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THE $\mathcal{I}$-ALMOST CONSTANT CONVERGENCE OF SEQUENCES OF REAL FUNCTIONS

Abstract

Let T be an infinite set and $\mathcal{I}$ be a fixed ideal on T. We introduce and study the notion of almost constant convergence of sequences $\{f_t : t \in T\}$ of real functions with respect to the ideal $\mathcal{I}$. This notion generalizes discrete convergence, transfinite convergence, and ω_2-convergence. In particular, we consider the question when a given family of functions (e.g., continuous, Baire class 1, Borel measurable, Lebesgue measurable, or functions with the Baire property) is closed with respect to this kind of convergence.

1 Notation

We will use the standard terminology and notation. In particular, ordinal numbers will be identified with the set of their predecessors and cardinal numbers with the initial ordinals. Thus, the first infinite cardinal ω is identified with the set of natural numbers. The cardinality of the set $\mathbb{R}$ of real numbers is denoted by $\mathfrak{c}$. The cardinality of a set A is denoted by $|A|$. For any infinite set X and a cardinal κ by $[X]^{<\kappa}$ we denote the family of all subsets of X of the size less than κ. In a similar way we define sets $[X]^{\kappa}$ and $[X]^{\leq\kappa}$.

Let $\mathcal{J}$ be a σ-ideal of subsets of $\mathbb{R}$. We shall denote by $\mathcal{B}_{\mathcal{J}}$ the σ-algebra generated by $\mathcal{J}$ and by the Borel sets $\mathcal{B}$ in $\mathbb{R}$. The σ-ideals that are the most interesting for us are the ideals $\mathcal{N}$ of Lebesgue measure zero subsets of $\mathbb{R}$ and $\mathcal{M}$ of meager subsets of $\mathbb{R}$. By $\mathcal{J}^c$ we will denote the family of all sets of the form $\mathbb{R} \setminus J$, where $J \in \mathcal{J}$. A σ-ideal $\mathcal{J}$ is c.c.c. when there is no uncountable family of pairwise disjoint sets in $\mathcal{B}_{\mathcal{J}} \setminus \mathcal{J}$.

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We will need also some cardinal numbers connected with an ideal $\mathcal{I}$ of subsets of T:

$$\mathrm{add}\,(\mathcal{I}) = \min\{|\mathcal{I}_0| \colon \mathcal{I}_0 \subset \mathcal{I} \ \&\ \bigcup \mathcal{I}_0 \notin \mathcal{I}\},$$

$$\mathrm{cov}\,(\mathcal{I}) = \min\{|\mathcal{I}_0| \colon \mathcal{I}_0 \subset \mathcal{I} \ \&\ \bigcup \mathcal{I}_0 = T\},$$

$$\mathrm{non}\,(\mathcal{I}) = \min\{|T_0| \colon T_0 \subset T \ \&\ T_0 \notin \mathcal{I}\}.$$

Let T be an infinite set and $\mathcal{I}$ be a proper ideal of subsets of T containing all singletons. For $\{x_t \colon t \in T\} \subset \mathbb{R}$ and $x \in \mathbb{R}$ we say that $\{x_t \colon t \in T\}$ converges $\mathcal{I}$-a.c. to x ($\mathcal{I}$-converges, $x_t \to_{\mathcal{I}} x$) if $\{t \in T \colon x_t \neq x\} \in \mathcal{I}$. A sequence $\{f_t \colon t \in T\} \subset \mathbb{R}^{\mathbb{R}}$ $\mathcal{I}$-converges to $f \in \mathbb{R}^{\mathbb{R}}$ ($f_t \to_{\mathcal{I}} f$) if $f_t(x) \to_{\mathcal{I}} f(x)$ for each $x \in \mathbb{R}$. (The idea of such convergence has been suggested to us by Dr. Kazimierz Wiśniewski.)

Remark 1.1. (1) If $x_t \to_{\mathcal{I}} x$ and $x_t \to_{\mathcal{I}} x'$, then $x = x'$.

(2) If $f_t \to_{\mathcal{I}} f$ and $f_t \to_{\mathcal{I}} f'$, then $f = f'$.

(3) If $\mathcal{I} \subset \mathcal{J}$ and $f_t \to_{\mathcal{I}} f$, then $f_t \to_{\mathcal{J}} f$.

If $T = \kappa$, $\mathcal{I}$ is the ideal of all subsets of κ of size less than κ, and $f_t \to_{\mathcal{I}} f$, then we say that $\{f_\alpha \colon \alpha < \kappa\}$ κ-converges to f. The following examples of κ-convergence of real functions have been considered in the bibliography.

- If $\kappa = \omega$, then the ω-convergence is equivalent to the discrete convergence. (See [CL].)
- If $\kappa = \omega_1$, then ω_1-a.c. convergence means the transfinite convergence. (See [WS].)
- The ω_2-convergence was considered in [PK].

Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. By $\mathrm{LIM}_{\mathcal{I}}(\mathcal{F})$ we denote the $\mathcal{I}$-a.c. closure of $\mathcal{F}$; i.e., the family of all $\mathcal{I}$-a.e. limits of sequences of functions from $\mathcal{F}$. In a similar way we define the family $\mathrm{LIM}_{\kappa}(\mathcal{F})$, i.e., the κ-closure of $\mathcal{F}$. Following [JL] we will use the following notation.

(1) $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is $\mathcal{I}$-closed if $\mathrm{LIM}_{\mathcal{I}}(\mathcal{F}) = \mathcal{F}$.

(2) $\{f_t \colon t \in T\}$ is $\mathcal{I}$-almost constant if there is a $I \in \mathcal{I}$ such that $f_t = f_{t'}$ for any $t, t' \in T \setminus I$.

(3) $\mathcal{F} \in \mathbb{R}^{\mathbb{R}}$ is strictly $\mathcal{I}$-closed if for all $\{f_t \colon t \in T\} \subset \mathcal{F}$, if $f_t \to_{\mathcal{I}} f$ for some $f \in \mathbb{R}^{\mathbb{R}}$, then $\{f_t \colon t \in T\}$ is $\mathcal{I}$-almost constant.

(4) A set $D \subset \mathbb{R}$ is determining for a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ if for all $f, g \in \mathcal{F}$, $f = g$ whenever $f|D = g|D$.

2 Properties

Remark 2.1. (1) If $\{f_t\colon t \in T\}$ is $\mathcal{I}$-almost constant, then it is $\mathcal{I}$-convergent.

(2) If $\mathcal{I} \subset \mathcal{J}$ and $\mathcal{F}$ is $\mathcal{J}$-closed, then $\mathcal{F}$ is $\mathcal{I}$-closed.

Lemma 2.2. *Suppose $f_t \to_{\mathcal{I}} f$ and $E \subset \mathbb{R}$.*

(1) *If $|E| < \operatorname{add}(\mathcal{I})$, then $\{f_t|E\colon t \in T\}$ is $\mathcal{I}$-almost constant.*

(2) *If $|E| < \operatorname{cov}(\mathcal{I})$, then there is $T_0 \notin \mathcal{I}$ with $f_t|E = f|E$ for each $t \in T_0$.*

PROOF. Let $I_x = \{t \in T\colon f_t(x) \neq f(x)\}$ for $x \in E$. Set $I = \bigcup_{x \in E} I_x$. Then $f_t|E = f|E$ for each $t \notin I$. Moreover, $I \in \mathcal{I}$ whenever $\operatorname{add}(\mathcal{I}) > |E|$, and $T_0 = T \setminus I \notin \mathcal{I}$ if $\operatorname{cov}(\mathcal{I}) > |E|$. □

Corollary 2.3. *If $\operatorname{cov}(\mathcal{I}) > \mathfrak{c}$, then each family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is $\mathcal{I}$-closed.*

Theorem 2.4. *If a family $\mathcal{F}$ has a determining set D with $|D| < \operatorname{add}(\mathcal{I})$, then $\mathcal{F}$ is strictly $\mathcal{I}$-closed.*

PROOF. Assume $\{f_t\colon t \in T\} \subset \mathcal{F}$, $f_t \to_{\mathcal{I}} f$, $D \subset \mathbb{R}$ is a determining set for $\mathcal{F}$ with $|D| < \operatorname{add}(\mathcal{I})$. By Lemma 2.2.1, there is a $I \in \mathcal{I}$ such that $f_t|D = f_{t'}|D$ for $t, t' \in T \setminus I$. Since D is determining, $f_t = f_{t'}$ and hence $\{f_t\colon t \in T\}$ is $\mathcal{I}$-almost constant. □

Corollary 2.5. (1) *If $\mathcal{I}$ is a σ-ideal and $\mathcal{F}$ is a family of functions having a countable determining set, then $\mathcal{F}$ is strictly $\mathcal{I}$-closed.*

(2) *The family $\mathcal{C}(\mathbb{R}, \mathbb{R})$ of all continuous functions is strictly $\mathcal{I}$-closed for any proper σ-ideal $\mathcal{I}$.*

(3) *Let $\mathcal{I}_0$ be the ideal of finite sets and T be uncountable. Then the family $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is $\mathcal{I}_0$-closed.*

Theorem 2.6. *Assume $\mathcal{F}$ satisfies the condition*

$$(\forall g \notin \mathcal{F})\ (\exists D \in [\mathbb{R}]^{<\operatorname{cov}(\mathcal{I})})\ (\forall f \in \mathcal{F})\ f|D \neq g|D.$$

Then $\mathcal{F}$ is $\mathcal{I}$-closed.

PROOF. Suppose $\mathcal{F}$ is not $\mathcal{I}$-closed. There exist $\{f_t\colon t \in T\} \subset \mathcal{F}$, $g \notin \mathcal{F}$ such that $f_t \to_{\mathcal{I}} g$. Since $g \notin \mathcal{F}$, there is $D \subset \mathbb{R}$ such that $|D| < \operatorname{cov}(\mathcal{I})$ and $f_t|D \neq g|D$ for every $t \in T$, contrary to Lemma 2.2.2. □

Corollary 2.7. *If $\operatorname{cov}(\mathcal{I}) > \omega$, then the first class of Baire B_1 is $\mathcal{I}$-closed.*

PROOF. Suppose $g \notin B_1$. Then there exist a non-empty perfect set $P \subset \mathbb{R}$ and reals $a < b$ such that the sets $A = P \cap [g < a]$ and $B = P \cap [g > b]$ are both dense in P. Let D be a countable set such that both $D \cap A$ and $D \cap B$ are dense in P. Then $f|\overline{D}$ has no continuity points for every $f \in \mathbb{R}^{\mathbb{R}}$ with $f|D = g|D$; so $f|D = g|D$ for no $f \in B_1$. By Theorem 2.6, the class B_1 is $\mathcal{I}$-closed. □

Theorem 2.8. *Suppose $\mathcal{I}$ is an ideal of subsets of T and $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$.*

(i) *If $\mathrm{add}(\mathcal{I}) = \mathrm{cov}(\mathcal{I}) = \kappa$, then $\mathrm{LIM}_{\mathcal{I}}(\mathcal{F}) \subset \mathrm{LIM}_{\kappa}(\mathcal{F})$.*

(ii) *If moreover $\mathcal{I}$ has a basis of size $\leq \kappa$, then $\mathrm{LIM}_{\mathcal{I}}(\mathcal{F}) = \mathrm{LIM}_{\kappa}(\mathcal{F})$.*

PROOF. Note that the assumption $\mathrm{add}(\mathcal{I}) = \mathrm{cov}(\mathcal{I}) = \kappa$ implies that there is a sequence $\{I_\alpha\colon\ \alpha < \kappa\} \subset \mathcal{I}$ such that $I_\alpha \subset I_\beta$ for $\alpha < \beta < \kappa$, $I_{\alpha+1} \setminus I_\alpha \neq \emptyset$ for each α, and $T = \bigcup_{\alpha<\kappa} I_\alpha$. If moreover $\mathcal{I}$ has a basis of size $\leq \kappa$, then we can assume that for each $I \in \mathcal{I}$ there is $\alpha < \kappa$ with $I \subset I_\alpha$. (A basis of $\mathcal{I}$ is a subfamily $\mathcal{I}_0 \subset \mathcal{I}$ such that for each $I \in \mathcal{I}$ there is $I_0 \in \mathcal{I}_0$ with $I \subset I_0$.)

(i) Let $\{f_t\colon\ t \in T\} \subset \mathcal{F}$ and $f_t \to_{\mathcal{I}} f$. For each $\alpha < \kappa$ fix $t_\alpha \in I_{\alpha+1} \setminus I_\alpha$. Then $\{f_{t_\alpha}\colon\ \alpha < \kappa\}$ κ-converges to f; so $f \in \mathrm{LIM}_{\kappa}(\mathcal{F})$. Indeed, for $x \in \mathbb{R}$ we have $f_t(x) \to_{\mathcal{I}} f(x)$. Thus $I = \{t \in T\colon\ f_t(x) \neq f(x)\} \in \mathcal{I}$. Consequently, $I \subset I_\alpha$ for some $\alpha < \kappa$ and $f_{t_\beta}(x) = f(x)$ for $\beta > \alpha$; so $f_{t_\alpha} \to_\kappa f(x)$.

(ii) Let $f \in \mathrm{LIM}_{\kappa}(\mathcal{F})$. There exists $\{f_\alpha\colon\ \alpha < \kappa\} \subset \mathcal{F}$ such that $f_\alpha \to_\kappa f$. For $t \in T$ define $f_t = f_\alpha$ for $t \in I_{\alpha+1} \setminus I_\alpha$. Then $f_t \to_{\mathcal{I}} f$. In fact, for $x \in \mathbb{R}$ there is $\alpha < \kappa$ with $f_\beta(x) = f(x)$ for $\beta > \alpha$. Thus $\{t \in T\colon\ f_t(x) \neq f(x)\} \subset I_{\alpha+1}$. □

Theorem 2.9. *Assume $\omega \leq \kappa < |T|$ and $\mathcal{I}_\kappa = [T]^{\leq\kappa}$. If the family $\{f_t\colon t \in T\}$ is $\mathcal{I}_\kappa$-almost constant, then it possesses a determining set D with $|D| \leq \kappa$.*

PROOF. Fix $I \in \mathcal{I}_\kappa$ with $f_t = f_{t'}$ for $t, t' \notin I$. If $I = \emptyset$, then each set D is determining for $\{f_t\colon t \in T\}$. If $I \neq \emptyset$, fix $t_0 \in T \setminus I$. For every pair $(t, t') \in (I \cup \{t_0\})^2$ such that $f_t \neq f_{t'}$ choose $x_{(t,t')} \in \mathbb{R}$ such that $f_t(x_{(t,t')}) \neq f_{t'}(x_{(t,t')})$. Let D be the set of all such $x_{(t,t')}$. Then $|D| \leq \kappa^2 = \kappa$ and D is determining for $\{f_t\colon t \in T\}$. □

Corollary 2.10. *Suppose κ is a cardinal number such that $\omega \leq \kappa < |T|$. A family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is strictly $\mathcal{I}_\kappa$-closed iff each $\mathcal{I}_\kappa$-convergent subfamily $\{f_t\colon t \in T\}$ has a determining set D such that $|D| \leq \kappa$.*

PROOF. The implication "⇒" follows from Theorem 2.9. The implication "⇐" is a consequence of Theorem 2.4, because $\mathrm{add}(\mathcal{I}_\kappa) = \kappa^+$. □

Theorem 2.11. *Suppose $\mathfrak{c}$ is a regular cardinal. A family $\mathcal{F}$ is $\mathfrak{c}$-closed iff*

$$(\forall g \notin \mathcal{F})\ (\exists D \in [\mathbb{R}]^{<\mathfrak{c}})\ (\forall f \in \mathcal{F})\ f|D \neq g|D$$

PROOF. "⇐" is a consequence of Theorem 2.6, because $\operatorname{cov}([\mathfrak{c}]^{<\mathfrak{c}}) = \mathfrak{c}$.

"⇒" Assume $g \in \mathbb{R}^{\mathbb{R}}$ and for each $D \in [\mathbb{R}]^{<\mathfrak{c}}$ there is $f \in \mathcal{F}$ such that $f|D = g|D$. Let $\mathbb{R} = \{x_\xi : \xi < \mathfrak{c}\}$, $D_\alpha = \{x_\xi : \xi < \alpha\}$ for $\alpha < \mathfrak{c}$, and $T = \mathfrak{c}$. For each $\alpha < \mathfrak{c}$ we can choose an $f_\alpha \in \mathcal{F}$ such that $f_\alpha|D_\alpha = g|D_\alpha$. Then $\{f_\alpha : \alpha < \mathfrak{c}\} \to_{\mathfrak{c}} g$ and therefore $g \in \mathcal{F}$. □

3 Approximately Continuous Functions and Derivatives

In this and the next section we will need the following notations. Let T be an uncountable set and let $\mathcal{I}$ be an ideal on T. We say that the family $\{A_t : t \in T\} \subset \mathcal{P}(\mathbb{R})$ is point-$\mathcal{I}$ disjoint if $\{t \in T : x \in A_t\} \in \mathcal{I}$ for each $x \in \mathbb{R}$ (cf. [DF], p. 7).

Fix an ideal $\mathcal{J}$ of subsets of $\mathbb{R}$. Following [CiL] (cf. [RZ]), we shall say that a set $D \subset T \times \mathbb{R}$ is a 0-1 set provided $D^y \in \mathcal{I}$ for every $y \in \mathbb{R}$ and $\mathbb{R} \setminus D_t \in \mathcal{J}$ for every $t \in T$. We will consider the following statements.

$S_{\mathcal{I},\mathcal{J}}$: There exists a 0-1 set in $T \times \mathbb{R}$.

$T_{\mathcal{I},\mathcal{J}}$: No family $\{A_t : t \in T\} \subset \mathcal{J}^c$ is point-$\mathcal{I}$ disjoint.

$T^*_{\mathcal{I},\mathcal{J}}$: No family $\{A_t : t \in T\} \subset \mathcal{B}_{\mathcal{J}} \setminus \mathcal{J}$ is point-$\mathcal{I}$ disjoint.

Lemma 3.1. *The following properties are equivalent:*

(i) $T_{\mathcal{I},\mathcal{J}}$;

(ii) $\neg S_{\mathcal{I},\mathcal{J}}$;

(iii) for any family $\{A_t : t \in T\} \subset \mathcal{J}^c$ there is $T_0 \subset \mathcal{P}(T) \setminus \mathcal{I}$ with $\bigcap_{t \in T_0} A_t \neq \emptyset$.

(iv) there is no family $\{J_t : t \in T\} \subset \mathcal{J}$ such that $\bigcup_{t \in T_0} J_t = \mathbb{R}$ for each $T_0 \notin \mathcal{J}$.

PROOF. The equivalence $(i) \Leftrightarrow (ii)$ follows from [RZ, Proposition 1.5]. $(ii) \Leftrightarrow (iii)$ and $(iii) \Leftrightarrow (iv)$ are obvious. □

Given a semigroup G of Borel functions from $\mathbb{R}$ to $\mathbb{R}$ and an ideal $\mathcal{J}$ on $\mathbb{R}$ we say that $\mathcal{J}$ is G-invariant if $g^{-1}(J) \in \mathcal{J}$ for any $g \in G$ and $J \in \mathcal{J}$; and $\mathcal{J}$ is G-ergodic when $\bigcup_{g \in G} g^{-1}(A) \in \mathcal{J}^c$ for every $A \in \mathcal{B}_{\mathcal{J}} \setminus \mathcal{J}$. Note that the σ-ideals $\mathcal{N}$ and $\mathcal{M}$ are invariant and ergodic under the group of rational translations.

Lemma 3.2. *Suppose $\mathcal{J}$ is a σ-ideal on $\mathbb{R}$ which is invariant and ergodic under some countable semigroup G. Then statements $T_{\mathcal{I},\mathcal{J}}$ and $T^*_{\mathcal{I},\mathcal{J}}$ are equivalent.*

PROOF. (Cf. [RZ, Lemma 2.8].) The implication "$\Leftarrow$" is clear. To prove "$\Rightarrow$" fix $\{A_t\colon t \in T\} \subset \mathcal{B}_{\mathcal{J}} \setminus \mathcal{J}$. For $t \in T$ put $B_t = \bigcup_{g\in G} g(A_t)$. Then $\{B_t\colon t \in T\} \subset \mathcal{J}^c$; so there is $T_0 \in \mathcal{P}(T) \setminus \mathcal{J}$ such that $\bigcap_{t\in T_0} B_t \neq \emptyset$. Fix $x \in \bigcap_{t\in T_0} B_t$. For each $t \in T_0$ there is $g_t \in G$ with $x \in g_t(A_t)$. Let $T_g = \{t \in T_0\colon g_t = g\}$. Since $T_0 \notin \mathcal{J}$, $\mathcal{J}$ is a σ-ideal and G is countable, there is $g \in G$ such that $T_g \notin \mathcal{J}$. Then $x \in g(\bigcap_{t\in T_g} A_t)$; so $\bigcap_{t\in T_g} A_t$ is non-empty. □

We say that a family $\mathcal{F} \subset B_1$ is $\mathcal{J}$-regular when:

(1) for each $J \in \mathcal{J}$ and for every Baire 1 function $f\colon \mathbb{R} \to \mathbb{R}$ there exists $\tilde{f} \in \mathcal{F}$ with $f|J = \tilde{f}|J$;

(2) for each $f_0, f_1 \in \mathcal{F}$, if $[f_0 \neq f_1] \in \mathcal{J}$, then $f_0 = f_1$.

Note that the families $\mathcal{C}_d$ of all approximately continuous functions and D of all derivatives are $\mathcal{N}$-regular. (See e.g., [MD, Theorems 3 and 4].)

Theorem 3.3. *Suppose $\mathcal{F} \subset B_1$ is $\mathcal{J}$-regular and $\mathcal{I}$ is an ideal on T. Then $T^*_{\mathcal{I},\mathcal{J}} \Rightarrow \mathrm{LIM}_{\mathcal{I}}(\mathcal{F}) = \mathcal{F}$.*

PROOF. Suppose $\{f_t\colon t \in T\} \subset \mathcal{F}$, $f_t \to_{\mathcal{I}} f$. For each $T \in T$ put $A_t = [f_t \neq f]$. By Corollary 2.7, $f \in B_1$; so $\{A_t\colon t \in T\} \subset \mathcal{B}_{\mathcal{J}}$.

First, note that $T_0 = \{t \in T\colon A_t \in \mathcal{J}\} \notin \mathcal{I}$. In fact, suppose $T_0 \in \mathcal{I}$. For any $t \in T$ set $B_t = \mathbb{R}$ if $t \in T_0$ and $B_t = A_t$ for $t \notin T_0$. By $T^*_{\mathcal{I},\mathcal{J}}$, there exists $T_1 \in \mathcal{P}(T) \setminus \mathcal{I}$ such that $\bigcap_{t\in T_1} B_t \neq \emptyset$. Then $T_2 = T_1 \setminus T_0 \notin \mathcal{I}$ and $\bigcap_{t\in T_2} A_t = \bigcap_{t\in T_1} B_t \neq \emptyset$. Fix $x \in \bigcap_{t\in T_2} A_t$. Then $T_2 \subset \{t \in T\colon f_t(x) \neq f(x)\} \notin \mathcal{I}$, contrary to $f_t \to_{\mathcal{I}} f$.

Thus $T_0 \notin \mathcal{I}$ and for any $t_0, t_1 \in T_0$ we have $[f_{t_0} \neq f_{t_1}] \subset [f_{t_0} \neq f] \cup [f_{t_1} \neq f] \in \mathcal{I}$; so $f_{t_0} = f_{t_1}$. This implies that $f = f_t$ for every $t \in T_0$; so $f \in \mathcal{F}$. □

Theorem 3.4. *Suppose $\mathcal{F} \subset B_1$ is $\mathcal{J}$-regular and $\mathcal{I}$ is an ideal on T. Then $\neg T_{\mathcal{I},\mathcal{J}} \Rightarrow \mathrm{LIM}_{\mathcal{I}}(\mathcal{F}) = B_1$.*

PROOF. Assume $\neg T_{\mathcal{I},\mathcal{J}}$. Then there exists a point-$\mathcal{I}$ disjoint family $\{A_t\colon t \in T\} \subset \mathcal{J}^c$. This means that $I_x = \{t \in T\colon x \in A_t\} \in \mathcal{I}$ for each $x \in \mathbb{R}$. Put $A^c_t = \mathbb{R} \setminus A_t$ for any $t \in T$. Corollary 2.7 implies $\mathrm{LIM}_{\mathcal{I}}(\mathcal{F}) \subset B_1$; so we have to show that $B_1 \subset \mathrm{LIM}_{\mathcal{I}}(\mathcal{F})$. Fix $f \in \mathcal{B}_1$. For any $t \in T$ there exists $f_t \in \mathcal{F}$ with $f_t|A^c_t = f|A^c_t$. Then $f_t \to_{\mathcal{I}} f$. Indeed, for any $x \in \mathbb{R}$, if $t \notin I_x$, then $x \in A^c_t$; so $f(x) = f_t(x)$, and therefore $\{t\colon f_t(x) \neq f(t)\} \subset I_x \in \mathcal{I}$. □

Corollary 3.5. *Suppose $\mathcal{J}$ is a σ-ideal on $\mathbb{R}$ which is invariant and ergodic under some countable semigroup G, and $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is $\mathcal{J}$-regular. Then for any σ-ideal $\mathcal{I}$ on T we have:*

- $\mathrm{LIM}_{\mathcal{I}}(\mathcal{F}) = \mathcal{F}$ *if $T_{\mathcal{I},\mathcal{J}}$ holds;*
- $\mathrm{LIM}_{\mathcal{I}}(\mathcal{F}) = B_1$ *if $T_{\mathcal{I},\mathcal{J}}$ does not hold.*

In particular, for families of approximately continuous functions and derivatives we have the following results.

Corollary 3.6. *For any σ-ideal $\mathcal{I}$ on T we have:*

- $\mathrm{LIM}_{\mathcal{I}}(\mathcal{C}_d) = \mathcal{C}_d$ *and* $\mathrm{LIM}_{\mathcal{I}}(D) = D$ *if $T_{\mathcal{I},\mathcal{N}}$ holds;*
- $\mathrm{LIM}_{\mathcal{I}}(\mathcal{C}_d) = \mathrm{LIM}_{\mathcal{I}}(D) = B_1$ *if $T_{\mathcal{I},\mathcal{N}}$ does not hold.*

Theorem 3.7. *Suppose $|T| > \omega$ and $\mathcal{I} = [T]^{<\omega}$. Then $T^*_{\mathcal{I},\mathcal{J}}$ holds for every c.c.c. σ-ideal $\mathcal{J}$ on $\mathbb{R}$.*

PROOF. This is a consequence of [DF, Lemma 1E(b)]. □

Corollary 3.8. *Suppose $|T| > \omega$ and $\mathcal{I} = [T]^{<\omega}$. Then both families $\mathcal{C}_d$ and D are $\mathcal{I}$-closed.*

Now assume that $\omega \leq \lambda < |T| < \mathfrak{c}$ and $\mathcal{I}_\lambda = [T]^{\leq\lambda}$. Then $\mathcal{I}$ is a σ-ideal and we again have only two possibilities.

- $\mathrm{LIM}_{\mathcal{I}_\lambda}(\mathcal{C}_d) = \mathcal{C}_d$ and $\mathrm{LIM}_{\mathcal{I}_\lambda}(D) = D$ if $T_{\mathcal{I}_\lambda,\mathcal{N}}$ holds;
- $\mathrm{LIM}_{\mathcal{I}_\lambda}(\mathcal{C}_d) = \mathrm{LIM}_{\mathcal{I}_\lambda}(D) = B_1$ if $T_{\mathcal{I}_\lambda,\mathcal{N}}$ does not hold.

In particular, if $\lambda = \omega$, then both possibilities may happen. (See [MD] in the case $T = \omega_1$.) First, it is easy to observe that

- if $\mathrm{cov}(\mathcal{N}) > \omega_1$, then $T_{\mathcal{I}_\omega,\mathcal{N}}$ holds.

(See [RZ, Lemma 2.7], or [DF, Lemma 1E(c)].) Now,

- If $\mathrm{cov}(\mathcal{N}) = \omega_1 = |T|$, then $T_{\mathcal{I}_\omega,\mathcal{N}}$ does not hold.

Indeed, in this case $\mathbb{R} = \bigcup_{\alpha<\omega_1} N_\alpha$, where $N_\alpha \in \mathcal{N}$ for $\alpha < \omega_1$ and $N_\alpha \subset N_\beta$ if $\alpha \leq \beta$. Then the family of complements of N_α's is a counterexample for $T_{\mathcal{I}_\omega,\mathcal{N}}$. The statement $T_{\mathcal{I}_\omega,\mathcal{N}}$ may does not hold also when $|T| > \omega_1$. This is a consequence of the following result. (Recall that a set $L \subset \mathbb{R}$ is a Lusin set if $|L \cap M| \leq \omega$ for $M \in \mathcal{M}$.)

Theorem 3.9. (I. Recław) *Suppose $\omega \le \lambda < |T| \le \mathfrak{c}$. If there exists a Lusin set L of the size $|T|$, then $S_{\mathcal{I}_\lambda,\mathcal{N}}$ holds.*

Proof. Fix a partition $\mathbb{R} = G \cup H$, $H \cap G = \emptyset$, $G \in \mathcal{M}$ and $H \in \mathcal{N}$. Let L be a Lusin set and $|L| = |T| = \kappa$. Set $B = \{(x,y)\colon\ x \in L \ \&\ y \in x + H\}$. Then $B_x \in \mathcal{N}$ for all $x \in L$ and for each $y \in \mathbb{R}$ we have $B^y = \{x \in L\colon x \in y - H\} \in [L]^{\le\omega} \subset \mathcal{I}_\lambda$. Thus $L \times \mathbb{R} \setminus B$ is a 0-1 set in $L \times \mathbb{R}$. □

Note that it is consistent that $\mathfrak{c} > \omega_1$ and there exists a Lusin set of the size continuum. (Indeed, the generic set of reals in a Cohen extension is a Lusin set. See e.g., [AM], p. 205.)

4 Measurable Functions

Let $\mathcal{J}$ be a c.c.c. σ-ideal. Throughout this section $\mathfrak{M}_\mathcal{J}$ will denote the family of all $\mathcal{B}_\mathcal{J}$ measurable functions.

For an ideal $\mathcal{J}$ set

$$\mathrm{shr}\,(\mathcal{J}) = \min\{\lambda\colon\ (\forall A \notin \mathcal{J})\ (\exists A_0 \subset A)\ (A_0 \notin \mathcal{J} \ \&\ |A_0| \le \lambda)\}$$

$$\mathrm{non}_{\,l}(\mathcal{J}) = \min\{\lambda\colon\ (\forall A \in \mathcal{B}_\mathcal{J} \setminus \mathcal{J})\ (\exists A_0 \subset A)\ (A_0 \notin \mathcal{J} \ \&\ |A_0| \le \lambda)\}$$

$$\mathrm{cov}_{\,l}(\mathcal{J}) = \min\{|\mathcal{I}|\colon\ (\mathcal{I} \subset \mathcal{J}) \ \&\ (\bigcup \mathcal{I} \in \mathcal{B}_\mathcal{J} \setminus \mathcal{J})\}$$

Note that for each ideal $\mathcal{J}$ (containing all singletons) we have $\mathrm{add}\,(\mathcal{J}) \le \mathrm{non}\,(\mathcal{J}) \le \mathrm{non}_{\,l}(\mathcal{J}) \le \mathrm{shr}\,(\mathcal{J})$ and $\mathrm{add}\,(\mathcal{J}) \le \mathrm{cov}_{\,l}(\mathcal{J}) \le \mathrm{cov}\,(\mathcal{J})$. (See [RZ].)

Lemma 4.1. ([RZ, Theorem 2.16, Claim 2]) *For any $g \in \mathbb{R}^\mathbb{R} \setminus \mathfrak{M}_\mathcal{J}$ there exists a set $D_g \subset \mathbb{R}$ such that $|D_g| \le \mathrm{shr}\,(\mathcal{J})$ and $f|D_g = g|D_g$ for no $f \in \mathfrak{M}_\mathcal{J}$.*

Theorem 4.2. *If $\mathrm{shr}\,(\mathcal{J}) < \mathrm{cov}\,(\mathcal{I})$, then the family $\mathfrak{M}_\mathcal{J}$ is $\mathcal{I}$-closed.*

Proof. This is an easy consequence of Lemmas 1.2 and 4.1. □

Theorem 4.3. $\neg T_{\mathcal{I},\mathcal{J}} \ \Rightarrow\ \mathrm{LIM}_\mathcal{I}(\mathfrak{M}_\mathcal{J}) = \mathbb{R}^\mathbb{R}$

Proof. Let $\{J_t\colon t \in T\} \subset \mathcal{J}$ be a counterexample on $T_{\mathcal{I},\mathcal{J}}$; so $\bigcap_{t \in T_0}(\mathbb{R} \setminus J_t) = \emptyset$ for each $T_0 \in \mathcal{P}(T) \setminus \mathcal{I}$. Fix $f \in \mathbb{R}^\mathbb{R}$ and define the function f_t for $t \in T$ by the formula $f_t|J_t = f|J_t$ and $f_t = 0$ on $\mathbb{R} \setminus J_t$. Then $f_t \in \mathfrak{M}_\mathcal{J}$ and $f_t \to_\mathcal{I} f$. □

Theorem 4.4. *If $\mathrm{add}\,(\mathcal{J}) = \mathrm{cov}\,(\mathcal{J}) = \mathrm{add}\,(\mathcal{I}) = \mathrm{cov}\,(\mathcal{I})$, then the condition $T_{\mathcal{I},\mathcal{J}}$ does not hold.*

Proof. Let $\mathrm{add}\,(\mathcal{J}) = \kappa$. There exist sequences $\{I_\alpha\colon \alpha < \kappa\} \subset \mathcal{I}$, $\{J_\alpha\colon \alpha < \kappa\} \subset \mathcal{J}$ such that $I_\alpha \subset I_\beta$ and $J_\alpha \subset J_\beta$ for $\alpha < \beta$, and $T = \bigcup_{\alpha<\kappa} I_\alpha$, $\mathbb{R} = \bigcup_{\alpha<\kappa} J_\alpha$. Then $B = \bigcup_{\alpha<\kappa} I_\alpha \times (\mathbb{R} \setminus J_\alpha)$ is a 0-1 set. □

Theorem 4.5. *Assume a σ-ideal $\mathcal{J}$ satisfies the condition $(*)$ every $B \in \mathcal{B}_{\mathcal{J}}$ contains a subset $S \notin \mathcal{B}_{\mathcal{J}}$. If $\operatorname{add}(\mathcal{J}) = \operatorname{cov}(\mathcal{I})$, then the family $\mathfrak{M}_{\mathcal{J}}$ is not $\mathcal{I}$-closed.*

PROOF. Let $\operatorname{cov}(\mathcal{I}) = \kappa$, $T = \bigcup_{\alpha<\kappa} I_\alpha$, $I_\alpha \in \mathcal{I}$, and sets I_α are pairwise disjoint. Since $\operatorname{add}(\mathcal{J}) = \kappa$, there exists a sequence $\{J_\alpha : \alpha < \kappa\} \subset \mathcal{J}$ such that $J_\alpha \subset J_\beta$ if $\alpha < \beta$ and $S = \bigcup_{\alpha<\kappa} J_\alpha \notin \mathcal{J}$. By $(*)$, we can assume that $S \notin \mathcal{B}_{\mathcal{J}}$. Let f_t be the characteristic function of the set J_α for $t \in I_\alpha$. Then $f_t \in \mathfrak{M}_{\mathcal{J}}$ for $t \in T$, and $\{f_t : t \in T\}$ $\mathcal{I}$-converges to the characteristic function of S; so the limit is not $\mathcal{B}_{\mathcal{J}}$-measurable. □

Lemma 4.6. ([RZ, Lemma 2.4.(i)]) *There exists a set $D \subset \mathbb{R}$ such that $|D| \leq \operatorname{non}_l(\mathcal{J})$ and $D \cap B \neq \emptyset$ for every $B \in \mathcal{B}_{\mathcal{J}} \setminus \mathcal{J}$.*

Theorem 4.7. *Suppose $\{f_t : t \in T\} \subset \mathfrak{M}_{\mathcal{J}}$, $f_t \to_{\mathcal{J}} f$.*

(1) *If $\operatorname{non}_l(\mathcal{J}) < \operatorname{add}(\mathcal{I})$, then there exists $I \in \mathcal{I}$ such that $[f_t \neq f_{t'}] \in \mathcal{J}$ for each $t, t' \in T \setminus I$.*

(2) *If $\operatorname{non}_l(\mathcal{J}) < \operatorname{cov}(\mathcal{I})$ and moreover $f \in \mathfrak{M}_{\mathcal{J}}$, then $\{t \in T : [f_t \neq f] \in \mathcal{J}\} \notin \mathcal{I}$.*

PROOF. Let D be as in Lemma 4.6, i.e., $|D| \leq \operatorname{non}_l(\mathcal{J})$ and D meets each set $B \in \mathcal{B}_{\mathcal{J}} \setminus \mathcal{J}$. Observe that if $f|D = g|D$ for some $f, g \in \mathfrak{M}_{\mathcal{J}}$, then $[f \neq g] \in \mathcal{J}$. Now, to prove (1) apply Lemma 2.2.(1) to find a $I \in \mathcal{I}$ such that $f_t|D = f_{t'}|D$ for each $t, t' \notin I$. Thus for $t, t' \notin I$ we have $[f_t \neq f_{t'}] \in \mathcal{J}$.

To prove (2) use Lemma 2.2.(2) and observe that $T_0 = \{t \in T : f_t|D = f|D\} \notin \mathcal{I}$; so $[f_t \neq f] \in \mathcal{J}$ for all $t \in T_0$. □

Corollary 4.8. *Suppose $\{f_t : t \in T\} \subset \mathfrak{M}_{\mathcal{J}}$, $f_t \to_{\mathcal{I}} f$. If $\operatorname{shr}(\mathcal{J}) < \operatorname{cov}(\mathcal{I})$, then $\{t \in T : [f_t \neq f] \in \mathcal{J}\} \notin \mathcal{I}$.*

PROOF. This is a consequence of Theorems 4.2 and 4.7.2 (and the inequality $\operatorname{non}_l(\mathcal{J}) \leq \operatorname{shr}(\mathcal{J})$). □

We will say that a σ-ideal is separable, if there exists a countable family $\mathcal{D} \subset \mathcal{B}_{\mathcal{J}} \setminus \mathcal{J}$ such that for each $B \in \mathcal{B}_{\mathcal{J}} \setminus \mathcal{J}$ there is $D \in \mathcal{D}$ with $D \setminus B \in \mathcal{J}$. Note that the ideal $\mathcal{M}$ is separable.

Theorem 4.9. *Assume $\{f_t : t \in T\} \in \mathfrak{M}_{\mathcal{J}}$, $f \in \mathfrak{M}_{\mathcal{J}}$ and $\mathcal{J}$ is a separable σ-ideal such that $\operatorname{cov}_l(\mathcal{J}) > \operatorname{shr}(\mathcal{I})$. If $f_t \to_{\mathcal{I}} f$, then $\{t \in T : [f_t \neq f] \notin \mathcal{J}\} \in \mathcal{I}$.*

PROOF. Suppose $T_0 = \{t \in T : [f_t \neq f] \notin \mathcal{J}\} \notin \mathcal{I}$. Choose $T_1 \subset T_0$, $|T_1| \leq \operatorname{shr}(\mathcal{I})$, $T_1 \notin \mathcal{I}$. Fix $t \in T_1$. Since $f_t, f \in \mathfrak{M}_{\mathcal{J}}$, there is $D_t \in \mathcal{D}$ such that $D_t \setminus [f_t \neq f] \in \mathcal{J}$. By the countability of $\mathcal{D}$, there exists a set $D \in \mathcal{D}$

such that $T_2 = \{t \in T_1 \colon D_t = D\} \notin \mathcal{I}$. Since $|T_2| \le \operatorname{shr}(\mathcal{I}) < \operatorname{cov}_l(\mathcal{J})$, $A = D \setminus \bigcup_{t\in T_2}[f_t = f] \notin \mathcal{J}$. Thus for $x \in A$ we have $T_2 \subset \{t \colon f_t(x) \neq f(x)\} \notin \mathcal{I}$, contrary to $f_t \to_{\mathcal{I}} f$. □

5 Problems

Fix an infinite set T. Let $\mathcal{I}_0$ denote the ideal of finite subsets of T.

Theorem 5.1. *For any σ-algebra $\mathcal{A}$ the family $\mathfrak{M}_{\mathcal{A}}$ of all $\mathcal{A}$-measurable functions is $\mathcal{I}_0$-closed.*

PROOF. Let $\{f_t \colon t \in T\} \subset \mathfrak{M}_{\mathcal{A}}$, $f \to_{\mathcal{I}_0} f$. Fix a one-to-one sequence $\{t_n \colon n < \omega\} \subset T$. Then the sequence $(f_{t_n})_n$ converges discretely (so, pointwise converges) to f. Thus f is $\mathfrak{M}_{\mathcal{A}}$-measurable. □

Corollary 5.2. (1) *The families: $\mathcal{B}$-of all Borel functions, $\mathfrak{M}_{\mathcal{M}}$-of all functions having the Baire property, $\mathfrak{M}_{\mathcal{N}}$-of all Lebesgue measurable functions are $\mathcal{I}_0$-closed.*

(2) *For each $\alpha < \omega_1$, if f is the $\mathcal{I}_0$-limit of a sequence $\{f_t \colon t \in T\} \subset B_\alpha$, then $f \in B_{\alpha+1}$.*

Observe that if T is countable, then no Baire class α is $\mathcal{I}_0$-closed. (See [CL].) But if $|T| > \omega$, then by Corollary 2.5.3, $\mathcal{C}(\mathbb{R},\mathbb{R})$ is $\mathcal{I}_0$-closed, and by Corollary 2.7, B_1 is $\mathcal{I}_0$-closed.

Theorem 5.3. *It is consistent that $T = \mathfrak{c} = \omega_2$ and the class B_2 is $\mathcal{I}_0$-closed.*

PROOF. This is a consequence of [PK, Theorem 5], where is shown that it is consistent that $\mathfrak{c} = \omega_2$ and the class B_2 is ω_2-closed. Indeed, we have $B_2 \subset \operatorname{LIM}_{\mathcal{I}_0}(B_2) \subset \operatorname{LIM}_{\omega_2}(B_2) = B_2$. □

Problem 1. Suppose T is uncountable and $\alpha > 1$. Is the class B_α $\mathcal{I}_0$-closed?

Now, consider the ideal $\mathcal{I}_\omega$ of countable sets and the family $\mathcal{B}$ of all Borel functions. It is easy to observe that if $|T| = \omega_1$ then $\operatorname{LIM}_{\mathcal{I}_\omega}(\mathcal{B}) = \mathbb{R}^{\mathbb{R}}$. We don't know what happens when $\omega_1 < |T| \le \mathfrak{c}$. Notice that this problem is equivalent to the following one.

Problem 2. Does there exist $B \subset T \times \mathbb{R}$ with the following properties:

(1) $B_t \in \mathcal{B}$ for each $t \in T$;

(2) for each $y \in \mathbb{R}$, either $B^y \in \mathcal{I}_\omega$ or $\mathbb{R} \setminus B^y \in \mathcal{I}_\omega$.

(3) $\{y \in \mathbb{R} \colon B^y \in \mathcal{I}_\omega\} \notin \mathcal{B}$?

Problem 3. Suppose that $\omega \le \lambda < \lambda^+ < \kappa \le \mathfrak{c}$, $|T| = \kappa$ and $\mathcal{I}_\lambda = [T]^{\le\lambda}$. Are the classes $\mathcal{B}$, B_α, $\alpha > 1$, $\mathcal{I}_\lambda$-closed?

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