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MARTIN'S AXIOM AND MAXIMAL ORTHOGONAL FAMILIES

Abstract

It is shown that Martin's Axiom for σ-centered partial orders implies that every maximal orthogonal family in $\mathbb{R}^{\mathbb{N}}$ is of size $2^{\aleph_0}$.

For $x, y \in \mathbb{R}^{\mathbb{N}}$ define the inner product $\langle x, y\rangle = \sum_{n=0}^{\infty} x(n)y(n)$ in the obvious way noting, however, that it may not be finite or, indeed, may not even exist. Nevertheless, if $\langle x, y\rangle$ converges and equals 0, then x and y are said to be orthogonal. A family $X \subseteq \mathbb{R}^{\mathbb{N}}$ will be said to be maximal orthogonal if any two of its elements are orthogonal and for every $y \in \mathbb{R}^{\mathbb{N}} \setminus X$, there is some $x \in X$ which is not orthogonal to y. In [1], various results are established which indicate a similarity between maximal orthogonal familes and maximal almost disjoint families of sets of integers. There is a key distinction though. While no infinite, countable family of subsets of the integers can be maximal almost disjoint, there are countably infinite maximal orthogonal families. In [1], the question of whether it is possible to construct a maximal orthogonal family of cardinality $\aleph_1$ without assuming any extra set theoretic axioms was posed. The following theorem establishes that this is not possible.

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Theorem 1. *Martin's Axiom for σ-centered partial orders implies that every uncountable, maximal orthogonal family in $\mathbb{R}^{\mathbb{N}}$ is of size $2^{\aleph_0}$.*

Proof. Let $X \subseteq \mathbb{R}^{\mathbb{N}}$ be an uncountable orthogonal family of cardinality less than $2^{\aleph_0}$. It will be shown that it can be extended to a larger orthogonal

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family. Before continuing, some notation and terminology will be established. Whenever a topology on $\mathbb{R}^{\mathbb{N}}$ is mentioned, this will refer to the usual product topology. Basic neighborhoods of $\mathbb{R}^{\mathbb{N}}$ will be taken to be sets of the form

$$\mathcal{V} = \{x \in \mathbb{R}^{\mathbb{N}} : (\forall i \leq k)(a_i < x(i) < b_i)\}$$

where the end points a_i and b_i are all rational. The integer k will be said to be the length of $\mathcal{V}$ and will be denoted by $l(\mathcal{V})$ (this violates the usual notation), while $\max_{i \leq k}(b_i - a_i)$ will be referred to as the width of $\mathcal{V}$ and will be denoted by $w(\mathcal{V})$.

Let $\mathbb{P}$ be the set of all triples $p = (\mathcal{V}, W, \eta)$ such that:

- $\mathcal{V}$ is a basic open subset of $\mathbb{R}^{\mathbb{N}}$
- W is a finite subset of X
- $0 < \eta \in \mathbb{Q}$ and $\eta \geq w(\mathcal{V})$
- if U is the set of all $x \in X \cap \mathcal{V}$ such that $|\sum_{i=0}^{k} w(i)x(i)| < \eta$ for any k greater than the length of $\mathcal{V}$ and any $w \in W$, then $|U| \geq \aleph_1$.

Define $\mathcal{V}(p) = \mathcal{V}$, $W(p) = W$, $\eta(p) = \eta$ and $U(p) = U$. Define $p \leq_{\mathbb{P}} p'$ if and only if

- $\mathcal{V}(p) \subseteq \mathcal{V}(p')$
- $W(p) \supseteq W(p')$
- $\eta(p) \leq \eta(p')$
- and for each $t \in \mathcal{V}(p)$ and each integer j such that $l(\mathcal{V}(p')) < j \leq l(\mathcal{V}(p))$ the inequality $|\sum_{i=0}^{j} t(i)w(i)| < \eta(p')$ holds for for every $w \in W(p')$.

Observe that $\mathbb{P}$ is σ-centered since, given any finite set of conditions $\mathcal{P} \subseteq \mathbb{P}$ such that $\mathcal{V}(p') = \mathcal{V}$ and $\eta(p) = \eta$ for each $p \in \mathcal{P}$, the triple $(\mathcal{V}, \bigcup_{p \in \mathcal{P}} W(p), \eta)$ is a lower bound for all of them.

It will be shown that the following sets are dense in $\mathbb{P}$:

- $A(x) = \{p \in \mathbb{P} : x \in W(p)\}$
- $B(x) = \{p \in \mathbb{P} : x \notin \overline{\mathcal{V}(p)}\}$
- $C(m) = \{p \in \mathbb{P} : \eta(p) < 1/m\}$
- $D(m) = \{p \in \mathbb{P} : l(\mathcal{V}(p)) > m\}$

where $x \in X$ and $m \in \mathbb{N}$. Given that this assertion can be established, let $G \subseteq \mathbb{P}$ be a filter such that $G \cap A(x) \cap B(x) \cap C(m) \cap D(m) \neq \emptyset$ for each $x \in X \cup \{\vec{0}\}$, where $\vec{0}$ denotes the constant zero function, and $m \in \mathbb{N}$. Using that $G \cap C(m) \cap D(m) \neq \emptyset$ for each $m \in \mathbb{N}$, let $x_G \in \mathbb{R}^{\mathbb{N}}$ be the unique sequence such that $x_G \in \mathcal{V}(p)$ for each $p \in G$. Observe that $x_G \neq x$ if $G \cap B(x) \neq \emptyset$. Hence $x_G \notin X$.

To see that $\langle x_G, x \rangle = 0$ for each $x \in X$, let $x \in X$ and $\epsilon > 0$ be given and choose $k \in \mathbb{N}$ such that $1/k < \epsilon$. Then, select $p \in G \cap A(x) \cap C(k)$. Now, given any j greater than the length of $\mathcal{V}(p)$ use that $G \cap D(j) \neq \emptyset$ to choose $p' \in G \cap D(j)$ such that $p' \leq_{\mathbb{P}} p$. It is an immediate consequence of the definition of $\leq_{\mathbb{P}}$ and the facts that $x_G \in \mathcal{V}(p')$, $x \in W(p) \subseteq W(p')$ and $l(\mathcal{V}(p)) \leq j \leq l(\mathcal{V}(p'))$ that $|\sum_{i=0}^{j} x_G(i)x(i)| < \eta(p) < 1/k < \epsilon$. Since ϵ was arbitrary, it follows that $\langle x_G, x \rangle = 0$.

So all that remains to be shown is that the sets $A(x)$, $B(x)$, $C(m)$ and $D(m)$ are dense for each $x \in X$ and $m \in \mathbb{N}$.

Claim 1. $C(m) \cap D(m)$ is dense for any $m \in \mathbb{N}$. Moreover, for any $p \in \mathbb{P}$ and any uncountable $Z \subseteq U(p)$ it is possible to find $q \leq p$ in $C(m) \cap D(m)$ such that $Z \cap U(q)$ is uncountable.

PROOF. Let $p \in \mathbb{P}$ and $Z \subseteq U(p)$ be uncountable. For each $x \in Z \setminus W(p)$, there is some $k(x) \geq m$ such that $|\sum_{i=0}^{j} w(i)x(i)| < 1/m$ for each $j \geq k(x)$ and $w \in W(p)$. Choose k such that $U = \{x \in Z : k(x) = k\}$ is uncountable. Since $\mathbb{R}^{\omega}$ has a countable base it is possible to find $x \in U$ which is a complete accumulation point of U. By the definition of $x \in U(p)$ it follows that $|\sum_{i=0}^{m} w(i)x(i)| < \eta(p)$ for every $w \in W(p)$ and $l(\mathcal{V}(p)) < m \leq k$. Therefore, there is some $\delta > 0$ such that for any sequence $\{t_j\}_{j=0}^{k}$, $|x(j) - t_j| < \delta$ for each $j \leq k$ and the inequality $|\sum_{i=0}^{m} w(i)t_i| < \eta(p)$ holds for every $w \in W(p)$, $l(\mathcal{V}(p)) < m \leq k$.

Let $\mathcal{W}$ be a neighborhood of x with length k but of width less than the minimum of δ and $1/m$. Let $q = (\mathcal{W}, W(p), 1/m)$ and note that $U \cap \mathcal{W} \subseteq U(q) \cap Z$ and $U \cap \mathcal{W}$ is uncountable since x was chosen to be a complete accumulation point of U. Hence $q \in \mathbb{P}$ is as required. It is also easily verified that the choice of δ guarantees that $q \leq_{\mathbb{P}} p$ and that $q \in C(m) \cap D(k) \subseteq C(m) \cap D(m)$. □

Claim 2. $A(x)$ is dense for any $x \in X$.

PROOF. Let $p \in \mathbb{P}$. Choose some integer $m \geq l(\mathcal{V}(p))$ such that if Z is defined to be the set of all $z \in U(p)$, $|\sum_{i=0}^{j} z(i)x(i)| < \eta(p)$ for each $j \geq m$, then $|Z| \geq \aleph_1$. Use the claim about the density of $C(m) \cap D(m)$ to find $q \leq p$ such that $Z \cap U(q)$ is uncountable and $l(\mathcal{V}(q)) \geq m$. It follows that there are uncountably many $z \in X \cap \mathcal{V}(q)$ such that $|\sum_{i=0}^{j} z(i)x(i)| < \eta(p)$ for each

$j \geq l(\mathcal{V}(q)) \geq m$. This, in conjunction with the fact that $p \in \mathbb{P}$, implies that $|\sum_{i=0}^{j} z(i)w(i)| < \eta(p)$ for each $j \geq l(\mathcal{V}(q))$ and $w \in W(p) \cup \{x\}$. Therefore, if q' is defined to be $(\mathcal{V}(q), W(p) \cup \{x\}, \eta(p))$, then $q' \in \mathbb{P} \cap A(x)$ and $q' \leq_{\mathbb{P}} p$. $\square$

Claim 3. $B(x)$ is dense for any $x \in X$.

PROOF. Let $p \in \mathbb{P}$. For each $z \in U(p) \setminus \{x\}$ choose a pair of integers $(m(z), e(z))$ such that $|x(m(z)) - z(m(z))| > 1/e(z)$ and let (m, e) be some pair of integers such that the set $Z = \{z \in U(p) : (m(z), e(z)) = (m, e)\}$ is uncountable. Let k be the maximum of m and e. It follows that for each $z \in Z$ no neighborhood $\mathcal{W}$ of z of length k and width $1/k$ contains x. Use the claim about the density of $C(k) \cap D(k)$ to find $q \leq p$ such that $Z \cap U(q) \neq \emptyset$ and $l(\mathcal{V}(q)) \geq k$. It follows $x \notin \mathcal{V}(q)$ and so $q \in B(x)$. $\square$

This concludes the proofs of the claims and, hence, the proof of the theorem. $\square$

References

[1] A. W. Miller and J. Steprāns, *Orthogonal families of real sequences*, J. Symbolic Logic, **63** (1998), 29–49.