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MEASURES OF ABERRANCY

Abstract

In much the same way that curvature provides a measure of the nonlinearity of a curve, aberrancy provides a measure of the noncircularity of a curve. Curvature can be defined in several ways, but they all result in the same formula. In contrast, different approaches to aberrancy yield different formulas. We consider a number of different approaches to aberrancy and show that there are some interesting and unexpected connections between them. This is a largely unexplored concept that can be used to generate projects for students in real analysis and calculus.

1 Introduction.

It is well-known that the curvature at each point of a line is zero, and that the curvature of a circle has the same value at each of its points, namely the reciprocal of the radius of the circle. In addition, curvature is invariant under translation and rotation; the measure of curvature does not depend on the orientation of the curve. Roughly speaking, as the magnitude of the curvature increases, the curve bends more. In this sense, curvature provides a measure of the nonlinearity of a curve. As we will see, the aberrancy at each point of a circle is zero, and aberrancy is also invariant under translation and rotation. As the magnitude of the aberrancy increases, the curve differs more from a circle. These facts reveal that aberrancy provides a measure of the noncircularity of a curve. Although the analogy (using a standard notation) "aberrancy : circle : : curvature : line" is not perfect, it goes a long way

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toward explaining the main idea behind this article. This analogy between aberrancy and curvature breaks down when considering various approaches to the two concepts. Curvature can be defined in several ways (see the next section), but each of these ways gives the same formula for curvature. In contrast to curvature, different approaches to aberrancy yield different formulas for the aberrancy of a function. We will consider a number of different approaches to aberrancy and show that there are some interesting and unexpected connections between them. The computational aspects of this material are suitable as extended projects for calculus students who have access to a computer algebra system, while the theoretical aspects provide a good challenge to students in a real analysis course.

2 Original Definition of Aberrancy.

We begin with a quick review of curvature for plane curves. Curvature is most often defined as the rate of change of the angle of inclination of the tangent line with respect to arc length. For a twice differentiable function f, the angle of inclination ϕ and arc length s satisfy

$$\phi = \arctan f', \quad \frac{ds}{dx} = \sqrt{1+f'^2}.$$

It follows that the curvature κ of f is given by

$$\kappa = \frac{d\phi}{ds} = \frac{d\phi}{dx} \div \frac{ds}{dx} = \frac{f''}{1+f'^2} \div \sqrt{1+f'^2} = \frac{f''}{\left(1+f'^2\right)^{3/2}}.$$

However, the following list gives several other ways to define curvature. Assume that f is a twice differentiable function defined on some open interval containing a point c.

1. Curvature is the reciprocal of the radius of the circle that best approximates the function f at c. This circle goes through $(c, f(c))$ and has the same slope (first derivative is $f'(c)$) and concavity (second derivative is $f''(c)$) as f.

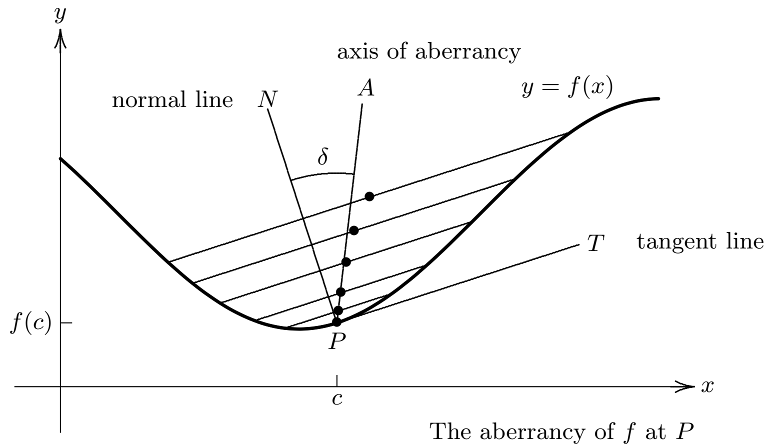
2. Let (u_ϵ, v_ϵ) be the point of intersection of the lines normal to the graph of $y = f(x)$ at the points $(c-\epsilon, f(c-\epsilon))$ and $(c+\epsilon, f(c+\epsilon))$. As $\epsilon \to 0$, the point (u_ϵ, v_ϵ) tends to a limiting point in the plane called the center of curvature of f at $(c, f(c))$; denote this point by (h, k). The curvature of f at c is then the reciprocal of the distance from $(c, f(c))$ to (h, k).

3. Let (u_ϵ, v_ϵ) be the center of the circle containing the points $(c-\epsilon, f(c-\epsilon))$, $(c, f(c))$, and $(c+\epsilon, f(c+\epsilon))$. As $\epsilon \to 0$, the point (u_ϵ, v_ϵ) moves to the center of curvature, and curvature is defined as in (2).

All of these formulations of curvature are equivalent for plane curves; that is, they all give the same formula for curvature. Methods 1 and 2 are not very difficult; the first involves implicit differentiation and the second (in more generality) is discussed in [2]. Method 3 is rather tedious, but the details do work out. For a discussion of the early history of curvature, including the various approaches to curvature considered by Newton, see [3]. The fact that curvature is invariant under translation and rotation is most apparent from its relationship to $d\phi/ds$.

We now turn to a discussion of aberrancy. Since aberrancy is not a familiar concept, we present its original definition in detail. (For the history of this concept, see Schot [7].) Suppose that f is a thrice differentiable function defined on an open interval I, and let c be a point in I for which $f''(c) \neq 0$. In other words, the curvature of f at $(c, f(c))$ is nonzero. This condition guarantees that a line that is close to and parallel to the tangent line of f at $(c, f(c))$ intersects the curve at least two times (as long as the line is on the appropriate side of the tangent line), once on each side of c. Let (u_ϵ, v_ϵ) be the midpoint of the resulting chord, where $\epsilon > 0$ represents the perpendicular distance from the parallel chord to the tangent line. As $\epsilon \to 0^+$, this point approaches the point $(c, f(c))$. However, the quantity $\big(v_\epsilon - f(c)\big)/\big(u_\epsilon - c\big)$, which represents the slope of the line through the points (u_ϵ, v_ϵ) and $(c, f(c))$, has a limit as $\epsilon \to 0^+$. Denote the value of this limit by the symbol S_c. The line $y = S_c(x-c) + f(c)$ is known as the **axis of aberrancy** of the curve at $(c, f(c))$. The tangent of the angle made by the axis of aberrancy and the normal line of f at $(c, f(c))$ is defined to be the **aberrancy** of f at c. The following graph indicates how the midpoints of the chords (represented by bullets •) approach the point of interest in a linear way. It also shows the axis of aberrancy and the angle δ used to determine aberrancy.

With a little thought, it should be clear that aberrancy is invariant under both translation and rotation (The chords and thus their midpoints are independent of the orientation of the curve.), and that the aberrancy of a circle is zero at every point on the circle. Aberrancy thus provides a numerical measure of the noncircularity of the graph of a function at a given point. Alternatively, aberrancy can be considered as a measure of the asymmetry of a curve about its normal line.



The aberrancy of f at P is defined to be $\tan\delta$.

If the curvature of f at $(c, f(c))$ is zero, then the aberrancy of f at $(c, f(c))$ is undefined. In this case, there may not be any parallel chords; for an example, consider the graph of $f(x) = x^3$ at the origin. Since aberrancy is invariant under translation and rotation, is zero at each point of a circle, and is undefined when curvature is zero, it is plausible that aberrancy is related to the quantity $d\rho/ds$, where ρ is the radius of curvature (which is infinite when the curvature is zero) and s is arc length. Since $\rho = 1/\kappa$, we find that

$$\begin{aligned}\frac{d\rho}{ds} &= \frac{d(1/\kappa)}{ds} = \frac{d(1/\kappa)}{dx} \div \frac{ds}{dx} \\ &= \frac{d}{dx}\,\frac{\left(1+f'^2\right)^{3/2}}{f''} \div \left(1+f'^2\right)^{1/2} = 3f' - \frac{f'''\left(1+f'^2\right)}{f''^2}.\end{aligned}$$

It is indeed the case that this quantity is related to aberrancy, and we will later show that (for this particular definition of aberrancy) the aberrancy $\mathcal{A}$ of f is given by

$$\mathcal{A} = f' - \frac{f'''\left(1+f'^2\right)}{3f''^2},$$

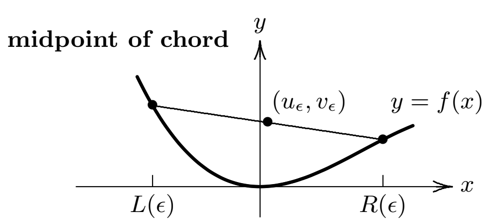
that is, $\mathcal{A} = \frac{1}{3} \cdot d\rho/ds$. We will see that each of the geometrical measures of aberrancy obtained in this paper is of the form $k \cdot d\rho/ds$ for some constant value of k. In other words, the quantity $d\rho/ds$ is the key measure of aberrancy; we are simply discovering geometrical interpretations for various multiples of this measure.

3 Defining Other Measures of Aberrancy.

We now want to consider other geometrical ways to measure aberrancy. To be a legitimate measure of aberrancy, the measure must be invariant under translation and rotation and give a value of zero at each point on a circle. Since the measure is invariant under translation, we will always assume that the origin is our point of interest, that is, we will assume that $f(0) = 0$. Looking back over the original definition of aberrancy, we find that two ingredients are required. First of all, for each $\epsilon > 0$ there exist numbers $L(\epsilon)$ and $R(\epsilon)$ such that $L(\epsilon) < 0 < R(\epsilon)$ and $\lim_{\epsilon\to 0^+} L(\epsilon) = 0 = \lim_{\epsilon\to 0^+} R(\epsilon)$. In the original definition, these numbers are determined by the intersection of the parallel chord with the graph of the function, but there are other ways to determine these values. Secondly, there must be a method for using the points $(L(\epsilon), f(L(\epsilon)))$ and $(R(\epsilon), f(R(\epsilon)))$ to determine a point (u_ϵ, v_ϵ). In the original definition, we use the midpoint of the chord, but once again there are other ways to determine this point. The slope of the axis of aberrancy is then given by $\lim_{\epsilon\to 0^+} v_\epsilon/u_\epsilon$ and the corresponding measure of aberrancy follows easily.

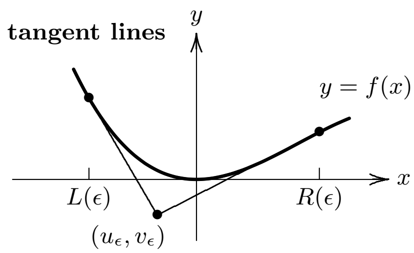
We will specify the functions L and R a little later. At present, we will simply assume that they are twice differentiable functions defined on a neighborhood of 0 and that they satisfy both $L(0) = 0 = R(0)$ and $L'(0) = -R'(0)$. These conditions guarantee that the limiting value of (u_ϵ, v_ϵ) is the origin and that the approaches on both sides of $(0, 0)$ have a certain degree of uniformity. Our focus now will be on using $L(\epsilon)$ and $R(\epsilon)$ to determine a point (u_ϵ, v_ϵ) in five different ways. We will illustrate each method with a graph, making the assumption that $f'(0) = 0$ and $f''(0) > 0$. The first assumption makes the graphs easier to read since the normal line is then the y-axis, while the second just involves replacing f with $-f$ if necessary.

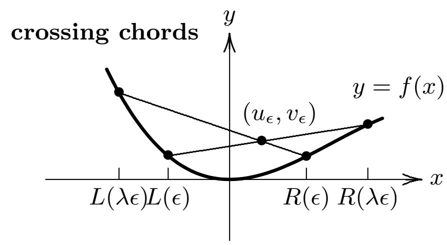
One method, as already indicated, for determining (u_ϵ, v_ϵ) is to take the midpoint of the chord joining the points $(L(\epsilon), f(L(\epsilon)))$ and $(R(\epsilon), f(R(\epsilon)))$; see the following graph.



We next consider two methods that involve the intersection of lines. The first is to let (u_ϵ, v_ϵ) be the intersection of the tangent lines of f at the points $(L(\epsilon), f(L(\epsilon)))$ and $(R(\epsilon), f(R(\epsilon)))$. The second is to fix a constant $\lambda > 1$

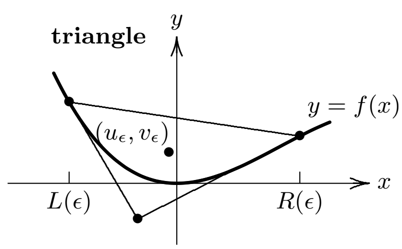
and let (u_ϵ, v_ϵ) be the intersection of the line through $(L(\epsilon), f(L(\epsilon)))$ and $(R(\lambda\epsilon), f(R(\lambda\epsilon)))$ with the line through $(L(\lambda\epsilon), f(L(\lambda\epsilon)))$ and $(R(\epsilon), f(R(\epsilon)))$. We will later show that the aberrancy measure obtained from this crossing chord method is independent of λ. By the way, the point (u_ϵ, v_ϵ) exists in both cases since f'' is nonzero on a neighborhood of 0. These two methods are illustrated in the following graphs.

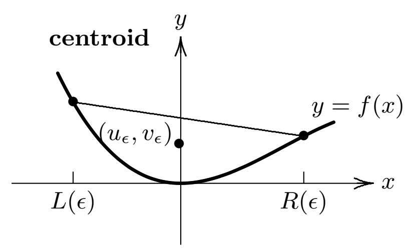




Although these two ways to determine a point may not be quite as natural as the midpoint method, they are certainly reasonable ways to generate a point.

The other two methods we will consider involve the centroid (or center of mass) of a certain region. For the first of these, let (u_ϵ, v_ϵ) be the centroid of the triangle with vertices $(L(\epsilon), f(L(\epsilon)))$, $(R(\epsilon), f(R(\epsilon)))$, and (U_ϵ, V_ϵ), where the latter point is the intersection of the tangent lines mentioned earlier. For the second, let (u_ϵ, v_ϵ) be the centroid of the region bounded by the graph of f and the chord passing through the points $(L(\epsilon), f(L(\epsilon)))$ and $(R(\epsilon), f(R(\epsilon)))$. These last two methods are illustrated in the following graphs.





We thus have five distinct ways for determining the point (u_ϵ, v_ϵ) from the functional values $L(\epsilon)$ and $R(\epsilon)$.

As with the original definition of aberrancy, the value of $\lim_{\epsilon\to 0^+} v_\epsilon/u_\epsilon$ gives the slope of the axis of aberrancy, and the tangent of the angle δ made by the normal line and the axis of aberrancy provides the measure of aberrancy. Under the assumption that $f'(0) = 0$, the normal line is the y-axis and thus $\tan\delta = \lim_{\epsilon\to 0^+} u_\epsilon/v_\epsilon$; the reciprocal of the slope of the axis of aberrancy is used since the angle is measured with respect to the y-axis, not the x-axis.

Determining the value of this quantity will be the subject of the next section. Some comments related to the $f'(0) = 0$ assumption will be made near the end of the paper.

4 Computing Measures of Aberrancy.

For each of the five methods discussed in the previous section, it is not difficult to write down the quantity u_ϵ/v_ϵ. However, determining the limit of this quantity as $\epsilon \to 0^+$ is a rather formidable task for most of the methods. In this section, we will simply develop the formulas for u_ϵ/v_ϵ and record the corresponding limits. The tedious calculations behind these values will be presented in the next section. As with the graphs in the previous section, we will continue to assume that $f'(0) = 0$. It turns out that the resulting limits only involve the quantities $f''(0)$, $f'''(0)$, $R'(0)$, $R''(0)$, $L'(0)$, and $L''(0)$. Hence, one way to proceed is to assume

$$f(x) = \frac{b}{2}\,x^2 + \frac{c}{6}\,x^3, \quad R(\epsilon) = q\,\epsilon + \frac{r}{2}\,\epsilon^2, \quad L(\epsilon) = -q\,\epsilon + \frac{t}{2}\,\epsilon^2,$$

then use a computer algebra system to simplify u_ϵ/v_ϵ. If this quantity is expressed as a ratio of two polynomials, which for some of the aberrancy measures is a nontrivial task, then the limit can be identified as the ratio of the coefficients of the smallest degree terms. (Reasons for the validity of this approach will become apparent in the next section.) Hence, the value of $\lim\limits_{\epsilon\to 0^+} u_\epsilon/v_\epsilon$ will be expressed in terms of b, c, q, r, and t. When we specify the functions R and L in a later section, we will find that the numbers q, r, and t depend on b and c.

Let $\mathcal{A}_{mc}$ denote the aberrancy measure obtained from the method involving the midpoint of the chord. Then u_ϵ/v_ϵ is simply the sum of the x-coordinates divided by the sum of the y-coordinates, and we find that

$$\mathcal{A}_{mc} = \lim_{\epsilon\to 0^+} \frac{L(\epsilon) + R(\epsilon)}{f(L(\epsilon)) + f(R(\epsilon))} = \frac{r+t}{2\,b\,q^2}.$$

Let $\mathcal{A}_{t\ell}$ denote the aberrancy measure obtained from the method involving the intersection of tangent lines. The equations of the tangent lines are

$$y = f'(L(\epsilon))(x - L(\epsilon)) + f(L(\epsilon)) = m_1 x + b_1,$$
$$y = f'(R(\epsilon))(x - R(\epsilon)) + f(R(\epsilon)) = m_2 x + b_2.$$

Since (u_ϵ, v_ϵ) is the point of intersection of these two lines, we find that

$$u_\epsilon = \frac{b_1 - b_2}{m_2 - m_1}, \quad v_\epsilon = m_1 u_\epsilon + b_1.$$

Because $\lim_{\epsilon \to 0^+} m_1 = 0$, it then follows that

$$\mathcal{A}_{t\ell} = \lim_{\epsilon \to 0^+} \frac{u_\epsilon}{m_1 u_\epsilon + b_1} = \lim_{\epsilon \to 0^+} \left(m_1 + \frac{b_1}{u_\epsilon} \right)^{-1} = \lim_{\epsilon \to 0^+} \frac{u_\epsilon}{b_1} = -\frac{r+t}{2\,b\,q^2} - \frac{2}{3} \cdot \frac{c}{b^2}.$$

Since the expression for u_ϵ is a fraction, it is necessary to simplify u_ϵ / b_1 before a computer algebra system is much help in evaluating the limit.

Let $\mathcal{A}_{cc}$ denote the aberrancy measure obtained from the method involving the intersection of crossing chords. The lines representing the two chords are

$$y = \frac{f(R(\lambda\epsilon)) - f(L(\epsilon))}{R(\lambda\epsilon) - L(\epsilon)}(x - L(\epsilon)) + f(L(\epsilon)) = m_3 x + b_3;$$
$$y = \frac{f(L(\lambda\epsilon)) - f(R(\epsilon))}{L(\lambda\epsilon) - R(\epsilon)}(x - R(\epsilon)) + f(R(\epsilon)) = m_4 x + b_4.$$

Using the same reasoning as the tangent line method, but with a bit more effort, we find that

$$\mathcal{A}_{cc} = \lim_{\epsilon \to 0^+} \frac{u_\epsilon}{m_3 u_\epsilon + b_3} = \lim_{\epsilon \to 0^+} \left(m_3 + \frac{b_3}{u_\epsilon} \right)^{-1} = \lim_{\epsilon \to 0^+} \frac{u_\epsilon}{b_3} = -\frac{r+t}{2\,b\,q^2} - \frac{2}{3} \cdot \frac{c}{b^2}.$$

Note that the constant λ does not appear in the final limit.

Let $\mathcal{A}_{tr}$ denote the aberrancy measure obtained from the method involving the centroid of the triangle. Denoting the intersection of the tangent lines by (U_ϵ, V_ϵ) and recalling that the centroid of a triangle is simply $(\overline{x}, \overline{y})$, where $\overline{x}$ and $\overline{y}$ are the averages of the x-coordinates and y-coordinates, respectively, of the three vertices, we obtain

$$\mathcal{A}_{tr} = \lim_{\epsilon \to 0^+} \frac{u_\epsilon}{v_\epsilon} = \lim_{\epsilon \to 0^+} \frac{U_\epsilon + L(\epsilon) + R(\epsilon)}{V_\epsilon + f(L(\epsilon)) + f(R(\epsilon))} = 3 \cdot \frac{r+t}{2\,b\,q^2} + \frac{2}{3} \cdot \frac{c}{b^2}.$$

Let $\mathcal{A}_{ce}$ denote the aberrancy measure obtained from the method involving the centroid of the region bounded by the chord and the graph of f. The linear function representing the chord is given by

$$\ell(x) = \frac{f(R(\epsilon)) - f(L(\epsilon))}{R(\epsilon) - L(\epsilon)}(x - R(\epsilon)) + f(R(\epsilon)).$$

Using standard formulas for finding the centroid of a region, we obtain

$$\mathcal{A}_{ce} = \lim_{\epsilon\to 0^+} \frac{u_\epsilon}{v_\epsilon} = \lim_{\epsilon\to 0^+} \frac{\int_{L(\epsilon)}^{R(\epsilon)} x\big(\ell(x) - f(x)\big)\, dx}{\frac{1}{2}\int_{L(\epsilon)}^{R(\epsilon)} \big(\ell(x)^2 - f(x)^2\big)\, dx} = \frac{5}{3}\cdot\frac{r+t}{2\,b\,q^2} + \frac{2}{9}\cdot\frac{c}{b^2}.$$

The simplest way to obtain this limit with a computer algebra system is to deal with each of the four integrals separately and then combine the results.

Recording these five measures of aberrancy in a single place, it is possible to see some simple relationships between them:

$$\begin{aligned}
\mathcal{A}_{mc} &= r + \frac{t}{2}\,b\,q^2;\\
\mathcal{A}_{t\ell} &= -r + \frac{t}{2}\,b\,q^2 - \frac{2}{3}\cdot\frac{c}{b^2} = -\mathcal{A}_{mc} - \frac{2}{3}\cdot\frac{c}{b^2};\\
\mathcal{A}_{cc} &= -\frac{r+t}{2\,b\,q^2} - \frac{2}{3}\cdot\frac{c}{b^2} = -\mathcal{A}_{mc} - \frac{2}{3}\cdot\frac{c}{b^2};\\
\mathcal{A}_{tr} &= 3\cdot\frac{r+t}{2\,b\,q^2} + \frac{2}{3}\cdot\frac{c}{b^2} = 3\mathcal{A}_{mc} + \frac{2}{3}\cdot\frac{c}{b^2};\\
\mathcal{A}_{ce} &= \frac{5}{3}\cdot\frac{r+t}{2\,b\,q^2} + \frac{2}{9}\cdot\frac{c}{b^2} = \frac{5}{3}\mathcal{A}_{mc} + \frac{2}{9}\cdot\frac{c}{b^2}.
\end{aligned}$$

As has been mentioned, we expect each of these aberrancy measures to yield a multiple of $d\rho/ds$. When $f(x) = (b/2)x^2 + (c/6)x^3$, the quantity $d\rho/ds$ at the point $(0,0)$ takes on the simple form $-c/b^2$. We thus expect the quantity $(r+t)/(2\,b\,q^2)$, which appears in all of the final limits, to be some multiple of c/b^2; this fact will be verified later. For now, we simply assume that this is the case and note some of the interesting relationships that exist between the various aberrancy measures. We will write $\mathcal{A}_{mc} = k_{mc}\cdot d\rho/ds$ and use a similar notation for the other aberrancy measures. Looking back over the data, we obtain the following relationships:

$$k_{t\ell} = -k_{mc} + \frac{2}{3};\quad k_{cc} = -k_{mc} + \frac{2}{3};\quad k_{tr} = 3k_{mc} - \frac{2}{3};\quad k_{ce} = \frac{5}{3}\,k_{mc} - \frac{2}{9}.$$

These equations show that once one type of aberrancy is known, all of the others follow. The relationships between the measures are linear and independent of the choices for R and L (assuming that these functions generate a measure that is independent of rotation). We have expressed all of the measures in terms of k_{mc}, but it is possible to use any one of the five measures as the base value. The two most surprising results are the fact that the sum of k_{mc} and

$k_{t\ell}$ is a constant and, given their rather different definitions, the fact that $k_{t\ell}$ and k_{cc} have the same value. Some other relationships to note are

$$k_{tr} = 2k_{mc} - k_{t\ell}; \quad k_{mc} = \frac{1}{2}\,(k_{t\ell} + k_{tr}); \quad k_{ce} = \frac{1}{3}\,(2k_{mc} + k_{tr}).$$

The last two show that $\mathcal{A}_{mc}$ is the average of $\mathcal{A}_{t\ell}$ and $\mathcal{A}_{tr}$, and that $\mathcal{A}_{ce}$ is a weighted average of $\mathcal{A}_{tr}$ and $\mathcal{A}_{mc}$. (The first formula is also a weighted average if you are willing to consider negative weights.)

5 Analytical Limit Calculations.

As mentioned in the previous section, the computations needed to determine $\lim_{\epsilon\to 0^+} u_\epsilon/v_\epsilon$ can be extremely tedious. One approach is to let a computer algebra system do all of the work, but then it is hard to see why the results come out the way they do. In this section, we will take a closer look at the calculations and obtain the results analytically rather than through a black box. This section can be skipped or skimmed without any loss of continuity. In keeping with the notation of the last section, we will let

$$\begin{aligned} &f'(0) = 0, \quad f''(0) = b, \quad f'''(0) = c,\\ &R'(0) = q, \quad R''(0) = r,\\ &L'(0) = -q, \quad L''(0) = t. \end{aligned}$$

(Recall also that $f(0)$, $L(0)$, and $R(0)$ are all zero.) We should mention that although the following computations appear intimidating, they are quite simple compared with some of our earlier attempts.

We first look at $\mathcal{A}_{mc}$ because it is the easiest of the aberrancy values to find. Using L'Hôpital's Rule and the definition of the derivative, we obtain

$$\begin{aligned} \mathcal{A}_{mc} &= \lim_{\epsilon\to 0^+} \frac{L(\epsilon)+R(\epsilon)}{f(L(\epsilon))+f(R(\epsilon))} = \lim_{\epsilon\to 0^+} \frac{L'(\epsilon)+R'(\epsilon)}{f'(L(\epsilon))L'(\epsilon)+f'(R(\epsilon))R'(\epsilon)}\\ &= \lim_{\epsilon\to 0^+} \frac{L'(\epsilon)+R'(\epsilon)}{\epsilon}\Big(\frac{f'(L(\epsilon))L'(\epsilon)+f'(R(\epsilon))R'(\epsilon)}{\epsilon}\Big)^{-1}\\ &= \frac{(L'+R')'(0)}{\big(L'\cdot(f'\circ L)+R'\cdot(f'\circ R)\big)'(0)} = \frac{r+t}{2\,b\,q^2}. \end{aligned}$$

Note that we must avoid using $\lim_{\epsilon\to 0^+}(L''(\epsilon)+R''(\epsilon))$ since R'' and L'' are not assumed to be continuous.

The calculations needed to determine $\mathcal{A}_{t\ell}$ indicate some of the reasons why the simple approach (that is, assume all of the functions involved are small degree polynomials) carried out earlier generates the correct answers. The main piece of information that we need is the following. If g is a thrice differentiable function defined on a neighborhood of 0, then

$$g(x) = g(0) + g'(0)\,x + \frac{g''(0)}{2}\,x^2 + \frac{g'''(0)}{6}\,x^3 + x^3\,Z_g(x),$$

where $\lim\limits_{x\to 0} Z_g(x) = 0$. Since g may not have a fourth derivative, the remainder $Z_g(x)$ may not be expressible in its usual Taylor's formula form. To verify that $\lim\limits_{x\to 0} Z_g(x) = 0$, it is necessary to prove that

$$\lim_{x\to 0} \frac{g(x) - \Big(g(0) + g'(0)\,x + \frac{1}{2}\,g''(0)\,x^2\Big)}{x^3} = \frac{g'''(0)}{6};$$

the elementary details are left to the reader. Similar expressions are valid for twice differentiable functions and so on. In our case, we have

$$f(x) = \frac{b}{2}\,x^2 + \frac{c}{6}\,x^3 + x^3\,Z_f(x), \quad f'(x) = b\,x + \frac{c}{2}\,x^2 + x^2\,Z_{f'}(x),$$

where $\lim\limits_{x\to 0} Z_f(x) = 0 = \lim\limits_{x\to 0} Z_{f'}(x)$. To find $\mathcal{A}_{t\ell}$, we must evaluate (see the previous section for an explanation of the notation)

$$\lim_{\epsilon\to 0^+} \frac{u_\epsilon}{v_\epsilon} = \lim_{\epsilon\to 0^+} \frac{u_\epsilon}{m_1 u_\epsilon + b_1} = \lim_{\epsilon\to 0^+} \frac{1}{m_1 + \frac{b_1}{u_\epsilon}} = \lim_{\epsilon\to 0^+} \frac{1}{m_1 + b_1\cdot\frac{m_2 - m_1}{b_1 - b_2}}.$$

Since $\lim\limits_{\epsilon\to 0^+} m_1 = f'(0) = 0 = \lim\limits_{\epsilon\to 0^+} m_2$, we find that (assuming the limits exist)

$$\lim_{\epsilon\to 0^+} \frac{u_\epsilon}{v_\epsilon} = \lim_{\epsilon\to 0^+} \frac{b_1 - b_2}{b_1(m_2 - m_1)}.$$

To evaluate this limit, we need to assemble a number of pieces. Using a combination of the definition of the derivative, L'Hôpital's Rule, and factoring, we find that

$$\lim_{\epsilon\to 0^+} \frac{R(\epsilon) - L(\epsilon)}{\epsilon} = 2q; \qquad \lim_{\epsilon\to 0^+} \frac{R(\epsilon)^2 - L(\epsilon)^2}{\epsilon^3} = q(r+t);$$

$$\lim_{\epsilon\to 0^+} \frac{R(\epsilon) + L(\epsilon)}{\epsilon^2} = \frac{r+t}{2}; \qquad \lim_{\epsilon\to 0^+} \frac{R(\epsilon)^3 - L(\epsilon)^3}{\epsilon^3} = 2q^3.$$

Omitting the argument (ϵ) to simplify the writing, we see that

$$f(R) - f(L) = \frac{b}{2}\big(R^2 - L^2\big) + \frac{c}{6}\big(R^3 - L^3\big) + R^3\, Z_f(R) - L^3\, Z_f(L),$$

and thus,

$$\begin{aligned}\lim_{\epsilon\to 0^+} & \frac{f(R) - f(L)}{\epsilon^3} \\ &= \lim_{\epsilon\to 0^+}\Big(\frac{b}{2}\cdot\frac{R^2 - L^2}{\epsilon^3} + \frac{c}{6}\cdot\frac{R^3 - L^3}{\epsilon^3} + \frac{R^3}{\epsilon^3}\cdot Z_f(R) - \frac{L^3}{\epsilon^3}\cdot Z_f(L)\Big) \\ &= \frac{b}{2}\cdot q(r+t) + \frac{c}{6}\cdot 2q^3 + q^3\cdot 0 - (-q)^3\cdot 0 = \frac{1}{2}\, b\, q(r+t) + \frac{1}{3}\, c\, q^3.\end{aligned}$$

For future reference, we will call the value of this limit B. Let p be the function defined by $p(x) = xf'(x)$ and note that

$$p(x) = bx^2 + \frac{c}{2}\, x^3 + x^3\, Z_{f'}(x).$$

Using the same method for p that we just used for f, we find that

$$\lim_{\epsilon\to 0^+}\frac{p(R(\epsilon)) - p(L(\epsilon))}{\epsilon^3} = b\, q(r+t) + c\, q^3 = 2B + \frac{1}{3}\, c\, q^3.$$

We can use these limits to compute the limits of the components of u_ϵ/v_ϵ:

$$\begin{aligned}\lim_{\epsilon\to 0^+}\frac{m_2 - m_1}{\epsilon} &= \lim_{\epsilon\to 0^+}\frac{f'(R(\epsilon)) - f'(L(\epsilon))}{\epsilon} = b\, q - b(-q) = 2b\, q; \\ \lim_{\epsilon\to 0^+}\frac{b_1}{\epsilon^2} &= \lim_{\epsilon\to 0^+}\frac{f(L(\epsilon)) - L(\epsilon)f'(L(\epsilon))}{\epsilon^2} \\ &= \frac{b}{2}\,(-q)^2 - (-q)b(-q) = -\frac{1}{2}\, b\, q^2; \\ \lim_{\epsilon\to 0^+}\frac{b_1 - b_2}{\epsilon^3} &= \lim_{\epsilon\to 0^+}\Big(\frac{p(R(\epsilon)) - p(L(\epsilon))}{\epsilon^3} - \frac{f(R(\epsilon)) - f(L(\epsilon))}{\epsilon^3}\Big) \\ &= B + \frac{1}{3}\, c\, q^3.\end{aligned}$$

Putting all of this information together gives

$$\lim_{\epsilon\to 0^+}\frac{u_\epsilon}{v_\epsilon} = \lim_{\epsilon\to 0^+}\Big(\frac{b_1 - b_2}{\epsilon^3}\cdot\frac{\epsilon^2}{b_1}\cdot\frac{\epsilon}{m_2 - m_1}\Big) = -\frac{B}{b^2 q^3} - \frac{1}{3}\cdot\frac{c}{b^2} = -\frac{r+t}{2b\, q^2} - \frac{2}{3}\cdot\frac{c}{b^2},$$

the value given in the previous section.

In spite of the fact that $\mathcal{A}_{cc} = \mathcal{A}_{t\ell}$, its computations are more complicated since the arguments for the functions R and L involve both ϵ and $\lambda\epsilon$. As with $\mathcal{A}_{t\ell}$, the limit that must be evaluated is

$$\lim_{\epsilon\to 0^+} \frac{b_3 - b_4}{b_3(m_4 - m_3)},$$

where

$$m_3 = \frac{f(R(\lambda\epsilon)) - f(L(\epsilon))}{R(\lambda\epsilon) - L(\epsilon)} = \frac{N_3}{D_3}; \qquad m_4 = \frac{f(L(\lambda\epsilon)) - f(R(\epsilon))}{L(\lambda\epsilon) - R(\epsilon)} = \frac{N_4}{D_4};$$
$$b_3 = f(L(\epsilon)) - m_3 L(\epsilon); \qquad b_4 = f(R(\epsilon)) - m_4 R(\epsilon).$$

A number of elementary computations (some involving data from the previous paragraphs) then yield

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \frac{m_3}{\epsilon} &= \lim_{\epsilon\to 0^+} \frac{N_3/\epsilon^2}{D_3/\epsilon} = \frac{(b/2)(\lambda^2-1)q^2}{(\lambda+1)q} = \frac{1}{2}\, b\, q\,(\lambda - 1) \\
&= -\lim_{\epsilon\to 0^+} \frac{m_4}{\epsilon}; \\
\lim_{\epsilon\to 0^+} \frac{D_3 + D_4}{\epsilon^2} &= \lim_{\epsilon\to 0^+} \Big(\frac{R(\lambda\epsilon) + L(\lambda\epsilon)}{\epsilon^2} - \frac{R(\epsilon) + L(\epsilon)}{\epsilon^2} \Big) = (\lambda^2 - 1)\,\frac{r+t}{2}; \\
\lim_{\epsilon\to 0^+} \frac{N_3 - N_4}{\epsilon^3} &= \lim_{\epsilon\to 0^+} \Big(\frac{f(R(\lambda\epsilon)) - f(L(\lambda\epsilon))}{\epsilon^3} + \frac{f(R(\epsilon)) - f(L(\epsilon))}{\epsilon^3} \Big) \\
&= (\lambda^3 + 1)B; \\
\lim_{\epsilon\to 0^+} \frac{m_3 + m_4}{\epsilon^2} &= \lim_{\epsilon\to 0^+} \frac{\epsilon}{D_3} \Big(\frac{N_3 - N_4}{\epsilon^3} + \frac{m_4}{\epsilon} \cdot \frac{D_3 + D_4}{\epsilon^2} \Big) \\
&= \frac{(\lambda^3+1)B + (b/2)q(1-\lambda)(\lambda^2-1)(r+t)/2}{(\lambda+1)q} \\
&= (\lambda^2 - \lambda + 1)\frac{B}{q} - \frac{1}{4}\,(\lambda-1)^2 b(r+t); \\
\lim_{\epsilon\to 0^+} \frac{b_3}{\epsilon^2} &= \lim_{\epsilon\to 0^+} \Big(\frac{f(L(\epsilon))}{\epsilon^2} - \frac{m_3}{\epsilon} \cdot \frac{L(\epsilon)}{\epsilon} \Big) \\
&= \frac{b}{2}\, q^2 - \frac{1}{2}\, b\, q(\lambda - 1)\cdot(-q) = \frac{1}{2}\, b\, q^2 \lambda;
\end{aligned}$$

$$\begin{aligned}
&\lim_{\epsilon\to 0^+} \frac{b_3-b_4}{\epsilon^3}\\
&= \lim_{\epsilon\to 0^+} \left(-\frac{f(R(\epsilon))-f(L(\epsilon))}{\epsilon^3} + \frac{m_4}{\epsilon}\cdot\frac{R(\epsilon)+L(\epsilon)}{\epsilon^2} - \frac{L(\epsilon)}{\epsilon}\cdot\frac{m_3+m_4}{\epsilon^2}\right)\\
&= -B + \frac{1}{2}\,b\,q(1-\lambda)\cdot\frac{r+t}{2} - (-q)\cdot\left((\lambda^2-\lambda+1)\frac{B}{q} - \frac{1}{4}\,(\lambda-1)^2 b(r+t)\right)\\
&= (\lambda^2-\lambda)\left(\frac{1}{4}\,b\,q(r+t) + \frac{1}{3}\,c\,q^3\right).
\end{aligned}$$

Putting all of the pieces together, we obtain

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \frac{u_\epsilon}{v_\epsilon} &= \lim_{\epsilon\to 0^+} \frac{b_3-b_4}{\epsilon^3}\cdot\left(\frac{b_3}{\epsilon^2}\cdot\frac{m_4-m_3}{\epsilon}\right)^{-1}\\
&= \frac{(\lambda^2-\lambda)\left(\frac{1}{4}\,b\,q(r+t)+\frac{1}{3}\,c\,q^3\right)}{\frac{1}{2}\,b\,q^2\lambda\cdot b\,q\,(1-\lambda)} = -\frac{r+t}{2\,b\,q^2} - \frac{2}{3}\cdot\frac{c}{b^2},
\end{aligned}$$

which is the same value obtained for $\mathcal{A}_{t\ell}$.

The calculations required to find $\mathcal{A}_{tr}$ are tedious if done directly, but it is possible to avoid most of them. Let (u_ϵ, v_ϵ) be the centroid of the triangle, let (U_ϵ, V_ϵ) be the intersection point of the tangent lines, and let (S_ϵ, T_ϵ) be the midpoint of the chord. Then

$$\frac{u_\epsilon}{v_\epsilon} = \frac{U_\epsilon+2S_\epsilon}{V_\epsilon+2T_\epsilon} = \frac{U_\epsilon+2S_\epsilon}{2T_\epsilon}\cdot\frac{2T_\epsilon}{U_\epsilon}\cdot\frac{U_\epsilon}{V_\epsilon+2T_\epsilon} = \left(\frac{2S_\epsilon}{2T_\epsilon}+\frac{U_\epsilon}{2T_\epsilon}\right)\cdot\frac{2T_\epsilon}{U_\epsilon}\cdot\left(\frac{2T_\epsilon}{U_\epsilon}+\frac{V_\epsilon}{U_\epsilon}\right)^{-1}.$$

The only new ratio in this expression is $U_\epsilon/(2T_\epsilon)$. Using the formulas for U_ϵ and T_ϵ recorded earlier, we find that

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \frac{U_\epsilon}{2T_\epsilon} &= \lim_{\epsilon\to 0^+} \frac{b_1-b_2}{(m_2-m_1)(f(R(\epsilon))+f(L(\epsilon)))}\\
&= \lim_{\epsilon\to 0^+} \frac{b_1-b_2}{b_1(m_2-m_1)}\cdot\frac{b_1/\epsilon^2}{(f(R(\epsilon))+f(L(\epsilon)))/\epsilon^2}\\
&= \mathcal{A}_{t\ell}\cdot\frac{-(b/2)q^2}{b\,q^2} = -\frac{\mathcal{A}_{t\ell}}{2}.
\end{aligned}$$

It follows that

$$\lim_{\epsilon\to 0^+}\frac{u_\epsilon}{v_\epsilon} = \Big(\mathcal{A}_{mc} - \frac{\mathcal{A}_{t\ell}}{2}\Big)\cdot\Big(-\frac{2}{\mathcal{A}_{t\ell}}\Big)\cdot\Big(-\frac{2}{\mathcal{A}_{t\ell}} + \frac{1}{\mathcal{A}_{t\ell}}\Big)^{-1} = 2\mathcal{A}_{mc} - \mathcal{A}_{t\ell},$$

the value obtained in the previous section.

Last, but not least, we turn to the computation of $\mathcal{A}_{ce}$. As noted earlier, the expression for u_ϵ/v_ϵ involves four integrals. We will consider each integral separately and find its limit when divided by ϵ^5; a fair amount of experimentation revealed that each integral was related to ϵ^5. Using the Fundamental Theorem of Calculus and L'Hôpital's Rule, we obtain

$$\begin{aligned}\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^5}\int_{L(\epsilon)}^{R(\epsilon)} f(x)^2\,dx &= \lim_{\epsilon\to 0^+}\frac{f(R(\epsilon))^2R'(\epsilon) - f(L(\epsilon))^2L'(\epsilon)}{5\epsilon^4}\\ &= \frac{1}{5}\lim_{\epsilon\to 0^+}\Big(R'(\epsilon)\Big(\frac{f(R(\epsilon))}{\epsilon^2}\Big)^2 - L'(\epsilon)\Big(\frac{f(L(\epsilon))}{\epsilon^2}\Big)^2\Big)\\ &= \frac{(bq^2/2)^2q - (bq^2/2)^2(-q)}{5} = \frac{1}{10}\,b^2q^5.\end{aligned}$$

For the other integral involving f, namely $\int_L^R xf(x)\,dx$, we first note that

$$\int_0^x tf(t)\,dt = \frac{b}{8}\,x^4 + \frac{c}{30}\,x^5 + x^5Z_1(x),$$

where $\lim_{x\to 0} Z_1(x) = 0$. It follows that

$$\int_L^R xf(x)\,dx = \frac{b}{8}\,(R^2+L^2)(R^2-L^2) + \frac{c}{30}\,(R^5-L^5) + R^5Z_1(R) - L^5Z_1(L),$$

and hence (using previous results),

$$\begin{aligned}\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^5}\int_{L(\epsilon)}^{R(\epsilon)} xf(x)\,dx &= \frac{b}{8}\,(2q^2)q(r+t) + \frac{c}{30}\,(2q^5)\\ &= \frac{1}{4}\,b\,q^3(r+t) + \frac{1}{15}\,c\,q^5 = q^2\Big(\frac{1}{2}\,B - \frac{1}{10}\,c\,q^3\Big).\end{aligned}$$

The equation of the line representing the chord is given by

$$\ell(x) = \frac{f(R(\epsilon)) - f(L(\epsilon))}{R(\epsilon) - L(\epsilon)}(x - R(\epsilon)) + f(R(\epsilon)) = m_5x + b_5.$$

Note that $b_5 = f(L) - m_5 L = f(R) - m_5 R$. Since

$$\int_L^R \ell(x)^2\,dx = \frac{1}{3m_5}\,(m_5x+b_5)^3\Big|_L^R = \frac{1}{3}\cdot\frac{f(R)^3-f(L)^3}{m_5}$$
$$= \frac{1}{3}\,(R-L)\big(f(R)^2+f(R)f(L)+f(L)^2\big),$$

we find that

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^5}\int_{L(\epsilon)}^{R(\epsilon)}\ell(x)^2\,dx = \frac{1}{3}\cdot 2q\cdot 3\Big(\frac{b}{2}\,q^2\Big)^2 = \frac{1}{2}\,b^2q^5.$$

For the remaining integral $\int_L^R x\ell(x)\,dx$, we first observe that

$$\lim_{\epsilon\to 0^+}\frac{m_5}{\epsilon^2} = \lim_{\epsilon\to 0^+}\frac{(f(R)-f(L))/\epsilon^3}{(R-L)/\epsilon} = \frac{B}{2q};$$
$$\lim_{\epsilon\to 0^+}\frac{b_5}{\epsilon^2} = \lim_{\epsilon\to 0^+}\Big(\frac{f(R)}{\epsilon^2} - R\cdot\frac{m_5}{\epsilon^2}\Big) = \frac{1}{2}\,b\,q^2 - 0\cdot\frac{B}{2q} = \frac{1}{2}\,b\,q^2;$$
$$\int_L^R x\ell(x)\,dx = \int_L^R x(m_5x+b_5)\,dx = \frac{1}{3}\,m_5(R^3-L^3) + \frac{1}{2}\,b_5(R^2-L^2).$$

From these equations, we find that

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon^5}\int_{L(\epsilon)}^{R(\epsilon)} x\ell(x)\,dx = \frac{1}{3}\cdot\frac{B}{2q}\cdot 2q^3 + \frac{1}{2}\cdot\frac{b\,q^2}{2}\cdot q(r+t) = q^2\Big(\frac{5}{6}\,B - \frac{1}{6}\,c\,q^3\Big).$$

Putting all of the components together, we end up with

$$\mathcal{A}_{ce} = \lim_{\epsilon\to 0^+}\frac{\int_{L(\epsilon)}^{R(\epsilon)} x\big(\ell(x)-f(x)\big)\,dx}{\frac{1}{2}\int_{L(\epsilon)}^{R(\epsilon)}\big(\ell(x)^2-f(x)^2\big)\,dx}$$
$$= \frac{q^2\Big(\frac{5}{6}\,B - \frac{1}{r6}\,c\,q^3\Big) - q^2\Big(\frac{1}{2}\,B - \frac{1}{10}\,c\,q^3\Big)}{\frac{1}{2}\Big(\frac{1}{2}\,b^2q^5 - \frac{1}{10}\,b^2q^5\Big)}$$
$$= \frac{q^2\Big(\frac{1}{3}\,B - \frac{1}{15}\,c\,q^3\Big)}{\frac{1}{5}\,b^2q^5} = \frac{5}{3}\cdot\frac{r+t}{2\,b\,q^2} + \frac{2}{9}\cdot\frac{c}{b^2},$$

as advertised in the previous section.

It is certainly clear that these computations, all of which involve elementary ideas, are quite tedious. Even though the ideas and computations are accessible to calculus students, they can be quite intimidating even for junior and senior level mathematics majors. Hopefully, plowing through some of these computations will give students an appreciation for some aspects of computational and theoretical mathematics.

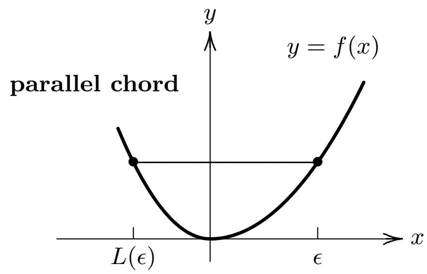
6 Specific Choices for Functions R and L.

Now that we have computed the measures of aberrancy in terms of generic functions R and L, it is time to specify what these functions are to give a legitimate measure of aberrancy. This means that the functions R and L must have the property that the aberrancy measure at a given point does not change if the curve is translated or rotated, and the corresponding measure of aberrancy should be zero for each point of a circle. Thus, our methods for choosing R and L must be independent of the orientation of the curve, and the generated point (u_ϵ, v_ϵ) must always land on the normal of a circle. The reader should check these properties for each of the choices for R and L that follow. In the original definition of aberrancy, the numbers $R(\epsilon)$ and $L(\epsilon)$ were determined by a chord parallel to the tangent line. We will consider this method first, then look at three other methods for obtaining R and L. Since each method of selecting R and L generates five measures of aberrancy, we will end up with a total of twenty measures of aberrancy. Note that once R and L are defined, all we need to do is compute $R'(0)$, $R''(0)$, and $L''(0)$. In our discussion, we will determine R and L under the assumption that $f'(0)$ is arbitrary and that $f''(0) > 0$. However, for simplicity, we will continue to sketch the illustrative graphs under the assumption that $f'(0) = 0$. After finding the general functions R and L, we will substitute a value of 0 for $f'(0)$.

For the parallel chord method of determining the functions R and L, we begin by defining $R(\epsilon) = \epsilon$ for each sufficiently small positive ϵ, then choosing $L(\epsilon)$ to be the greatest negative number (that is, closest to 0) that satisfies

$$\frac{f(\epsilon) - f(L(\epsilon))}{\epsilon - L(\epsilon)} = f'(0).$$

Note that 0 is the point guaranteed by the Mean Value Theorem for the function f on the interval $[L(\epsilon), \epsilon]$ and that the chord joining the points $(\epsilon, f(\epsilon))$ and $(L(\epsilon), f(L(\epsilon)))$ is parallel to the tangent line of f at 0. The following graph illustrates the case in which $f'(0) = 0$.



It can be shown that L is a well-defined twice differentiable function that satisfies

$$\lim_{\epsilon\to 0^+} L(\epsilon) = 0, \ \lim_{\epsilon\to 0^+} L'(\epsilon) = -1, \quad \lim_{\epsilon\to 0^+} L''(\epsilon) = -\frac{2f'''(0)}{3f''(0)}.$$

A rigorous proof of these results is a bit involved so we will simply refer the reader to [5]. However, a simple way to obtain these values is to assume that both f and L have Maclaurin series expansions:

$$f(\epsilon) = a\,\epsilon + \frac{b}{2}\,\epsilon^2 + \frac{c}{6}\,\epsilon^3 + \cdots, \quad L(\epsilon) = -q\,\epsilon + \frac{t}{2}\,\epsilon^2 + \cdots.$$

Since the function L must satisfy the equation

$$f(\epsilon) - f'(0)\epsilon = f(L(\epsilon)) - f'(0)L(\epsilon),$$

equating polynomial coefficients in the expanded form of this equation makes it possible to find q and t in terms of b and c; the values obtained yield the limits listed earlier. In any event, for the parallel chord method, we find that

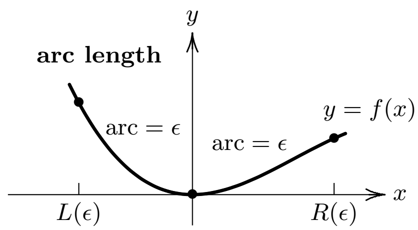
$$R(\epsilon) = \epsilon, \quad L(\epsilon) = -\epsilon - \frac{4c}{3b}\,\epsilon^2 + \epsilon^2\, Z_L(\epsilon),$$

where $\lim_{\epsilon\to 0^+} Z_L(\epsilon) = 0$. Putting in the relevant values for q, r, and t yields

$$\frac{r+t}{2\,b\,q^2} = -\frac{1}{3}\cdot\frac{c}{b^2}.$$

(Recall from an earlier section that $(r+t)/(2\,b\,q^2)$ is the key quantity.) Thus, the parallel chord method for determining $\mathcal{A}_{mc}$, which we will denote by ${}^{p}\!\mathcal{A}_{mc}$, gives a value of $\frac{1}{3}\cdot d\rho/ds$. Once this value is known, all of the other aberrancy measures using this choice of R and L are determined. However, we will wait until the end of this section to record all of the values.

The next method that we will consider involves arc length. The idea is to travel the same distance away from the point $(0,0)$ along the arc of the curve and define R and L to be the corresponding x-coordinates:



The arc length function s of f, given by $s(x) = \int_0^x \sqrt{1+f'(t)^2}\,dt$ for x in a neighborhood of the origin, is strictly increasing on its domain. Let R be the inverse of s and define L by $L(\epsilon) = R(-\epsilon)$. The functions R and L have the desired properties since the points $(L(\epsilon), f(L(\epsilon)))$, and $(R(\epsilon), f(R(\epsilon)))$ are exactly the same distance (namely ϵ) from $(0,0)$ along the curve $y = f(x)$. Using the Fundamental Theorem of Calculus and the formula for the derivative of an inverse function, we obtain

$$R'(\epsilon) = \frac{1}{s'(R(\epsilon))} = \frac{1}{\sqrt{1+f'(R(\epsilon))^2}}, \quad R''(\epsilon) = -\frac{f'(R(\epsilon))f''(R(\epsilon))}{\left(1+f'(R(\epsilon))^2\right)^2},$$

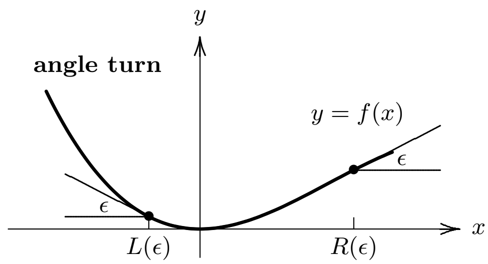
and thus

$$R'(0) = \frac{1}{\sqrt{1+f'(0)^2}} = -L'(0), \quad R''(0) = -\frac{f'(0)f''(0)}{\left(1+f'(0)^2\right)^2} = L''(0).$$

Letting $f'(0) = 0$ gives $q = 1$ and $r = 0 = t$. It then follows that the arc length method for determining $\mathcal{A}_{mc}$, which we will denote by ${}^{s}\!\mathcal{A}_{mc}$, is 0. According to this method, the graph of every thrice differentiable function looks locally like a circle or, to say it another way, the axis of aberrancy always coincides with the normal line. Hence, this particular aberrancy measure does not distinguish between curves.

The third method that we consider involves the amount of turning done by the tangent lines. For each small positive number ϵ, let $L(\epsilon)$ be the largest negative number for which the angle between the tangent line to f at $(L(\epsilon), f(L(\epsilon)))$ and the tangent line to f at $(0,0)$ is ϵ, and let $R(\epsilon)$ be the smallest positive number for which the angle between the tangent line to f at $(R(\epsilon), f(R(\epsilon)))$ and the tangent line to f at $(0,0)$ is ϵ. The values for L and

R are indicated in the following graph under the assumption that $f'(0) = 0$.



Since we are assuming that $f''(0) \neq 0$, the function f' is strictly monotone on a neighborhood of 0 and thus has an inverse g. If T is the function defined by $T(\epsilon) = \tan(\arctan f'(0) + \epsilon)$, then $R(\epsilon) = g(T(\epsilon))$ and $L(\epsilon) = R(-\epsilon)$. To compute the necessary derivatives, we need the chain rule and the properties of inverse functions. First of all,

$$\begin{aligned}
T'(\epsilon) &= \sec^2(\arctan f'(0) + \epsilon), \\
T''(\epsilon) &= 2\sec^2(\arctan f'(0) + \epsilon)\tan(\arctan f'(0) + \epsilon), \\
g'(x) &= \frac{1}{f''(g(x))}, \\
g''(x) &= -\frac{f'''(g(x))}{f''(g(x))^3}, \\
R'(\epsilon) &= g'(T(\epsilon))T'(\epsilon), \\
R''(\epsilon) &= g'(T(\epsilon))T''(\epsilon) + g''(T(\epsilon))T'(\epsilon)^2.
\end{aligned}$$

From these formulas, we find that

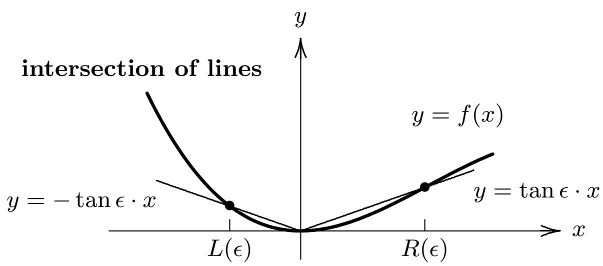
$$\begin{aligned}
R'(0) &= g'(T(0))T'(0) = g'(f'(0))\sec^2(\arctan f'(0)) = \frac{1 + f'(0)^2}{f''(0)}; \\
R''(0) &= g'(f'(0))T''(0) + g''(f'(0))T'(0)^2 \\
&= \frac{2f'(0)(1 + f'(0)^2)}{f''(0)} - \frac{f'''(0)}{f''(0)^3}\left(1 + f'(0)^2\right)^2.
\end{aligned}$$

Now set $f'(0) = 0$ (note how messy the computations are without this assumption), $f''(0) = b$, and $f'''(0) = c$ to obtain $q = R'(0) = 1/b$ and $r = R''(0) = -c/b^3 = t$. It follows that

$$\frac{r + t}{2\,b\,q^2} = -\frac{c}{b^2}.$$

Thus, the angle turn of the tangent method for determining $\mathcal{A}_{mc}$, which we will denote by ${}^{a}\!\mathcal{A}_{mc}$, gives a value of $1 \cdot d\rho/ds$. Hence, this geometric measure corresponds exactly with $d\rho/ds$.

The fourth method that we will consider involves the intersection of the curve with lines emanating from the origin. For each small positive number ϵ, let $T(\epsilon) = \tan\bigl(\arctan f'(0) + \epsilon\bigr)$ (as in the last paragraph), let $R(\epsilon)$ be the smallest positive number that satisfies $f(x) = T(\epsilon)\,x$, and let $L(\epsilon)$ be the largest negative number that satisfies $f(x) = T(-\epsilon)\,x$. In addition, we define $R(0) = 0 = L(0)$. The fact that $f''(0) > 0$ guarantees that the numbers $R(\epsilon)$ and $L(\epsilon)$ exist for sufficiently small $\epsilon > 0$. The values for L and R are indicated in the following graph under the assumption that $f'(0) = 0$.



Let h be the function defined by $h(x) = f(x)/x$ for $x \neq 0$ and $h(0) = f'(0)$. Then

$$h(x) = f'(0) + \frac{b}{2}\,x + \frac{c}{6}\,x^2 + x^2 Z_f(x),$$

where $\lim_{x\to 0} Z_f(x) = 0$. It follows that

$$\begin{aligned}
h'(x) &= \frac{xf'(x) - f(x)}{x^2} = \frac{b}{2} + \frac{c}{3}\,x + xZ_{f'}(x) - xZ_f(x);\\
h'(0) &= \lim_{x\to 0} \frac{h(x) - h(0)}{x} = \lim_{x\to 0}\Bigl(\frac{b}{2} + \frac{c}{6}\,x + xZ_f(x)\Bigr) = \frac{b}{2};\\
h''(0) &= \lim_{x\to 0} \frac{h'(x) - h'(0)}{x} = \lim_{x\to 0}\Bigl(\frac{c}{3} + Z_{f'}(x) - Z_f(x)\Bigr) = \frac{c}{3}.
\end{aligned}$$

Since h' is continuous and $h'(0) > 0$ (recall that we are assuming $b = f''(0)$ is positive), it follows that h' is positive on a neighborhood of 0. Thus, h is increasing on a neighborhood of 0 and has an inverse, call it H. The functions R and L satisfy $R(\epsilon) = H(T(\epsilon))$ and $L(\epsilon) = H(T(-\epsilon)) = R(-\epsilon)$. Using the derivative of an inverse function and the chain rule, we find that

$$R'(\epsilon) = \frac{T'(\epsilon)}{h'(H(T(\epsilon)))} = \frac{T'(\epsilon)}{h'(R(\epsilon))},$$
$$R''(\epsilon) = \frac{h'(R(\epsilon))T''(\epsilon) - T'(\epsilon)h''(R(\epsilon))R'(\epsilon)}{h'(R(\epsilon))^2}.$$

These derivatives then give general formulas for $R'(0)$ and $R''(0)$:

$$R'(0) = \frac{T'(0)}{h'(0)} = \frac{2}{b}\,T'(0),$$
$$R''(0) = \frac{h'(0)T''(0) - T'(0)h''(0)R'(0)}{h'(0)^2} = \frac{4}{b^2}\Big(\frac{b}{2}\,T''(0) - \frac{2c}{3b}\,T'(0)^2\Big).$$

For the special case in which $f'(0) = 0$, it is easy to verify that $T'(0) = 1$ and $T''(0) = 0$. We then have $q = R'(0) = 2/b$ and $r = R''(0) = -8c/(3b^3) = t$, and it follows that

$$\frac{r+t}{2\,b\,q^2} = -\frac{8\,c}{3\,b^3}\cdot\frac{b}{4} = -\frac{2}{3}\cdot\frac{c}{b^2}.$$

Thus, the intersection of lines method for determining $\mathcal{A}_{mc}$, denoted by ${}^{i}\!\mathcal{A}_{mc}$, gives a value of $\frac{2}{3}\cdot d\rho/ds$.

The following table records all the measures of aberrancy discussed thus far. The entries in the table give the appropriate multiple of $d\rho/ds$ for the given measure. The methods for choosing a point (u_ϵ, v_ϵ) appear as rows and the methods for choosing functions R and L appear as columns, each with the notation developed in the paper.

	${}^{s}\!\mathcal{A}$	${}^{p}\!\mathcal{A}$	${}^{i}\!\mathcal{A}$	${}^{a}\!\mathcal{A}$
$\mathcal{A}_{mc}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\mathcal{A}_{t\ell}$	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
$\mathcal{A}_{cc}$	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$
$\mathcal{A}_{tr}$	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{7}{3}$
$\mathcal{A}_{ce}$	$-\frac{2}{9}$	$\frac{1}{3}$	$\frac{8}{9}$	$\frac{13}{9}$

One of the most fascinating features of this table is that the parallel chord column is constant. It is safe to assume that Carnot and Transon did not use this fact to make their decision on how to define aberrancy, but the stability of parallel chord aberrancy certainly shows that they made a good choice. It is interesting to note that all of the rows form arithmetic sequences (since the relationships are all linear) and that the values in the $\mathcal{A}_{mc}$ row are all the multiples of $\frac{1}{3}$ that lie between 0 and 1. The fact that two of the columns contain both positive and negative values is also a bit intriguing, as is the fact that two of the columns contain 0 entries. Only ${}^{a}\mathcal{A}_{mc}$ gives a geometric measure of aberrancy that corresponds exactly with $d\rho/ds$.

7 Infinite Collections of Aberrancy Measures.

Two other methods for using the numbers $R(\epsilon)$ and $L(\epsilon)$ to determine a point (u_ϵ, v_ϵ) are the following. For the first, use the crossing chords method to determine a point (U_ϵ, V_ϵ) as in the main discussion, then let (u_ϵ, v_ϵ) be the centroid of the triangle with vertices $(L(\epsilon), f(L(\epsilon)))$, $(R(\epsilon), f(R(\epsilon)))$, and (U_ϵ, V_ϵ). In contrast to the crossing chord method, the aberrancy measure from the cross triangle, which we will denote by $\mathcal{A}_{ct}^{\lambda}$, depends on λ, that is, each value of λ gives a different measure. Using the same technique as for the other triangle method, we find that

$$\mathcal{A}_{ct}^{\lambda} = \frac{2\mathcal{A}_{mc} + \lambda\mathcal{A}_{cc}}{2+\lambda}.$$

Note that $\mathcal{A}_{ct}^{\lambda}$ is a weighted average of $\mathcal{A}_{mc}$ and $\mathcal{A}_{cc}$, but that the weights depend on λ.

For the second method, let λ be a fixed positive number and let (u_ϵ, v_ϵ) be the point of intersection of the lines

$$y = f'(L(\lambda\epsilon))(x - R(\epsilon)) + f(R(\epsilon)), \quad y = f'(R(\lambda\epsilon))(x - L(\epsilon)) + f(L(\epsilon)).$$

Since a switching of the tangent line slopes occurs, we will denote this aberrancy measure by $\mathcal{A}_{sw}^{\lambda}$. We leave it to the reader to verify that

$$k_{sw}^{\lambda} = \frac{-\lambda^2+\lambda+1}{\lambda(2\lambda+1)}\, k_{mc} + \frac{3\lambda^2-1}{3\lambda(2\lambda+1)}.$$

As with all the other measures, the constant k_{sw}^{λ} depends linearly on k_{mc}. Note that this measure of aberrancy also depends on the choice of the parameter λ.

These results give us further rows in our table of aberrancy measures (recorded as multiples of $d\rho/ds$). In addition to the values dependent on

λ, we also record two very interesting special cases. The number ϕ represents the golden mean $(1+\sqrt{5})/2$, which is the positive solution to $\phi^2 = \phi + 1$.

	${}^{s}\!\mathcal{A}$	${}^{p}\!\mathcal{A}$	${}^{i}\!\mathcal{A}$	${}^{a}\!\mathcal{A}$
$\mathcal{A}_{ct}^{\lambda}$	$\frac{2\lambda}{3(\lambda+2)}$	$\frac{1}{3}$	$\frac{4}{3(\lambda+2)}$	$\frac{6-\lambda}{3(\lambda+2)}$
$\mathcal{A}_{ct}^{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\mathcal{A}_{sw}^{\lambda}$	$\frac{3\lambda^2-1}{3\lambda(2\lambda+1)}$	$\frac{1}{3}$	$\frac{(\lambda+1)^2}{3\lambda(2\lambda+1)}$	$\frac{3\lambda+2}{3\lambda(2\lambda+1)}$
$\mathcal{A}_{sw}^{\phi}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

There are many interesting features to note about these numbers. First of all, the constant for ${}^{p}\!\mathcal{A}$ is still $\frac{1}{3}$ for each of the new measures, independent of the choice of λ. Secondly, and more importantly, two specific values for λ generate a constant row. The value $\lambda = 2$ seems a natural choice for $\mathcal{A}_{ct}^{\lambda}$; it is convenient that it gives a constant row. However, the appearance of the golden mean as the value of λ for which $\mathcal{A}_{sw}^{\lambda}$ has a constant value is rather remarkable. A third feature is that ${}^{a}\!\mathcal{A}_{ct}^{6} = 0$, which puts a zero in a column that did not have a zero before. Some other random observations are the following:

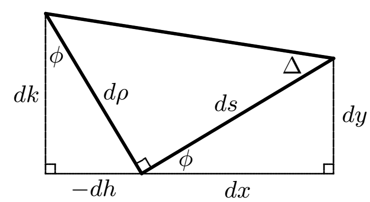
1. Although we initially restricted λ to be a number greater than 1 for the triangle method, other values for λ can be used. As the reader may verify, $\mathcal{A}_{ct}^{0} = \mathcal{A}_{mc}$, $\mathcal{A}_{ct}^{-1} = \mathcal{A}_{tr}$, $\mathcal{A}_{ct}^{-1/2} = \mathcal{A}_{ce}$, and $\lim\limits_{\lambda\to\infty} \mathcal{A}_{ct}^{\lambda} = \mathcal{A}_{t\ell}$. When $\lambda = 0$, the centroid of the triangle falls on the same line as the midpoint of the chord (the centroid of a triangle lies at the common intersection of the three medians). When $\lambda = -1$ and ϵ is small, the chords essentially become tangent lines. This explains the first two results; the other two results do not appear to have simple geometric interpretations.

2. From a graph, it appears that there should be some value of λ for which $\mathcal{A}_{sw}^{\lambda}$ is equivalent to $\mathcal{A}_{cc}$; it was this possibility that motivated us to look at this method. However, this does not occur unless $\lambda = -1$, which (in addition to being out of the range of the parameter values) essentially returns us to the tangent line method. For certain positive values of λ, the measure $\mathcal{A}_{sw}^{\lambda}$ becomes equal to other aberrancy measures. For instance, $\mathcal{A}_{sw}^{1/\sqrt{3}} = \mathcal{A}_{mc}$. The reader can easily find values for λ that give $\mathcal{A}_{tr}$ and $\mathcal{A}_{ce}$.

8 Interpreting Measures of Aberrancy.

We have now looked at a vast number of measures of aberrancy. Since the functions L, R and the points (u_ϵ, v_ϵ) each have geometric interpretations, and since aberrancy is related to the angle between the axis of aberrancy and the normal line, each measure has a geometric interpretation. But why are the various measures of aberrancy related to one another in such simple and interesting ways? Although we have been unable to find satisfactory explanations for these relationships, we consider a few fresh ways to look at the interconnections between them.

We begin with a diagram that illustrates the quantity $d\rho/ds$:



Since this diagram involves differentials, it must be interpreted a bit loosely, but it does reveal some interesting information. The right triangle involving dx, dy, and ds is the usual differential triangle, where ϕ is the angle of inclination of the tangent line, while the right triangle involving dh, dk, and $d\rho$ is the differential triangle for the evolute of the curve, where (h, k) represents the center of curvature. (Recall that the evolute is the curve passing through all of the centers of curvature of the original curve.) Since the differentials in the diagram represent lengths and are thus positive, we use $-dh$ for the base of the triangle since dh is negative for the situation considered here. For the record, we have assumed that dy and $d\rho$ are positive for this diagram; the reader is encouraged to draw similar diagrams when one or both of these quantities is negative. When using differentials to represent portions of curves, the curves are considered as being nearly straight lines. Hence, the side ds is a small portion of the original curve, and the side $d\rho$ is a small portion of the evolute. Since the tangent lines of the evolute are normal lines for the original curve, the sides ds and $d\rho$ meet at right angles. (In order for ds and $d\rho$ to meet at a point, the two differential triangles must be placed together as in the diagram; this involves a translation of the evolute graph.) This fact also explains the appearance of the angle ϕ in the evolute differential triangle. We now have a geometric interpretation for the quantity $d\rho/ds$; it is $\tan\Delta$, where Δ is the angle indicated in the diagram. Noting that the two differential triangles are similar reveals an interesting fact:

$$\frac{-dh}{dy} = \frac{d\rho}{ds} \;\Leftrightarrow\; \frac{dh}{dx} = -\frac{dy}{dx}\cdot\frac{d\rho}{ds}; \qquad \frac{dk}{dx} = \frac{d\rho}{ds}.$$

Therefore, the rate of change of the position of the center of curvature is related to aberrancy. We stumbled upon this fact after computing the derivatives of $h(x)$ and $k(x)$, where

$$h(x) = x - \frac{f'(x)}{f''(x)}\bigl(1 + f'(x)^2\bigr), \quad k(x) = f(x) + \frac{1}{f''(x)}\bigl(1 + f'(x)^2\bigr)$$

are the standard formulas for the center of curvature for a curve of the form $y = f(x)$. The appearance of $d\rho/ds$ in $h'(x)$ and $k'(x)$ was rather mysterious until we noted these similar triangles.

We next consider the equation $k_{mc} + k_{t\ell} = \frac{2}{3}$, which is equivalent to the equation $\mathcal{A}_{mc} + \mathcal{A}_{t\ell} = \frac{2}{3}\cdot d\rho/ds$. We have not found a good geometric explanation for this relationship, but it is interesting to write the equation in another form. Let's adopt the notation $\mathcal{A}_{mc} = \tan\delta_{mc}$, etc., and let $\tan\delta = \frac{1}{3}\cdot d\rho/ds$. It follows that

$$\tan\delta = \frac{1}{2}\bigl(\tan\delta_{mc} + \tan\delta_{t\ell}\bigr) \quad\Leftrightarrow\quad \delta = \arctan\Bigl(\frac{\tan\delta_{mc} + \tan\delta_{t\ell}}{2}\Bigr),$$

revealing that the angle δ is a type of mean of the angles δ_{mc} and $\delta_{t\ell}$ (see [6]). It is possible to find similar equations for the other aberrancy measures, all revealing that $\tan\delta$ is some sort of weighted average of other tangents of angles of aberrancy; we leave such exploration to the interested reader.

The work in this paper has involved a general thrice differentiable function f, but consider for a moment the quadratic polynomial $f(x) = px^2 + qx + r$, where $p \neq 0$, and let $c \in \mathbb{R}$. The chords parallel to the tangent line at $(c, f(c))$ have the form $y = f'(c)(x - c) + f(c) + \epsilon$, and these chords will intersect the parabola as long as ϵ has the same sign as p. The x coordinates of the two points of intersection satisfy the equation

$$px^2 + qx + r = (2pc + q)(x - c) + (pc^2 + qc + r) + \epsilon,$$

which can be written in standard form as

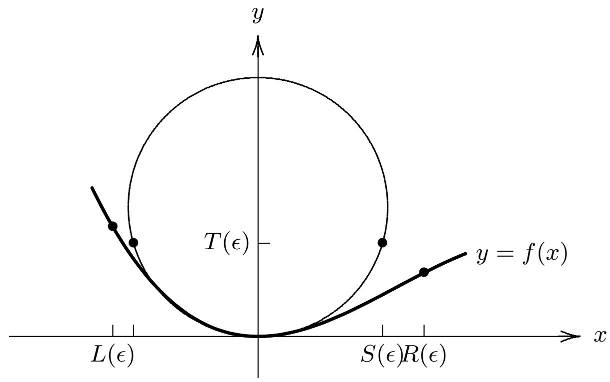
$$x^2 - 2cx + c^2 - \frac{\epsilon}{p} = 0.$$

The sum of the roots of this equation is $2c$. That is, the midpoint of any chord of a quadratic polynomial lies directly above or below the point on the parabola where the tangent line is parallel to the chord. Equivalently, the point guaranteed by the Mean Value Theorem for a quadratic polynomial is the midpoint of the interval. Hence, the axis of aberrancy is a vertical line, and the angle formed by the normal line and the axis of aberrancy is the same as the angle formed by the tangent line and the x-axis. In other words, using the notation developed in this section, ${}^{p}\delta_{mc} = \phi = \delta$. Since $d\rho/ds$ at the point $(c, f(c))$ in this case is $3f'(c) = 3\tan\phi$, this fact provides one explanation as to why $\frac{1}{3} \cdot d\rho/ds$ is the key measure of aberrancy.

The idea of using a parabola to study aberrancy has some merits. In keeping with the notation of the paper, consider the function f defined by $f(x) = ax + (b/2)x^2$, where $a \neq 0$ and $b > 0$, and focus on the behavior at the origin. It is relatively easy to find all of the aberrancy measures for this function, and all of the computations can be done by hand. (Since the normal line is no longer the y-axis, it is necessary to add a step in order to find $\tan\delta_{mc}$, etc.) We encourage the reader to carry out some of these computations because it helps clarify equations such as $k_{mc} + k_{t\ell} = \frac{2}{3}$ and $k_{cc} = k_{t\ell}$. For instance, the tangent lines to f at $(L, f(L))$ and $(R, f(R))$ intersect when $x = (L + R)/2$; this is why ${}^{p}\mathcal{A}_{mc} = {}^{p}\mathcal{A}_{t\ell}$. Also, the slope of the axis of aberrancy for the midpoint method is $a + \mu$, for some value of μ, while the slope of the axis of aberrancy for the tangent line method is $a - \mu$; this helps explain why $k_{mc} + k_{t\ell} = \frac{2}{3}$. Since it is comparatively easy to compute the measures of aberrancy for parabolas, we can use this as an indirect way to compute the aberrancy of a general curve. First, find the rotated parabola that best approximates the graph of a general thrice differentiable function f at a point $(c, f(c))$. To find such a parabola, the first three derivatives of both f and the parabola must be the same at $(c, f(c))$. The aberrancy of f at the given point is then the aberrancy of the approximating parabola at this point. This approach to aberrancy is mentioned in Schot's article; the details can be found in [5]. Although it has some merit, we do not find this indirect approach to aberrancy very satisfying.

9 An Affine Approach to Aberrancy.

As we have noted, aberrancy can be interpreted as a measure of how much a curve varies from a circle. The appropriate circle for comparison purposes in this situation is the circle of curvature at the given point. Adopting our earlier notation and conventions ($f'(0) = 0$, $f''(0) > 0$), each $\epsilon > 0$ generates two points on the curve, namely $(L(\epsilon), f(L(\epsilon)))$ and $(R(\epsilon), f(R(\epsilon)))$. Let $(-S(\epsilon), T(\epsilon))$ and $(S(\epsilon), T(\epsilon))$ be the corresponding points on the circle of curvature, where we have taken advantage of the symmetry of the circle. The points on the circle, that is, the functions S and T, will vary with the method for choosing L and R. One arrangement for these points is depicted in the following figure. (Once again, the origin is our point of interest.)



Suppose that we find a linear transformation that maps the points on the circle to the corresponding points on the curve and then see what happens to this transformation as ϵ decreases to 0. (The transformation is linear since the origin remains fixed; in a moment we will consider a situation in which this does not occur.) It seems plausible that the limiting transformation should tell us something about the aberrancy of the curve. As we will soon see, this approach gives us the same measure of aberrancy as the midpoint of the chord approach.

To carry out this process, we need to find a matrix with the following property:

$$\begin{bmatrix} A_\epsilon & B_\epsilon \\ C_\epsilon & D_\epsilon \end{bmatrix} \begin{bmatrix} -S(\epsilon) & S(\epsilon) \\ T(\epsilon) & T(\epsilon) \end{bmatrix} = \begin{bmatrix} L(\epsilon) & R(\epsilon) \\ f(L(\epsilon)) & f(R(\epsilon)) \end{bmatrix}.$$

Since the matrix corresponding to the circle points is invertible, we can easily solve this equation (we will suppress the (ϵ) notation for ease of reading):

$$\begin{bmatrix} A_\epsilon & B_\epsilon \\ C_\epsilon & D_\epsilon \end{bmatrix} = \frac{1}{2ST} \begin{bmatrix} L & R \\ f(L) & f(R) \end{bmatrix} \begin{bmatrix} -T & S \\ T & S \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \frac{R-L}{S} & \frac{R+L}{T} \\ \frac{f(R)-f(L)}{S} & \frac{f(R)+f(L)}{T} \end{bmatrix}$$

We claim that

$$\lim_{\epsilon \to 0^+} \begin{bmatrix} A_\epsilon & B_\epsilon \\ C_\epsilon & D_\epsilon \end{bmatrix} = \begin{bmatrix} 1 & \mathcal{A}_{mc} \\ 0 & 1 \end{bmatrix},$$

where the type of aberrancy agrees with the previous results and depends on the method for choosing the points on the curve. The limiting transformation corresponds to a skewing of the plane. It leaves the x-axis alone and rotates the y-axis (the normal line for the circle and the curve) onto the axis of aberrancy.

We will leave most of the details to the reader. The first thing to do is to find the appropriate formulas for $S(\epsilon)$ and $T(\epsilon)$. Since the circle of curvature has radius $1/b$ (recall that $f''(0) = b \neq 0$) and center $(0, 1/b)$, its equation can be written as $b(x^2 + y^2) = 2y$. Using the methods for finding the points on the curve and simple properties of circles, we find the following values for $S(\epsilon)$ and $T(\epsilon)$:

arc length: $S(\epsilon) = \frac{1}{b} \sin b\epsilon, \qquad T(\epsilon) = \frac{1}{b}\left(1 - \cos b\epsilon\right);$

parallel chord: $S(\epsilon) = \epsilon, \qquad T(\epsilon) = \frac{1}{b}\left(1 - \sqrt{1 - b^2\epsilon^2}\right);$

intersection of lines: $S(\epsilon) = \frac{1}{b} \sin 2\epsilon, \qquad T(\epsilon) = \frac{1}{b} \sin 2\epsilon \tan \epsilon;$

angle turn: $S(\epsilon) = \frac{1}{b} \sin \epsilon; \qquad T(\epsilon) = \frac{1}{b}\left(1 - \cos \epsilon\right).$

Note that $S(\epsilon)$ is of order ϵ and that $T(\epsilon)$ is of order ϵ^2 in all cases. As an example of the sort of computations needed, we will find $\lim\limits_{\epsilon \to 0^+} B_\epsilon$. Adopting the notation from previous sections, we find that

$$\lim_{\epsilon \to 0^+} B_\epsilon = \frac{1}{2} \lim_{\epsilon \to 0^+} \frac{R(\epsilon) + L(\epsilon)}{\epsilon^2} \div \lim_{\epsilon \to 0^+} \frac{T(\epsilon)}{\epsilon^2} = \frac{r + t}{4} \div \lim_{\epsilon \to 0^+} \frac{T(\epsilon)}{\epsilon^2}.$$

Locating the r and t values determined earlier and computing the elementary T limits yield the results

arc length: $\lim\limits_{\epsilon \to 0^+} B_\epsilon = 0 \cdot \frac{-c}{b^2};$

parallel chord: $\lim\limits_{\epsilon \to 0^+} B_\epsilon = \frac{1}{3} \cdot \frac{-c}{b^2};$

intersection of lines: $\lim\limits_{\epsilon \to 0^+} B_\epsilon = \frac{2}{3} \cdot \frac{-c}{b^2};$

angle turn: $\lim\limits_{\epsilon \to 0^+} B_\epsilon = 1 \cdot \frac{-c}{b^2}.$

Since $-c/b^2$ is $d\rho/ds$ for the current situation, these findings are in agreement with our previous results.

We close this section by mentioning another possibility. In our earlier work, we used the points $(L(\epsilon), f(L(\epsilon)))$ and $(R(\epsilon), f(R(\epsilon)))$ to generate a third point $(U(\epsilon), V(\epsilon))$. Let $(0, Z(\epsilon))$ denote the corresponding point for the circle of curvature. (Note that our assumption that $f'(0) = 0$ guarantees that this point will always fall on the y-axis.) We can now seek an affine mapping of the form

$$\begin{bmatrix} A_\epsilon & B_\epsilon \\ C_\epsilon & D_\epsilon \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} E_\epsilon \\ F_\epsilon \end{bmatrix} \quad \text{with} \quad \begin{array}{rcl} (-S(\epsilon), T(\epsilon)) & \mapsto & (L(\epsilon), f(L(\epsilon))), \\ (0, Z(\epsilon)) & \mapsto & (U(\epsilon), V(\epsilon)), \\ (S(\epsilon), T(\epsilon)) & \mapsto & (R(\epsilon), f(R(\epsilon))), \end{array}$$

and then determine the limit of the six terms as ϵ decreases to 0. It should then be possible to compare the resulting measure of aberrancy expressed by the limiting affine transformation (it will be linear) with the previous measures. We leave the pursuit of this approach (as well as others in the same vein) to the interested reader.

10 Some Comments on the Assumption $f'(0) = 0$.

If the reader is a bit troubled by the assumption that $f'(0) = 0$, then the following observations may be useful. We initially did all of our calculations under the assumption that $f'(0) \neq 0$. However, this increases the level of difficulty in the calculations quite a bit. We will consider two ways to validate the assumption that $f'(0) = 0$. The first method is elementary but tedious. Suppose that f does not have a horizontal tangent line at $(0, 0)$, that is, suppose that $f'(0) = a \neq 0$. We will rotate the axes so that the tangent line becomes horizontal, apply our previous results to the rotated curve, then express the answer in terms of the derivatives of the original function f. To make the appropriate change of variables to rotate the axes, let $\theta = \arctan a$ and

$$x = w\cos\theta - z\sin\theta = \alpha w - \beta z, \qquad y = w\sin\theta + z\cos\theta = \beta w + \alpha z,$$

where $\alpha = \cos\theta$ and $\beta = \sin\theta$. The equation $y = f(x)$ then assumes the form $\beta w + \alpha z = f(\alpha w - \beta z)$, which defines z as an implicit function of w. It is easy to verify that z has a horizontal tangent line at the origin. We leave it to the reader to use implicit differentiation to find the first three derivatives of z with respect to w, then show that

$$-\frac{\left.\dfrac{d^3z}{dw^3}\right|_{(0,0)}}{\left(\left.\dfrac{d^2z}{dw^2}\right|_{(0,0)}\right)^2} = 3a - \frac{c(a^2+1)}{b^2}.$$

This last value is $d\rho/ds$ for the function f at $(0, 0)$.

The second method is a bit more sophisticated since it requires knowledge of the unit tangent and unit normal vectors for a curve defined parametrically. Consider a thrice differentiable curve C in the plane and assume that C passes through the origin. It is then possible to represent C parametrically using arc length s (measured from the origin) as parameter. That is, the curve C is defined by $x = f(s)$ and $y = g(s)$, where f and g are thrice differentiable functions that satisfy $f(0) = 0 = g(0)$, and s represents arc length. Since arc length is the parameter, the unit tangent vector for the curve C is given by

$$\mathbf{T}(s) = \begin{pmatrix} f'(s) \\ g'(s) \end{pmatrix}.$$

The curvature κ of C is the rate of change of the direction of $\mathbf{T}$ with respect to arc length, that is, $\kappa(s) = |\mathbf{T}'(s)|$. (Note that $\kappa(s)$ is nonnegative for all values of s when viewed in this way.) It is easy to verify that $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal; simply differentiate the equation $\mathbf{T} \cdot \mathbf{T} = 1$. Consequently, the principal unit normal $\mathbf{N}$ is defined by $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$. The reader can verify that the following equations are valid:

$$\mathbf{T}'(s) = \begin{pmatrix} f''(s) \\ g''(s) \end{pmatrix}, \quad \kappa(s) = \sqrt{f''(s)^2 + g''(s)^2}, \quad \text{and} \quad \mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s).$$

Using expressions for thrice differentiable functions discussed earlier, it then follows that

$$\begin{aligned}
&\begin{pmatrix} f(s) \\ g(s) \end{pmatrix} \\
&\quad = s \begin{pmatrix} f'(0) \\ g'(0) \end{pmatrix} + \frac{s^2}{2} \begin{pmatrix} f''(0) \\ g''(0) \end{pmatrix} + \frac{s^3}{6} \begin{pmatrix} f'''(0) \\ g'''(0) \end{pmatrix} + s^3 \begin{pmatrix} Z_f(s) \\ Z_g(s) \end{pmatrix} \\
&\quad = s\,\mathbf{T}(0) + \frac{s^2}{2}\,\mathbf{T}'(0) + \frac{s^3}{6}\,\mathbf{T}''(0) + s^3 \begin{pmatrix} Z_f(s) \\ Z_g(s) \end{pmatrix} \\
&\quad = s\,\mathbf{T}(0) + \frac{s^2}{2}\,\kappa(0)\,\mathbf{N}(0) + \frac{s^3}{6}\big(\kappa'(0)\,\mathbf{N}(0) - \kappa(0)^2\,\mathbf{T}(0)\big) + s^3 \begin{pmatrix} Z_f(s) \\ Z_g(s) \end{pmatrix} \\
&\quad = \Big(s - \frac{s^3}{6}\,\kappa(0)^2 + s^3 Z_1^*(s)\Big)\mathbf{T}(0) + \Big(\frac{s^2}{2}\,\kappa(0) + \frac{s^3}{6}\,\kappa'(0) + s^3 Z_2^*(s)\Big)\mathbf{N}(0) \\
&\quad \approx \Big(s - \frac{s^3}{6}\,\kappa(0)^2\Big)\mathbf{T}(0) + \Big(\frac{s^2}{2}\,\kappa(0) + \frac{s^3}{6}\,\kappa'(0)\Big)\mathbf{N}(0),
\end{aligned}$$

where the approximation is justified by virtue of $\lim_{s \to 0} Z_1^*(s) = 0 = \lim_{s \to 0} Z_2^*(s)$.

In the **T**-**N** coordinate system, the tangent line is horizontal, and the coordinates of the curve are essentially

$$x(s) = s - \frac{\kappa(0)^2}{6}\, s^3, \qquad y(s) = \frac{\kappa(0)}{2}\, s^2 + \frac{k'(0)}{6}\, s^3.$$

Since the tangent line is horizontal, we can use our previous work to determine that aberrancy involves the quantity $-(d^3y/dx^3)/(d^2y/dx^2)^2$ evaluated at the origin. Computing the derivatives, we find that

$$\frac{d^2y}{dx^2}\Big|_{s=0} = \kappa(0), \quad \frac{d^3y}{dx^3}\Big|_{s=0} = \kappa'(0), \quad -\frac{d^3y/dx^3}{(d^2y/dx^2)^2}\Big|_{s=0} = -\frac{\kappa'(0)}{\kappa(0)^2} = \frac{d\rho}{ds}\Big|_{s=0},$$

(note that $\kappa' = d\kappa/ds$) a result consistent with all of our previous efforts.

11 Concluding Remarks.

We have not considered every possible way to measure the aberrancy of a function. There are other methods for obtaining the functions R and L, as well as other methods for determining the point (u_ϵ, v_ϵ). For instance, we may pick any point on the normal line (the center of curvature is a natural choice), then consider the two lines through this point that make an angle of ϵ with the normal line. The numbers $R(\epsilon)$ and $L(\epsilon)$ are the x-coordinates of the intersection of these lines with the curve. Other ways to determine the point (u_ϵ, v_ϵ) are related to our triangle method. We have used the centroid, but there are also (to name a few) the circumcenter, orthocenter, and incenter. Therefore, with each of the two triangles we have already considered, we could generate a point (u_ϵ, v_ϵ) in many different ways. We have studied several (but certainly not all) of these, but obtained no new or interesting results.

The curvature of a line is zero at each point on the line, and the curvature of a circle has the same constant value at each of its points. Similarly, the aberrancy (using any measure) of a circle is zero at each of its points. In fact, we designed each measure of aberrancy so that this result holds. A natural problem is to determine which curves have the property that $d\rho/ds$ is constant. It turns out that the only curves with this property are logarithmic spirals; curves which have the form $r = c\, e^{k\theta}$ in polar coordinates. Note that a circle is a member of this class of curves. A discussion of the aberrancy of logarithmic spirals can be found in Schot [7]. An extension of this problem is to seek a curve with a specified function for $d\rho/ds$. Rather than give an answer to this problem, we leave the reader with the following thoughts. Suppose that g is a differentiable function and that z is a fixed number. Consider the curve C

defined by the parametric equations

$$x(s) = \int_z^s \cos g(t)\, dt, \qquad y(s) = \int_z^s \sin g(t)\, dt,$$

where, as the reader may verify, the parameter s represents arc length. Using the standard formula for the curvature of a curve defined parametrically, we find that the curvature κ of C satisfies $\kappa(s) = g'(s)$. If $d\rho/ds$ is specified as a function of s, then $\rho(s)$ and thus $\kappa(s)$ can be determined; the parametric equations just given then yield a curve with the given $d\rho/ds$. As a first example, the reader can show that the parametric equations

$$x(s) = \int_1^s \cos(\ln t)\, dt, \qquad y(s) = \int_1^s \sin(\ln t)\, dt,$$

define a logarithmic spiral (which may be translated and/or rotated from its standard form) for which $d\rho/ds = 1$. As a second example, the reader is invited to look for a curve that satisfies $d\rho/ds = 2s$.

The fascinating aspect of aberrancy is the relationship between the various geometrical measures. For many familiar functions, there is not much of interest to note about the particular values of its aberrancy at various points along the curve. However, this is an unexplored area of mathematics so there may be some gems waiting to be discovered. One simple curve that does have some interesting features is the ellipse. We encourage the reader to determine the aberrancy of an ellipse and to determine the points on the ellipse at which $d\rho/ds$ has its greatest magnitude.

Other than its relationship to conic sections, the concept of aberrancy has not been studied much. In addition to the article by Schot [7], the interested reader may consult Boyer [1] and Walker [8]. Burgette and Gordon [2] consider the arc length approach to aberrancy; the research on other aberrancy measures began with some suggestions from Burgette. One intriguing problem concerning aberrancy is to determine the number of points at which a polynomial of degree n can have zero aberrancy. Since the zeros of $d\rho/ds$ and $d\kappa/dx$ are related, this problem can be considered as a search for points where the curvature of f has an extreme value. A partial solution to this problem can be found in [4].

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