\pi_2} \|F(I)\| < 5\epsilon.\end{aligned}$$

Note that the second term on the right-hand side of the above inequality is equal to

$$\sum_i \sum_j \sum_{\substack{x \in Y_i \setminus (Y_1 \cup \dots \cup Y_{i-1}) \\ m(x) = n(j,j)}} \|F_{m(x),Y_i}(I) - F_{Y_i}(I)\|.$$

If $m(x) = n(j,j)$ and $x \in Y_i \setminus (Y_1 \cup \cdots \cup Y_{i-1})$, then $j \geq i$. Here $n(j,j) = n(i, k_{(j)})$ for some $k_{(j)}$. Therefore, the above sum is less than $\sum_i \sum_j \epsilon 2^{-i-k_{(j)}}$. The conclusion follows from Lemma 4.2. □

References

[1] B. Bongiorno, L Di Piazza and K. Musial, *An alternate approach to the McShane integral*, Real Anal. Exchange **25** (1999/2000), 829–848.

[2] S. Cao, *The Henstock Integral for Banach-Valued Functions*, SEA Bull. Math. **16** (1992), 35–40.

[3] S. Cao, *On the Henstock-Bochner integral*, SEA Bill. Math. Special Issue (1993) 1–3.

[4] C. T. Seng and L. P. Yee, *Nonabsolute integration using Vitali covers*, New Zealand Journal of Mathematics, **23** (1994), 25–36.

[5] J. Diestel and J. J. Uhl Jr., *Vector measure, Maths Surveys*, **No. 15**, Amer. Math. Soc. 1977.

[6] J. C. Feauveau, *A generalized Riemann integral for Banach-valued functions*, Real Anal. Exchange **25** (1999/2000) 919–930.

[7] J. C. Feauveau, *Approximation theorems for generalized Riemann integrals*, Real Anal. Exchange **26** (2000/2001), 471–484.

[8] M. Federson, *The Fundamental theorem of Calculus for multidimensional Banach space-valued Henstock vector integrals*, Real Anal. Exchange, **25** (2000), 469–480.

[9] D. H. Fremlin and J. Mendoza, *On the integration of vector-valued functions*, Illinois J. Math. **38** (1994), 127–147.

[10] D. H. Fremlin, *The generalized McShane integral*, Illinois J. Math. **39** (1995), 39–67.

[11] J. L. Gamez and J. Mendoza, *On Denjoy-Dunford and Denjoy-Pettis integrals*, Studia Math. **130** (1998), 115–133.

[12] R. A. Gordon, *The McShane integral of Banach-valued functions*, Illinois J. Math. **34** (1990), 557–567.

[13] R. A. Gordon, *The Denjoy extension of the Bochner, Pettis, and Dunford integrals*, Studia Math. **92** (1989), 73–91.

[14] R. Henstock, *Generalized integrals of vector-valued functions*, Proc. Lond. Math. Soc. **19** (1969), 509–536.

[15] L. I. Paredes and C. T. Seng, *Controlled Convergence Theorem for Banach-valued HL integrals*, Scientiae Mathematicae Japonicae (On Line), **6** (2002), 261–271.

[16] I. P. Lorna, L. P. Yee and C. T. Seng, *Banach-valued HL Multiple integrals*, Research Report 788, Department of Mathematics, NUS, May 2002.

[17] L. P. Yee, *Lanzhou Lecture on Henstock Integration*, World Scientific, Singapore, 1989.

[18] L. P. Yee, *Multiple integrals*, unpublished.

[19] L. P. Yee and C. T. Seng, *A Short Proof of the Controlled Convergence Theorem for Henstock Integrals*, Bull. Lord. Math. Soc. **19** (1987), 60–62.

[20] L. P. Yee and R. Vyborny, *The Integral: An Easy Approach after Kurzweil and Henstock*, Cambrige University Press, 2000.

[21] T. J. Morrison, *A note on the Denjoy integrability of abstractly-valued functions*, Proc. Amer. Math. Soc. **61** (1976), 385–386.

[22] S. Nakanishi, *The Henstock integral for functions with values in nuclear spaces and the Henstock Lemma*, J. Math. Study, **27** (1994), 133–141.

[23] S. Nakanishi, *Riemann type integrals for functions with values in nuclear spaces and their properties*, Math. Japonica, **47** (1998), 367–396.

[24] L. Di. Piazza and K. Musial, *A characterization of variationally McShane integrable Banach-space valued functions*, Illinois J Math. **45** (2001), 279–289.

[25] R. M. Rey and P. Y. Lee, *A representation theorem for the space of Henstock-Bochner integrable functions*, SEA Bull. Math. Special Issue (1993), 129–136.

[26] V. A. Skvortsov and A. P. Solodov, *A variational integral for Banach-valued functions*, Real Anal. Exchange **24** (1998/1999), 799–806.

[27] A. P. Solodov, *On conditions of differentiability almost everywhere for absolutely continuous Banach-valued function*, Moscow Univ. Math. Bull **54** (1999), 29–32.

[28] C. Swartz, *A gliding hump property for the Henstock-kurzweil integral*, SEA Bull. Math. **22** (1998), 437–443.

[29] W. Congxin and Y. Xiaobo, *A Riemann-type definition of the Bochner integral*, J. Math. Study **27** (1994), 32–36.

[30] Y. Guoju, L. P. Yee and W. Congxin, *Convergence theorems of the Denjoy-Bochner*, Denjoy-Pettis and Denjoy-Dunford integrals, SEA Bull. Math. **23** (1999), 135–143.

[31] Y. Guoju and A. Tianqing, *On Henstock-Dunford and Henstock-Pettis integrals*, Int. J. Math. Math. Sci. **25** (2001), 467–478.