RESEARCH

Real Analysis Exchange
ISSN:0147-1937
Vol. 32(1), 2006/2007, pp. 171–178

F. Mazzone,* Dpto. de Matemática, Universidad Nacional de Río Cuarto, (5800) Río Cuarto, Argentina. email: `fmazzone@exa.unrc.edu.ar`
E. Schwindt,† Dpto. de Matemática, Universidad Nacional de Río Cuarto, (5800) Río Cuarto, Argentina. email: `leris98@yahoo.com.ar`

A MINIMAX FORMULA FOR THE BEST NATURAL $C([0,1])$-APPROXIMATE BY NONDECREASING FUNCTIONS

Abstract

Let f be a function in $C([0,1])$. We denote by f_p the best approximant to f in $L_p([0,1])$ by nondecreasing functions. It is well known that the limit $f_* := \lim_{p\to\infty} f_p$ exists and f_* is a best approximant to f in $C([0,1])$ by nondecreasing functions. In this paper we show an explicit formula for the function f_* and we prove some additional minimization properties of f_*.

1 Introduction.

Set $S := \{(a,b) \in [0,1]^2 : a < b\}$. For $f \in L_p([0,1])$, $(a,b) \in S$ and $1 < p < \infty$, we denote by $m_p^f(a,b) = m_p(a,b)$ the unique constant which is the best $L_p([a,b])$-approximant to f by constant functions. We note that m_p is characterized by the equality

$$\int_a^b \varphi_p(f - m_p(a,b))dx = 0, \tag{1}$$

where $\varphi_p(y) := |y|^{p-1}\mathrm{sign}(y)$. Similarly, for $f \in C([0,1])$ we define $m_\infty^f(a,b) = m_\infty(a,b)$ replacing the space $L_p([0,1])$ by the space $C([0,1])$ in the previous definition. In this case, we have

$$m_\infty(a,b) = \frac{1}{2}\left(\max_{[a,b]} f + \min_{[a,b]} f\right).$$

Key Words: Monotone best approximants, isotonic approximation, isotonic regression
Mathematical Reviews subject classification: 41A30
Received by the editors May 19, 2006
Communicated by: Alexander Olevskii
*Supported by SECyT-UNRC, ANCPyT PICT 03-12358 and CONICET
†supported by SECyT-UNRC

It is easy to show that $m_p : S \to \mathbb{R}$ is continuous for $1 < p \leq \infty$. If $I = [a, b]$ we write $m_p(I) := m_p(a, b)$.

For $1 < p \leq \infty$ and $x \in (0, 1)$, we will consider the following functions.

$$f_p(x) := \sup_{a<x} \inf_{b>x} m_p(a, b)$$

$$f^p(x) := \inf_{b>x} \sup_{a<x} m_p(a, b)$$

We will need the followings elementary properties.

Theorem 1.1. *For $1 < p \leq \infty$, we have*

1. *f_p, f^p are a nondecreasing functions.*
2. *$f_p \leq f^p$ for all $x \in (0, 1)$.*

Proof. If $x, y \in (0, 1)$ with $x < y$, then

$$\sup_{a<x} \inf_{b>x} m_p(a, b) \leq \sup_{a<x} \inf_{b>y} m_p(a, b) \leq \sup_{a<y} \inf_{b>y} m_p(a, b).$$

Therefore, $f_p(x) \leq f_p(y)$. Analogously, we can prove that f^p is a nondecreasing function.

In order to prove 2, we consider $x \in (0, 1)$. Since $m_p(a, b) \geq \inf_{c>x} m_p(a, c)$ for every $a < x < b$, we have

$$\sup_{a<x} m_p(a, b) \geq \sup_{a<x} \inf_{c>x} m_p(a, c) = f_p(x).$$

Consequently,

$$f^p(x) = \inf_{b>x} \sup_{a<x} m_p(a, b) \geq f_p(x).$$

The proof of Theorem 1.1 is now complete. □

Since f_p, f^p are nondecreasing and bounded functions, we can extend continuously these functions to the points 0 and 1. Henceforth, we assume f_p, f^p are defined on $[0, 1]$.

We also consider the sets

$$\mathcal{F}_p = \{x \in (0, 1) : m_p(a, x) \leq m_p(x, b),\ \forall a \in [0, x) \text{ and } \forall b \in (x, 1]\}.$$

It is easily seen that for $1 < p \leq \infty$ and $0 < a < b < 1$, the sets $\mathcal{F}_p \cap [a, b]$ are compact subsets of $[0, 1]$.

Let $(a,b) \in S$. As is usual, we say that a nondecreasing function $g \in L_p([a,b])$ is a best $L_p([a,b])$-approximant to $f \in L_p([a,b])$ by nondecreasing functions iff

$$\int_a^b |f-g|^p dx \leq \int_a^b |f-h|^p dx$$

for every nondecreasing function $h \in L_p([a,b])$. Analogously, we say that a nondecreasing function g is a best $C([a,b])$-approximant to $f \in C([a,b])$ by nondecreasing functions iff

$$\max_{a\leq x\leq b} |f-g| \leq \max_{a\leq x\leq b} |f-h|$$

for every nondecreasing function h (we note that a best $C([0,1])$-approximant is not assumed to be continuous). Existence of best approximants by nondecreasing functions has been proven in [9, 14]. Moreover, best $L_p([0,1])$-aproximants by nondecreasing functions are unique when $1 < p < \infty$ (see [9]). However, uniqueness is not even true in $C([0,1])$ (see [14]).

The problem of best approximation by monotone functions has been studied extensively in the literature. For example, in nonparametric regression it is considered the problem of isotonic regression. That means regression by nondecreasing functions defined on a finite and partially ordered set (see [2, 13]). Locally isotonic regression was applied in [1, 12] to signal and video processing. Best approximation by monotone functions defined on a set $\Omega \subset \mathbb{R}^n$ was considered by several authors (see [7, 10, 14] for $\Omega \subseteq \mathbb{R}$ a interval, [3, 5, 6, 8] for $\Omega = (0,1)^n$ and [11] for $\Omega \subset \mathbb{R}^n$ an open and bounded set).

It was proved in [4] that for $f \in C([0,1])$ the best $L_p([0,1])$-approximants to f by nondecreasing functions converge uniformly as $p \to \infty$ to f_*, a best $C([0,1])$-approximant to f by nondecreasing functions. The first goal of this paper is to show that $f_* = f_\infty = f^\infty$. In the discrete case, this type of results were obtained using others techniques by V. Ubhaya in [15]. The second objective is to prove that the best approximant f_∞ has an extra minimization property. More precisely, we will show that $f_\infty(x) = f(x)$ for every $x \in \mathcal{F}_\infty$, and that if $a, b \in \mathcal{F}_\infty$ with $a < b$, then f_∞ is a best $C([a,b])$-approximant to f by nondecreasing functions. We call f_∞ the best natural $C([0,1])$-approximant to f by nondecreasing functions.

2 Minimax Formulas for Best Natural Nondecreasing Approximants.

We start by proving a minimax formula for best L_p-approximants, $1 < p < \infty$.

Theorem 2.1. *Let $f \in L_p([0,1])$ for $1 < p < \infty$. Then $f_p = f^p$ a.e. and f_p is the best $L_p([0,1])$-approximant to f by nondecreasing functions.*

Proof. Let g be the best $L_p([0,1])$-approximant to f by nondecreasing functions. The function g is defined almost everywhere. Let $x \in (0,1)$ be a continuity point of g and we put $\alpha = g(x)$. We take $\delta > 0$. From [11, Theorem 3.2], we obtain

$$\int_{\{g \geq \alpha - \delta\} \cap (0,b)} \varphi_p(f - \alpha + \delta) dx \geq 0 \tag{2}$$

for every $b > x$. Since φ_p is strictly increasing, for $1 < p < \infty$, inequality (2) and equation (1) imply that $m_p(\{g \geq \alpha - \delta\} \cap (0,b)) \geq \alpha - \delta$ for every $b > x$. Therefore $\inf_{b>x} m_p(\{g \geq \alpha - \delta\} \cap (0,b)) \geq \alpha - \delta$. We observe that $\{g \geq \alpha - \delta\}$ is an interval with left end point less than x. Hence,

$$f_p(x) = \sup_{a<x} \inf_{b>x} m_p(a,b) \geq g(x) - \delta.$$

Since δ is a positive and arbitrary point and g is continuous a.e., we obtain $g(x) \leq f_p(x)$ a.e.. In a similar way, we can prove that $f^p(x) \leq g(x)$ a.e.. This completes the proof. $\square$

Lemma 2.2. *Let $f \in C([0,1])$. Then*

$$\lim_{p \to \infty} m_p(a,b) = m_\infty(a,b), \tag{3}$$

uniformly in $0 \leq a < b \leq 1$.

Proof. The equality (3) is a well known result when a and b are fixed numbers. We will prove that the limit is uniform in a and b. Suppose to the contrary that there exist $\epsilon > 0$, a sequence p_k tending to ∞, and sequences $a_k < b_k$ such that

$$|m^f_{p_k}(a_k,b_k) - m^f_\infty(a_k,b_k)| \geq \epsilon. \tag{4}$$

We define the functions $f_k(x) := f(a_k + (b_k - a_k)x)$. We observe that

$$m^f_p(a_k,b_k) = m^{f_k}_p(0,1) \tag{5}$$

for every $1 < p \leq \infty$. Since f is a uniformly continuous and bounded function, $\{f_k\}$ is an equicontinuous and bounded sequence. From the Arzela-Ascoli Theorem, we get a function $g \in C([0,1])$ and a subsequence of $\{f_k\}$ which converges to g in $C([0,1])$. For the sake of simplicity, we assume that f_k converges to g in $C([0,1])$. We take k such that

$$\sup_{x\in[0,1]} |f_k - g| < \frac{\epsilon}{3} \quad \text{and} \quad |m^g_{p_k}(0,1) - m^g_\infty(0,1)| < \frac{\epsilon}{3}. \tag{6}$$

It is easy to check that the first inequality in (6) implies

$$|m_p^{f_k}(0,1) - m_p^g(0,1)| < \frac{\epsilon}{3} \tag{7}$$

for every $1 < p \leq \infty$. Now (4), (5), (7) and the second inequality in (6) lead to a contradiction. $\square$

We now establish our first main result.

Theorem 2.3. *Let $f \in C([0,1])$. Then $f^\infty = f_\infty = \lim_{p\to\infty} f_p$, where the limit is considered in the $C([0,1])$ norm, and f_∞ is a best $C([0,1])$-approximant to f by nondecreasing functions.*

PROOF. The equality $f^\infty = f_\infty = \lim_{p\to\infty} f_p$ is a consequence of Lemma 2.2, the minimax formulae for f^p and f_p, and Theorem 2.1. Using the results in [4], and Theorem 2.1 again, we conclude that f_∞ is a best approximant to f by nondecreasing functions. $\square$

Corollary 2.4. *The function $f_\infty = \lim_{p\to\infty} f_p$ is continuous when f is continuous.*

PROOF. It is an immediate consequence of [4, Corollary 2]. $\square$

3 A Minimization Property of f_∞.

We shall need the following elementary observation, which can be easily proved. If $f \in C([0,1])$ and $0 \leq a < x < b \leq 1$, then

$$\min\{m_\infty(a,x), m_\infty(x,b)\} \leq m_\infty(a,b) \leq \max\{m_\infty(a,x), m_\infty(x,b)\}. \tag{8}$$

The following is our second main theorem.

Theorem 3.1. *Let $f \in C([0,1])$. Then*

1. *$f(x) = f_\infty(x)$ for every $x \in \mathcal{F}_\infty$.*

2. *if α and β are in $\mathcal{F}_\infty$ with $\alpha < \beta$ then f^∞ is the best natural $C([\alpha,\beta])$-approximant to f by nondecreasing functions.*

3. *f^∞ is constant in each connected component of $(0,1) \setminus \mathcal{F}_\infty$.*

Proof. We have $\inf_{b>x} m_\infty(x,b) \leq f(x) \leq \sup_{a<x} m_\infty(a,x)$ for $x \in (0,1)$ and $\inf_{b>x} m_\infty(x,b) \leq f_\infty(x) \leq f^\infty(x) \leq \sup_{a<x} m_\infty(a,x)$ for $x \in (0,1)$. Since $x \in \mathcal{F}_\infty$, $\sup_{a<x} m_\infty(a,x) = \inf_{b>x} m_\infty(x,b)$. Therefore, 1 is true. In order to prove 2, we take $\alpha, \beta \in \mathcal{F}_\infty$, with $\alpha < \beta$, and $x \in (\alpha, \beta)$. We consider $a, b \in (0,1)$ such that $a < x < b$. Suppose $a < \alpha$, then, as $\alpha \in \mathcal{F}_\infty$, (8) implies $m_\infty(a,b) \leq m_\infty(\alpha,b)$. Therefore, $\sup_{a<x} m_\infty(a,b) = \sup_{\alpha \leq a<x} m_\infty(a,b)$. Similarly, we can prove that $\inf_{b>x} \sup_{\alpha \leq a<x} m_\infty(a,b) = \inf_{\beta \geq b>x} \sup_{\alpha \leq a<x} m_\infty(a,b)$. Thus, the restriction of f^∞ to the interval $[\alpha,\beta]$ is the function f^∞ relative to $[\alpha,\beta]$. Hence, applying Theorem 2.3 to this interval, we obtain 2. We now prove 3. Let I be a connected component of $(0,1) \setminus \mathcal{F}_\infty$. Since the set $\mathcal{F}_\infty$ is relatively closed in $(0,1)$, $I = (a_0, b_0)$ with $0 \leq a_0 < b_0 \leq 1$. We suppose that f^∞ is not constant on (a_0, b_0). Then there exists $m \in \mathbb{N}$ such that f^∞ is not constant on $[a_0 + \frac{1}{m}, b_0 - \frac{1}{m}]$. Let $\{p_n\}$ be a sequence with $p_n \to \infty$ when $n \to \infty$. Since f^{p_n} converges to f^∞, we can suppose that f^{p_n} is not constant on $[a_0 + \frac{1}{m}, b_0 - \frac{1}{m}]$. Then, for each n there exists some $\alpha_n \in \mathbb{R}$ such that the left end point of the interval $\{f^{p_n} \geq \alpha_n\}$ falls in the interval $[a_0 + \frac{1}{m}, b_0 - \frac{1}{m}]$. We call this point x_n. From [11, Theorem 3.2] we get

$$\int_{x_n}^{b} \varphi_{p_n}(f - \alpha_n) dx \geq 0 \quad \text{and} \quad \int_{a}^{x_n} \varphi_{p_n}(f - \alpha_n) dx \leq 0$$

for every $a \in [0, x_n)$ and every $b \in (x_n, 1]$. The previous inequalities and equality (1) imply that $m_{p_n}(a, x_n) \leq \alpha_n \leq m_{p_n}(x_n, b)$ for all $a \in [0, x_n)$ and all $b \in (x_n, 1]$. Therefore, $x_n \in \mathcal{F}_{p_n}$. Let x be an accumulation point of the sequence x_n. Using Lemma 2.2 and the continuity of the function m_p we get that $x \in \mathcal{F}_\infty \cap (a_0, b_0)$ which is a contradiction with the fact that (a_0, b_0) is a connected component of $(0,1) \setminus \mathcal{F}_\infty$. $\square$

References

[1] T. Acton, A. Restrepo, *Locally Monotonic Models for Image and Video Processing*, Proc. IEEE lnt. Conf. on Image Processing, ICIP 99, Kobe, Lapan, 429–433, (1999). `http://viva.ee.virginia.edu/publications/c_lomo.pdf`.

[2] C. Chi-In, *The Min-Max Algorithm and Isotonic Regression*, Annals of Statistics, **11** (1983), 467–477.

[3] R. Darst, R. Huotari, *Monotone L_1-Approximation on the Unit n-Cube* , Proceedings of the American Mathematical Society, **95** (1985), 425–428.

[4] R. Darst, S. Sahab, *Approximation of Continuous and Quasi-Continuous Functions by Monotone Functions*, Journal of Approximation Theory, **38** (1983), 9–27.

[5] R. Darst, Shusheng Fu, *Approximation of Integrable, Approximately Continuous Functions on* $(0,1)^n$ *by Nondecreasing Functions*, Real Analisys Exchange, **14** (1988/89), 167–174.

[6] R. Darst, Shusheng Fu, *Best* L_1*-Approximation of* L_1*-Approximation Continuous Functions on* $(0,1)^n$ *by Nondecreasing Functions*, Proceedings of the American Mathematical Society, **97** (1986), 262–264.

[7] R. Huotari, A. Meyerowitz, M. Sheard, *Best Monotone Approximations in* $L_1[0,1]$, Journal of Approximation Theory, **47** (1986), 85–91.

[8] M. Iturrieta, F. Zó, *Best Monotone* L_φ*-Approximant in Several Variables*, Approximation Theory and Applications, **14** (1998), 1–10.

[9] D. Landers, L. Rogge, *Best Approximants in* L_ϕ*-Spaces*, Z. Wahrsch. verw. Gebiete, **51** (1980), 215–237.

[10] M. Marano, J. Quesada, L_φ*-Approximation by Non-decreasing Functions on the Interval*, Constructive Approximation, **13** (1997), 177–186.

[11] F. Mazzone, H. Cuenya, *A Characterization of Best* φ*-Approximants with Applications to Multidimensional Isotonic Approximation*, Constructive Approximation, **21** (2005), 207–223.

[12] A. Restrepo, *Locally Monotone Regression*, IEEE Transaction on Signal Processing, **41** (1993), 2796–2810.

[13] T. Robertson, F. Wright, R. Dykstra, *Order Restricted Statistical Inference*, London, Wiley (1988).

[14] V. Ubhaya, *Isotone Optimization, I*, Journal of Approximation Theory, **12** (1974), 146–159.

[15] V. Ubhaya, *Isotone Optimization, II*, Journal of Approximation Theory, **12** (1974), 315–331.