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SLOW POINTS FOR FUNCTIONS IN THE ZYGMUND CLASS Λ_d^*

Abstract

Using probabilistic methods, we find the exact Hausdorff measure function and dimension of sets of dyadic Lipschitz points (i.e., slow points) for functions belonging to particular Zygmund-type classes. We then explore, in depth, the relationship between sets of slow points and sets of standard Lipschitz points, both in the particular case of the van der Waerden–Takagi function and for more general dyadic Zygmund functions.

1 Zygmund Classes.

In 1945, A. Zygmund identified several classes of functions that inhabit an intermediate ground between the spaces of continuous functions and differentiable functions. Given a continuous periodic real-valued function f, we say that f is in the Zygmund class Λ^* if

$$f(x) - \frac{1}{2}\left[f(x+h) + f(x-h)\right] = O(h) \tag{1.1}$$

as $h \to 0$, uniformly in x. We obtain the related class λ^* by replacing the $O(h)$ condition with $o(h)$ in (1.1). Zygmund referred to a function $f \in \lambda^*$ as "smooth" and showed that, although it does not have to be differentiable everywhere, $f'(x)$ must exist for uncountably many points in any subinterval of the domain. By contrast, a function $f \in \Lambda^*$ does not have to "smooth" as

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it can have corners (but no cusps), and there exist examples that are nowhere differentiable. On the other hand, for functions in Λ^*, Zygmund's argument does imply the existence of a dense set of points in any subinterval where a Lipschitz condition is satisfied.

An important set of examples in this area are lacunary Fourier series of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{2^n} \cos(2^n x),$$

which is in Λ^* if and only if $\{a_n\}$ is a bounded sequence. Setting $a_n = 1$ gives the Weierstrass function which Hardy showed to be nowhere differentiable in 1916. Replacing the cosine function in this construction with the"sawtooth" function

$$h(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1/2 \\ 1-x & \text{if } 1/2 < x < 1 \end{cases}$$

which we take to be periodic with period 1 produces the van der Waerden–Takagi function (see Figure 2)

$$w(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x). \tag{1.2}$$

This function is also nowhere differentiable but it fails to satisfy the definition for Λ^* because of cusps that appear at each dyadic rational. It is, however, an important example of a function in the larger space Λ^*_d or "dyadic lambda star," as it is called.

To define this space of functions formally, we first restrict our attention to continuous real-valued functions of period 1. Referring back to equation (1.1), we let I be the interval $[a, b)$, where we set $a = x - h$ and $b = x + h$. Then, $f \in \Lambda^*$ if

$$\|f\|^* = \sup_I \left(\frac{|\Delta_2(f, I)|}{|I|} \right) < \infty \tag{1.3}$$

where $\Delta_2(f, I)$ is the second-order difference

$$\Delta_2(f, I) = f\left(\frac{a+b}{2}\right) - \frac{1}{2}[f(a) + f(b)].$$

For the dyadic analogue Λ^*_d, we follow the same procedure but restrict the supremum in (1.3) to be over only dyadic intervals

$$I_{n,k} = [k/2^n, (k+1)/2^n) \text{ where } n \geq 0,\ 0 \leq k < 2^n.$$

Setting $f_{n,k} = \Delta_2(f, I_{n,k})$ gives us the formulation $f \in \Lambda_d^*$ provided

$$\|f\|_d^* = \sup_n \sup_{0\le k<2^n} |2^n f_{n,k}| < \infty. \tag{1.4}$$

The focus on dyadic intervals opens up the following crucial link to the theory of stochastic processes. For each x and n, choose k so that $x \in I_{n,k}$. Then let $D_n f(x)$ be the dyadic difference quotient

$$D_n f(x) = \frac{\Delta_1(f, I_{n,k})}{|I_{n,k}|}$$

where $\Delta_1(f, I_{n,k})$ is the first-order difference

$$\Delta_1(f, I_{n,k}) = f\left(\frac{k+1}{2^n}\right) - f\left(\frac{k}{2^n}\right).$$

The key observation, expanded below and explored at length in [AP2], is that for $f \in \Lambda_d^*$, the sequence $\{D_n f(x)\}$ is a martingale when considered as a sequence of random variables on [0,1] (with Lebesgue measure) whose jumps,

$$D_{n+1} f(x) - D_n f(x),$$

are uniformly bounded.

Particular attention in this paper is given to functions where the jumps have modulus one because, in this case, the martingale $\{D_n f(x)\}$ is a classical random walk. The van der Waerden function is a primary example of such a function, and we'll use $\Lambda_{d,1}^*$ to denote the subset of all functions in Λ_d^* satisfying

$$|D_{n+1} f(x) - D_n f(x)| = 1 \text{ for all } x \in [0,1).$$

The fact that $\{D_n f(x)\}$ exhibits the properties of a random walk allows us to determine the growth rate of $\{D_n f(x)\}$ for almost all $x \in [0,1)$. Specifically, the law of the iterated logarithm tells us that

$$\limsup_{n\to\infty} |D_n f(x)|/\sqrt{2n \log\log n} = 1$$

almost surely; i.e., for a.e. $x \in [0,1)$, but there are sets of measure zero where $D_n f(x)$ behaves in exceptional ways. Of special interest here are sets of points where the difference quotients grow more slowly than expected.

Definition 1. For a fixed natural number K, set

$$S_K = S_K(f) = \{x \in [0,1) : |D_n f(x)| \le K \text{ for all } n\}.$$

Given a particular function f, a point $x \in [0,1)$ is called a *slow point with constant* K if $x \in S_K$, or simply a *slow point* if $x \in S_K$ for some K.

In terms of the graph of f, the condition $x \in S_K(f)$, or equivalently $|D_n f(x)| \leq K$, corresponds to saying that the function f satisfies a dyadic Lipschitz condition at x. It is this condition, and its relationship to the regular Lipschitz condition, that form the main thrust of our study.

The present paper offers a thorough analysis of the sets $S_K(f)$ in the context of Λ_d^* functions, treating questions both of size and smoothness. These questions are analogous to the questions studied by Davis, Kahane, Perkins, Orey and Taylor about “slow” (and “fast”) points for Brownian motion. After reviewing some representation theorems for the spaces Λ_d^* and $\Lambda_{d,1}^*$ in Section 2, we move on in Section 3 to prove a result describing the Hausdorff measure function and dimension of the sets $S_K(f)$ for $f \in \Lambda_{d,1}^*$. The situation for functions in Λ_d^* is more complicated (see, for example, the remarks at the end of Section 3.) In Section 4, we turn to questions of smoothness. Looking specifically at the van der Waerden–Takagi function, we prove that all of the slow points on this function satisfy a regular Lipschitz condition and provide sharp estimates for the constants. This turns out not to be the case for the general Λ_d^* function, however. In Section 5, we begin with examples of functions in Λ_d^* containing slow points that are not Lipschitz points, and then proceed with an analysis of how pervasive this phenomenon is. As it turns out, a significant number points in $S_K(f)$ are Lipschitz points for all $f \in \Lambda_{d,1}^*$. In particular, we show that the dimension of the set of Lipschitz points in $S_K(f)$ is equal to the dimension of $S_K(f)$. An interesting footnote to this result is the discovery of a curious universal set that, despite having measure zero, manages to contain slow points that are also regular Lipschitz points for every function in $\Lambda_{d,1}^*$. The final section of the paper contains a result that is a counterbalance to the dimension result of Section 5. By introducing random coefficients, we show that, from a probabilistic point of view, slow points that also satisfy a regular Lipschitz condition are in fact the rare exception.

The central insight underlying all of the main results is the interplay between the geometry of the graphs of the Λ_d^* functions and the martingale theory used to analyze the difference quotients. This relationship is at its most elegant when the martingale in question is a classical random walk, and this is essentially why the restriction to the subclass $\Lambda_{d,1}^*$ leads so efficiently to the sharp results of Theorem 5, Theorem 14, and Theorem 15. Extending these theorems to the entire space Λ_d^* is the next part of the story. The examples of results in this direction that we include (Theorem 6, Lemma 13) suggest that we should not expect anything as tidy or thorough as what we have found in this special case.

2 The Structure of Λ_d^* and $\Lambda_{d,1}^*$.

Let C be the space of continuous functions on $[0,1)$ that are continuous and periodic with period 1. Given $f \in C$, let $f_N(x)$ be the continuous function that is linear on each $I_{N,k}$ and satisfies

$$f_N(x) = f(x) \text{ whenever } x = k/2^N$$

where $0 \le k < 2^N$. Using the convention that we take right-hand derivatives at the corners of f_N, we make the important observation that

$$f_N'(x) = D_N f(x).$$

Inspired by the construction of the van der Waerden–Takagi function as a sum of "sawteeth," we offer a more explicit description of the linear interpolation $f_N(x)$ in terms of dyadic "triangle" functions. Let $t(x)$ be the (non-periodic) function

$$t(x) = \begin{cases} h(x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and for each dyadic interval $I_{n,k} = [a_n, b_n]$, set

$$t_{n,k}(x) = \frac{1}{2^n} t\left(\frac{x - a_n}{2^{-n}}\right).$$

Referring the reader to [AP2] for the necessary details,[1] it turns out that

$$f_{N+1}(x) = f_0 + \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} a_{n,k} t_{n,k}(x) \tag{2.1}$$

where f_0 is constant, $a_{n,k} = 2^{n+1} f_{n,k}$ and $f_{n,k} = \Delta_2(f, I_{n,k})$ as in (1.4). The fact that $f \in C$ is enough to conclude that $f_N \to f$ uniformly, and thus we have the representation

$$f(x) = f_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} a_{n,k} t_{n,k}(x). \tag{2.2}$$

Notice that we obtain the van der Waerden–Takagi function $w(x)$ by setting $f_0 = 0$ and $a_{n,k} = 1$ for every n and $0 \le k < 2^n$.

[1] In [AP2], the authors use the Schauder "triangle" functions $\phi_{n,k}(x) = 2^{n+1} t_{n,k}(x)$ which satisfy $\max \phi_{n,k} = 1$. In the present context, using $t_{n,k}$ is slightly more convenient for characterizing the space $\Lambda_{d,1}^*$.

Theorem 2. *Let $f \in C$ and let $a_{n,k}$ be defined as in (2.1). Then*

(a) $f \in \Lambda_d^*$ if and only if the coefficients $\{a_{n,k}\}$ are bounded.

(b) $f \in \Lambda_{d,1}^*$ if and only if $|a_{n,k}| = 1$ for all n and $0 \le k < 2^n$.

Proof. To prove (a), we just rewrite (1.4) in terms of the coefficients $\{a_{n,k}\}$ to get

$$\|f\|_d^* = \frac{1}{2} \sup_n \sup_{0\le k<2^n} |a_{n,k}|. \tag{2.3}$$

To prove (b), let $x \in I_{n,k}$ be given and let x^* be some point in the opposite half of $I_{n,k}$ after it is bisected. Then

$$\begin{aligned} D_{n+1}f(x) - D_n f(x) &= f'_{n+1}(x) - f'_n(x) \\ &= f'_{n+1}(x) - \frac{1}{2}\left[f'_{n+1}(x) + f'_{n+1}(x^*)\right] \\ &= \frac{1}{2}\left[f'_{n+1}(x) - f'_{n+1}(x^*)\right] = 2\left[\frac{\Delta_2(f, I_{n,k})}{|I_{n,k}|}\right] = a_{n,k}, \end{aligned}$$

and the result follows. □

Implicit in equation (2.3) is the following relationship between the rate of convergence in (2.2) and the Λ_d^* norm.

Lemma 3. *If $f \in \Lambda_d^*$ and f_N is given by (2.1), then*

$$|f(x) - f_N(x)| \le 2\|f\|_d^*\, 2^{-N}.$$

Proof. For a given x and fixed n, the intervals $I_{n,k}$ for $0 \le k < 2^n$ are disjoint and exactly one contains the point x. This allows us to write

$$\begin{aligned} |f(x) - f_N(x)| &= \left|\sum_{n=N}^{\infty} \sum_{k=0}^{2^n-1} a_{n,k} t_{n,k}(x)\right| \\ &\le \sum_{n=N}^{\infty} |a_{n,k}| \frac{1}{2^{n+1}} \le 2\|f\|_d^* \sum_{n=N}^{\infty} \frac{1}{2^{n+1}} = 2\|f\|_d^* 2^{-N}. \end{aligned}$$ □

Looking at the graph of a function $f \in \Lambda_d^*$ over a fixed dyadic interval $I_{N,k}$, Lemma 3 asserts that all the points on the graph will fall inside the parallelogram with vertical sides obtained by adding and subtracting $2\,\|f\|_d^* 2^{-N}$ to the linear component of $f_N(x)$ over $I_{N,k}$. (See Figure 1.) This kind of geometric intuition makes the following lemma seem quite plausible.

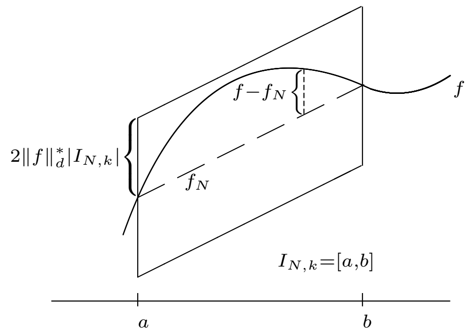

Figure 1: A geometric interpretation of Lemma 3.

Lemma 4. *If $x \in S_K(f)$ for a function $f \in \Lambda_d^*$, and $x, y \in I_{n,k}$, then*

$$|f(y) - f(x)| \leq (K + 4\|f\|_d^*)2^{-n}.$$

Proof. Because $x \in S_K(f)$, we know $|f_n(y) - f_n(x)| \leq K2^{-n}$. Then, by Lemma 3, we get

$$\begin{aligned}|f(y) - f(x)| &\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \\ &\leq (2\|f\|_d^* + K + 2\|f\|_d^*)2^{-n}.\end{aligned}$$ □

3 Exact Hausdorff Measures for Sets of Slow Points.

The probabilistic arguments in the next result require that we fix $f \in \Lambda_{d,1}^*$ so that $D_n f(x)$ is a classical random walk. To simplify notation, we shall begin to write S_K in place of $S_K(f)$ when no confusion is likely.

Theorem 5. *The set S_K for a function $f \in \Lambda_{d,1}^*$ has Hausdorff dimension $1 + log_2 r$ where $r = cos\left(\frac{\pi}{2(K+1)}\right)$ is the dominant eigenvalue for the random walk that vanishes the first time it leaves the interval $[-K, K]$.*

Proof. For each $n \in \mathbb{N}$, set

$$S_{K,n} = \{x \in [0, 1) : |D_m f(x)| \leq K \text{ for all } m \leq n\},$$

and observe that

$$S_K = \bigcap_{n=1}^{\infty} S_{K,n}.$$

Using $m(\cdot)$ to denote Lebesque measure, we set $p_{K,n} = m(S_{K,n})$, which we can interpret to be the probability that the random walk $D_n f(x)$ stays bounded by K during the first n steps.

Now $S_{K,n}$ consists of a disjoint union of dyadic intervals of length $1/2^n$. If $N_{K,n}$ equals the number of intervals that make up $S_{K,n}$, then it follows that $p_{K,n} = N_{K,n}2^{-n}$. With an eye toward computing the Hausdorff dimension of S_K, we set $\alpha = 1 + \log_2 r$ and use the previous fact to see that

$$\sum_{I \subseteq S_{K,n}} |I|^\alpha = N_{K,n}\frac{1}{2^{n\alpha}} = p_{K,n}\frac{1}{2^{n(\alpha-1)}}, \tag{3.1}$$

where the sum is over the dyadic intervals that constitute $S_{K,n}$.

To get our hands on the probabilities $p_{K,n}$, let's consider a modified random walk that starts at zero and vanishes the first time it leaves the interval $[-K, K]$. To model this, define the sub-stochastic matrix

$$Q_K = (q_{ij})_{i,j=-K}^{K},$$

by setting

$$q_{ij} = \begin{cases} 1/2 & \text{if } i = j \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathbf{v_0} = (v_i)_{i=-K}^{K}$ is the initial distribution (in this case $v_0 = 1$ and $v_i = 0$ for $i \neq 0$), then $p_{K,n}$ can be computed by summing the entries of $Q_K^n\mathbf{v_0}$. But Q_K can be diagonalized, and by the theorems of Perron and Frobenius it has a dominant eigenvalue $r > 0$, from which it follows that

$$c_0 r^n \leq p_{K,n} \leq c_1 r^n, \tag{3.2}$$

for strictly positive constants c_0 and c_1 which can be chosen independently of the initial distribution $\mathbf{v_0}$.

Returning to the calculation started above in (3.1), we now have

$$\sum_{I \subseteq S_{K,n}} |I|^\alpha \leq 2^{(1-\alpha)n}(c_1 r^n) = c_1\left(2^{1-\alpha}r\right)^n = c_1 < \infty,$$

and, given that $S_K \subseteq S_{K,n}$, we may conclude that $\dim(S_K) \leq \alpha = 1 + \log_2 r$.

To produce the reverse inequality it suffices, by Frostman's Lemma, to construct a non-trivial measure μ_K satisfying

$$\mu_K(I) \leq c|I|^\alpha \tag{3.3}$$

for all dyadic intervals $I = I_{n,k}$. The strategy is to define μ_K as the weak limit of the sequence of measures $\mu_{K,n}$ where for each n we define

$$\mu_{K,n}(A) = \frac{m(A \cap S_{K,n})}{m(S_{K,n})}$$

for Borel sets $A \subseteq [0,1)$. From the probabilistic point of view, $\mu_{K,n}(A)$ is the conditional probability of finding a particular path in A given that the path has stayed within $[-K,K]$ through the first n steps.

Let $I_{n_0,k}$ be an arbitrary dyadic interval of the n_0th generation. Then for $n \geq n_0$, write

$$\mu_{K,n}(I_{n_0,k}) = \frac{m(I_{n_0,k} \cap S_{K,n})}{m(S_{K,n})} = \frac{m(I_{n_0,k})}{m(S_{K,n})} \left[\frac{m(S_{K,n} \cap I_{n_0,k})}{m(I_{n_0,k})} \right], \tag{3.4}$$

where the bracketed quantity at the far right is now the conditional probability that the random walk stays within $[-K,K]$ during the first n steps given that the path lies in the interval $I_{n_0,k}$. If, at the n_0th step, $I_{n_0,k}$ is not contained in S_{K,n_0}, then this probability is zero. On the other hand, if $I_{n_0,k} \subseteq S_{K,n_0}$, then, as before, we can compute the probability of remaining in $[-K,K]$ by summing the entries of $Q^{n-n_0}\mathbf{v_0}$ (for a different initial distribution vector $\mathbf{v_0}$), and the conclusion is again that

$$\frac{m(S_{K,n} \cap I_{n_0,k})}{m(I_{n_0,k})} \leq c_1 r^{n-n_0}. \tag{3.5}$$

Combining (3.4), (3.5), and the first inequality in (3.2) now yields

$$\mu_{K,n}(I_{n_0,k}) \leq \frac{2^{-n_0}}{p_{K,n}} \left(c_1 r^{n-n_0} \right) \leq \left(\frac{c_1}{c_0} \right) \frac{1}{2^{n_0} r^{n_0}} = \left(\frac{c_1}{c_0} \right) |I_{n_0,k}|^{\alpha},$$

which holds as long as $n \geq n_0$. The fact that the approximating sets $S_{K,n}$ are nested makes it straightforward to verify that $\mu_K = \text{wk-}\lim_{n\to\infty} \mu_{K,n}$ exists and satisfies (3.3) as required.

All that remains is to compute the eigenvalue r. The eigenvalue equation

$$Q\mathbf{v} = r\mathbf{v} \text{ where } \mathbf{v} = (v_j)_{j=-K}^{K}$$

reduces to the system

$$\frac{1}{2} v_{j-1} + \frac{1}{2} v_{j+1} = r v_j, \; -K \leq j \leq K,$$

with the added convention that $v_{K+1} = v_{-K-1} = 0$. The form of this equation suggests setting $v_j = \cos \lambda j$ which immediately yields the solution $r = \cos \lambda$ provided $\cos \lambda(K+1) = 0$, or

$$\lambda = \frac{\pi}{2(K+1)} + \frac{m\pi}{K+1}, \quad m \in \mathbb{Z}.$$

Because r is the largest positive eigenvalue for Q, we take $m = 0$ and thus $r = \cos\left(\frac{\pi}{2(K+1)}\right)$ as desired. □

Theorem 5 gives a satisfying description of the size of $S_K(f)$ for functions $f \in \Lambda^*_{d,1}$ and, as the next proposition illustrates, it can also be used for estimating the size of $S_K(f)$ for special classes of functions in Λ^*_d. We point out that the full story for functions in Λ^*_d has not yet been determined. However, we offer the following observations.

By Theorem 2, every $f \in \Lambda^*_d$ has the form

$$f(x) = f_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} a_{n,k} t_{n,k}(x)$$

where $a_{n,k} = D_{n+1}f(x) - D_n f(x)$ are the "jumps" in the martingale $D_n f(x)$. For each $f \in \Lambda^*_d$, define $\tilde{f} \in \Lambda^*_{d,1}$ by

$$\tilde{f}(x) = f_0 + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \tilde{a}_{n,k} t_{n,k}(x)$$

where $\tilde{a}_{n,k} = a_{n,k}/|a_{n,k}|$ when $a_{n,k} \neq 0$ and $\tilde{a}_{n,k} = 1$ when $a_{n,k} = 0$.

It would be natural to conjecture that the size of of the set $S_K(f)$ decreases as the coefficients increase, and thus conclude $S_K(\tilde{f}) \subseteq S_K(f)$ for a function f with jumps of size less than one. This is not the case, as the following modest alteration to the van der Waerden-Takagi function shows. Let

$$w^*(x) = \frac{1}{2}h(x) + \sum_{n=1}^{\infty} \frac{1}{2^n} h(2^n x).$$

For this function, the first step of the associated random walk is $\pm 1/2$ and all the subsequent steps are ± 1. Considering $K = 1$, we can see that the set $S_1(w^*)$ consists of the two points $x = 1/3,\ 2/3$. However, for the the van der Waerden-Takagi function $w(x)$, the set $S_1(w)$ has dimension $1/2$ by Theorem 5. In this example, the dimension of the set S_1 is determined not by the size of the coefficients, but by complicated arithmetical considerations.

In the positive direction, the following theorem shows that we can say something conclusive when the coefficients decrease monotonically.

Theorem 6. *Let $f \in \Lambda_d^*$ satisfy $\|f\|_d^* \leq 1/2$. If for every $x = \bigcap_{n=0}^{\infty} I_{n,k_n}$ the sequence $(|a_{n,k_n}|)$ is decreasing, then $S_K(\tilde{f}) \subseteq S_K(f)$.*

Proof. Using the summation by parts formula we can write

$$\begin{aligned} D_n f(x) &= \sum_{j=0}^{n-1} (D_{j+1} f(x) - D_j f(x)) = \sum_{j=0}^{n-1} |a_{j,k_j}|\, \tilde{a}_{j,k_j} \\ &= D_n \tilde{f}(x) |a_{n,k_n}| + \sum_{j=0}^{n-1} D_{j+1} \tilde{f}(x) \left(|a_{j,k_j}| - |a_{j+1,k_{j+1}}| \right). \end{aligned}$$

Now $|a_{n,k}| \leq 2\|f\|_d^* \leq 1$ for all n and k, and assuming that $x \in S_K(\tilde{f})$, we may conclude

$$|D_n f(x)| \leq K |a_{n,k_n}| + K \sum_{j=0}^{n-1} \left(|a_{j,k_j}| - |a_{j+1,k_{j+1}}| \right) = K |a_{0,k_0}| \leq K,$$

so that $x \in S_K(f)$ as well. □

The argument in Theorem 6 can also be used to show that if x is slow for $\tilde{f}$, then x is slow for f whenever the sequence a_{n,k_n} that coincides with the point x can be written as the difference of monotone decreasing sequences.

4 The Van der Waerden–Takagi Function.

For a function $f \in \Lambda_d^*$ and constant $P > 0$, define $L_P = L_P(f)$ to be the set of points that satisfy a regular Lipschitz condition with constant P:

$$L_P = L_P(f) = \{x : |f(x) - f(y)| \leq P|x - y| \text{ for all } y \neq x\}.$$

Slow points are points that satisfy a dyadic Lipschitz condition:

$$S_K = S_K(f) = \{x : |f((k+1)/2^n) - f(k/2^n)| \leq K 2^{-n} \text{ for all } I_{n,k} \ni x\}.$$

The remainder of this paper is devoted to exploring the relationship between these two classes of sets in the context of functions in Λ_d^* and $\Lambda_{d,1}^*$. We begin with the important specific case of the van der Waerden–Takagi function $w(x)$ defined above in (1.2) and sketched in Figure 2. For this particular function, we show that all slow points are regular Lipschitz points.

Theorem 7. *For the van der Waerden–Takagi function $w(x)$, slow points with constant $K \geq 3$ satisfy a regular Lipschitz condition with constant $3K - \log_2(K) - 3$. Slow points with constant $K = 1$ or 2 require a Lipschitz constant of $K + 1$.*

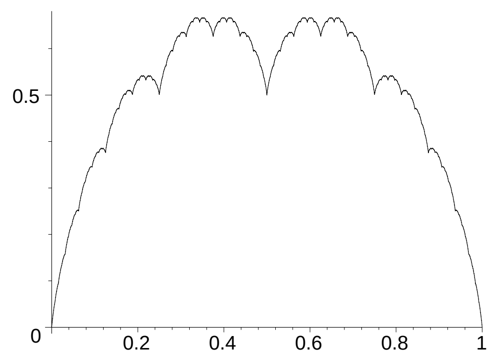

Figure 2: The van der Waerden–Takagi function $w(x) = \sum_{n=0}^{\infty}(1/2^n)h(2^n x)$.

The argument for Theorem 7 is organized into a sequence of lemmas. The first of these has a generalization to an arbitrary Λ_d^* function, but the others are specific to $w(x)$. For notational convenience, we let

$$m_f(x,y) = \frac{f(y) - f(x)}{y - x}.$$

For each x, let $T_{K,x}$ be given by

$$T_{K,x} = \begin{cases} m+1 & \text{if } x \in S_{K,m} \backslash S_{K,m+1} \\ \infty & \text{if } x \in S_K \end{cases}$$

which we can interpret to be the time at which $D_n w(x)$ first leaves the interval $[-K, K]$. Now consider the sequence of functions

$$w_{K,n}(x) = \sum_{j=0}^{\min\{n, T_{K,x}\}} \frac{1}{2^j} h(2^j x).$$

Lemma 8. *(a) The limit function*

$$w_K(x) = \lim_{n\to\infty} w_{K,n}(x)$$

satisfies $|m_{w_K}(x,y)| \le K+1$ *for all* $x \ne y$ *in the domain.*

(b) If $z \in S_K$*, then* $w_K(z) = w(z)$*.*

(c) If $x, z \in S_K$*, then* $|m_w(x,z)| \le K+1$*.*

Proof. Each $w_{K,n}$ is a continuous, piece-wise linear function consisting of segments all with slope less than $K+1$ in absolute value. Because $|w_{K,n+1}(x) - w_{K,n}(x)| \le 2^{-n}$, the sequence converges uniformly and (a) follows. Statement (b) is evident from the construction of $w_{K,n}(x)$, and (c) is a consequence of (a) and (b). □

The Lipschitz condition

$$|m_w(z,x)| = \left|\frac{w(x)-w(z)}{x-z}\right| \le M$$

is equivalent to the two inequalities

$$w(x) \le w(z) + M|x-z| \text{ and } w(x) \ge w(z) - M|x-z|, \tag{4.1}$$

which amounts to saying that w satisfies a Lipschitz condition "from above" and "from below," respectively.

Lemma 9. *If* $z \in S_K$*, then* $w(x) \ge w(z) - (K+1)|x-z|$ *for all* $x \ne z$*.*

Proof. This follows from the fact that the sequence $w_{K,n}(x)$ is increasing and bounded above by $w(x)$. □

Obtaining the other inequality in (4.1) is more involved. Given a point $x \in [0,1]$, the dyadic expansion

$$x = .a_1a_2a_3\ldots = \sum_{j=1}^{\infty} \frac{a_j}{2^j} \text{ where } a_j \in \{0,1\}$$

is useful because $D_nf(x) = \sum_{j=1}^{n}(-1)^{a_j}$, (provided we use the representation ending in repeating 1s for dyadic rational points.) For instance, it should be evident that the point

$$x_1 = .010101\cdots = \frac{1}{4}\Big(\frac{1}{1-1/4}\Big) = \frac{1}{3}$$

satisfies $x_1 \in S_1$ as well as $x_1 = \inf S_1$. In addition, $y_1 = w(x_1) = 2/3$ is the maximum value for $w(x)$. In fact, $w(x)$ attains this maximum if and only if $x \in S_1$, which makes the Lipschitz estimate "from above" trivial in the case $K = 1$.

In general, the points (x_k, y_k) on the graph of $w(x)$ given by

$$\begin{aligned} x_k &= .0\ldots01010\ldots = \frac{0}{2} + \cdots + \frac{0}{2^k} + \frac{1}{2^{k+1}} + \frac{0}{2^{k+2}} + \frac{1}{2^{k+3}} + \frac{0}{2^{k+4}} + \cdots \\ &= \frac{1}{2^{k+1}}\Big(\frac{1}{1-1/4}\Big) = \frac{1}{3\cdot 2^{k-1}} \\ y_k &= w(x_k) = \frac{k+1}{3\cdot 2^{k-1}} \end{aligned}$$

have special significance.

Lemma 10. *For each $k \in \mathbb{N}$, the point (x_k, y_k) described above satisfies the following:*

(a) $x_k \in S_k$ and $x_k = \inf S_k$

(b) (x_k, y_k) falls on an extreme point of the closed convex hull of the graph of $w(x)$

(c) $y_k = w(x_k) \le w(z)$ for all $z \in S_k$.

Proof. Part (a) follows by observing $\{D_n f(x_k)\} = \{1, 2, \ldots, k, k-1, k, \ldots\}$ and realizing that the dyadic expansion for x_k begins with as many zeros as possible.

For (b), we return to the observation that $w(x)$ attains its maximum value of $2/3$ if and only if $x \in S_1$. This implies that the line segment from $(1/3, 2/3)$ to $(2/3, 2/3)$ is part of the boundary of the convex hull of the graph of $w(x)$ over the interval $[0, 1]$. Now fix $k \in \mathbb{N}$, and notice that if we restrict our attention to the interval $[0, 1/2^k]$, then

$$w(x) - kx = \sum_{n=k}^{\infty} \frac{1}{2^n} h(2^n x) = w(2^k x)$$

is a copy of the original function $w(x)$ scaled by a factor of $1/2^k$. Thus, $w(x) - kx$ has a maximum value of $2/(3\cdot 2^k)$ occurring between $x = 1/(3\cdot 2^k)$ and $x = 2/(3\cdot 2^k)$. It follows that the line segment connecting the points $\big(\frac{1}{3\cdot 2^k}, \frac{k+2}{3\cdot 2^k}\big)$ and $\big(\frac{1}{3\cdot 2^{k-1}}, \frac{k+1}{3\cdot 2^{k-1}}\big)$ is part of the boundary of the convex hull of the graph, proving part (b).

We prove (c) by induction. The case $k = 1$ is clear, so assume $w(x_k) \le w(x)$ for all $x \in S_k$, and let $z \in S_{k+1}$ be arbitrary. Using the symmetry of the sawtooth function $h(x)$, we can find a new point $z^+ \in S_{k+1}$ such that

$D_n w(z^+) \geq 0$ for all n and $w(z^+) = w(z)$. Necessarily, $z^+ < 1/2$, and by considering the dyadic expansion for z^+, we can show that $2z^+ \in S_k$. But using the induction hypothesis and the fact that $x_{k+1} \leq z^+$, we see that

$$\begin{aligned} w(z) = w(z^+) &= h(z^+) + \sum_{n=1}^{\infty} \frac{1}{2^n} h(2^n z^+) = h(z^+) + \frac{1}{2} w(2z^+) \\ &\geq h(x_{k+1}) + \frac{1}{2} w(x_k) = w(x_{k+1}) \end{aligned}$$

which completes the proof. □

Lemma 11. *If $z \in S_K \cap [0, 1/2]$ and $x \in [1/2, 1]$, then $m_w(z, x) \leq K - \log_2(K+1)$.*

Proof. Straightforward geometric considerations show that we need only worry about the case where $z \in [1/4, 1/2]$ and $x \in [1/2, 3/4]$, and over each of these intervals we find a (scaled and translated) copy of the original graph of $w(x)$. This allows us to appeal to Lemma 10 to assert that $m_w(z, x)$ attains its maximum as a function of x for some $x' \in S_k$ where $k \leq K$. At this point, we could appeal to lemma 8(c) and content ourselves with a Lipschitz constant of $K+1$, but Lemma 10 gives us enough information to do a bit better. In particular, for $x \in [1/2, 1]$, we have

$$\begin{aligned} m_w(z, x) \leq m_w(z, x') &= \frac{w(x') - w(z)}{(x' - 1/2) + (1/2 - z)} \\ &\leq \frac{(k+1)/(3 \cdot 2^{k-1}) - (K+1)/(3 \cdot 2^{K-1})}{1/(3 \cdot 2^{k-1}) + 1/(3 \cdot 2^{K-1})} \\ &= \frac{(k+1)2^K - (K+1)2^k}{2^K + 2^k}. \end{aligned}$$

This last estimate is less than $K - \log_2(K+1)$ if and only if

$$r(k) = 2^K[K - \log_2(K+1) - (k+1)] + 2^k[2K - \log_2(K+1) + 1] \geq 0 \tag{4.2}$$

for all $k \leq K$. The case where $K = 1$ is clear. If $K \geq 2$, some calculus shows that the function $r(k)$ attains its minimum at the point k_0 satisfying $2^{k_0} = \frac{2^K}{(2K - \log_2(K+1)+1)\ln 2}$. Substituting back into (4.2) yields

$$2^K \left[\log_2 \left(\frac{2K - \log_2(K+1) + 1}{K+1} \right) + \left(\frac{1}{\ln 2} - 1 + \log_2(\ln 2) \right) \right] \geq 0$$

which is equivalent to

$$\frac{2K - \log_2(K+1) + 1}{K+1} \geq \frac{2^{(1-\frac{1}{\ln 2})}}{\ln 2}.$$

This inequality is easy to check when $K = 2$, and the expression on the left is increasing with increasing K, so the proof is complete. $\square$

The bound in Lemma 11 is essentially sharp. If we let

$m_K = \sup_{x,z}\{m_w(z,x) : z \in S_K \cap [0, 1/2], x \in [1/2, 1]\}$ and $a_K = K - \log_2(K+1)$,

then an argument similar to the one in Lemma 11 shows that $a_K - m_K$ is bounded by $\frac{1}{\ln 2} + \log_2(\ln 2)$ and, moreover, that $\lim_{K\to\infty}(a_K - m_K) = \frac{1}{\ln 2} + \log_2(\ln 2)$.

Having accumulated a list of useful facts, we are now ready to prove that all slow points on $w(x)$ satisfy a regular Lipschitz condition.

Proof of Theorem 7. Fix $z \in S_K$ and let $x \neq z$. Lemma 9 gives us

$$w(x) \geq w(z) - (K+1)|x - z|,$$

which is stronger than we need to get the Lipschitz "from below" estimate required for Theorem 7, so it just remains to prove the other inequality in (4.1).

Let $I = I_{n',k'}$ be the smallest dyadic interval containing both x and z, and consider the function

$$\tilde{w}(x) = \sum_{n=n'}^{\infty} \frac{1}{2^n} h(2^n x) = w(x) - l(x).$$

On the interval I, $l(x) = w(x) - \tilde{w}(x)$ is linear and, because $z \in S_K$, the slope necessarily satisfies $|l'(x)| \leq K$. In the case $|l'(x)| < K$, consider the modified function $\tilde{w}(x)$. The point z is still a slow point with respect to this function, but it is slow with constant $K + (K-1)$. By rescaling so that I corresponds to $[0, 1]$, we can apply Lemma 11 to assert that

$$\tilde{w}(x) \leq \tilde{w}(z) + (2K - \log_2(K) - 2)|x - z|.$$

Finally, because $w(x) = l(x) + \tilde{w}(x)$ and $|l'(x)| \leq K - 1$, we conclude that

$$w(x) \leq w(z) + (3K - \log_2(K) - 3)|x - z|,$$

as desired.

If $|l'(x)| = K$, then the slow point z must necessarily fall on the "high" side of the interval I and not too close to the midpoint. Using the ideas in Lemma 10(a) and (c), we can prove something as strong as $w(x) \le w(z)$, and the proof is complete. $\square$

Another example of some interest is the *alternating* van der Waerden–Takagi function

$$v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} h(2^n x),$$

sketched in Figure 3. By applying the technique in Lemmas 8 and 9 to the odd and even partial sums of $v(x)$, we can show that every slow point of constant K satisfies a Lipschitz condition of constant $K+2$. We also add that for each n, the discontinuities of $D_n v(x)$ are bounded in modulus by 4, from which it follows that $v \in \Lambda^*$ (see, e.g., the remark at the bottom of page 564 of [AP2]).

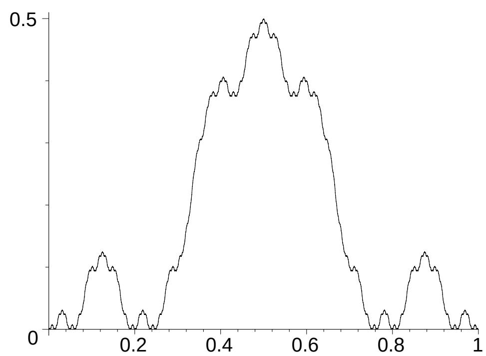

Figure 3: The alternating van der Waerden–Takagi function where $S_K \subseteq L_{K+2}$

5 Lipschitz Points for General Functions in Λ_d^* and $\Lambda_{d,1}^*$.

We begin this section with examples of functions in Λ_d^* that contain slow points that do not satisfy a regular Lipschitz condition.

Given a dyadic interval $I = [a, b) = [k/2^n, (k+1)/2^n)$, let

$$w_I(x) = 1_I(x) \cdot \frac{1}{2^n} w\left(\frac{x-a}{2^{-n}}\right),$$

which we can verify is a Λ_d^* function and supported on the interval I.

Example 12. (a) One easy example of a non-Lipschitz slow point can be found by considering a particular function $w_I(x)$ where, say, $I = [0, 1/2)$. Because we have defined $D_n w_I(x)$ so that it is continuous on the right, $D_n w_I(1/2) = 0$ for all n and thus $x = 1/2$ is a slow point. On the other hand, for the original van der Waerden–Takagi function, $w(1/2^m) = w(1 - 1/2^m) = m/2^m$. This yields a modulus of continuity of $\delta \log(1/\delta)$ at 0 and 1, and the resulting cusps are inherited by W_I at the endpoints of the interval I. To be more precise, observe that

$$\frac{w_I(1/2 - 1/2^m)}{1/2^m} = \frac{w(1 - 1/2^{m-1})}{1/2^{m-1}} = m - 1,$$

which is unbounded as $m \to \infty$, and consequently $x = 1/2$ is not a Lipschitz point.

(b) The previous example is a bit unsatisfactory because the point in question is a dyadic rational point. However, it is possible to modify this construction to produce a Λ_d^* function with a *non-dyadic* slow point that does not satisfy a Lipschitz condition. The example we seek is of the form

$$f(x) = \sum_{m=1}^{\infty} w_{I_m}(x)$$

where $\{I_m\}$ is a sequence of disjoint dyadic intervals which are monotone in the the sense that I_{m+1} always falls to the right of I_m. To define the sequence inductively, set $I_1 = [0, 1/2)$ and let $y_1 = 1$. Now, assuming we have an interval $I_m = [a_m, b_m) = [k_{n_m}/2^{n_m}, (k_{n_m} + 1)/2^{n_m})$ and a point $y_m > b_m$, first pick a new point y_{m+1} satisfying

(i) $b_m < y_{m+1} \le y_m$, and

(ii) $y_{m+1} - b_m \le \frac{1}{2^m}|I_m| = \frac{1}{2^{m+n_m}}$.

Now pick I_{m+1} to be a dyadic interval falling in the open interval (b_m, y_{m+1}). The fact that $\{I_m\}$ are all disjoint implies $f \in \Lambda_d^*$, and the monotonicity gives us

$$0 \le a - b_m < \frac{1}{2^{m+n_m}} \tag{5.1}$$

where $a = \lim a_m$ is, as we now show, a non-Lipschitz slow point that is also irrational.

The point a is slow because $f(x) = 0$ over every dyadic interval containing a. To see that it is not Lipschitz, consider the sequence $\alpha_m = b_m - 1/2^{m+n_m}$ which converges to a, and observe that

$$\left|\frac{f(a) - f(\alpha_m)}{a - \alpha_m}\right| = \frac{w_{I_m}(\alpha_m)}{(a - b_m) + 1/2^{m+n_m}} \geq \frac{1/2^{n_m} w\left(1 - 1/2^m\right)}{(1/2^{m+n_m}) + 1/2^{m+n_m}}$$
$$= \frac{m/2^m}{1/2^m + 1/2^m} = \frac{m}{2},$$

which is unbounded as $m \to \infty$. Finally, note that if a were rational, then substituting $a = p/q$ and $b_m = (k_{n_m} + 1)/2^{n_m}$ into (5.1) would yield

$$0 \leq p\, 2^{n_m} - (k_{n_m} + 1)q < q/2^m,$$

which is impossible when p and q are integers and m is sufficiently large.

The previous examples prompt us to explore what extra assumptions we might add in order to conclude that a slow point is also a Lipschitz point. One result (which we state without proof) is that if $f \in \Lambda^*$, then every slow point is Lipschitz. Another approach that focuses on functions in Λ_d^* is to insist that our slow point falls relatively close to the middle of each dyadic interval in which it falls.

Fix $m \in \mathbb{N}$ and, for each dyadic interval $I_{n,k}$, let $M_{n,k}^m$ be the closed middle proportion of the interval given by

$$M_{n,k}^m = \left[\frac{k}{2^n} + \frac{1}{2^{n+m+1}}, \frac{k+1}{2^n} - \frac{1}{2^{n+m+1}}\right].$$

Then let

$$M_n^m = \bigcup_{k=0}^{2^n-1} M_{n,k}^m \text{ and } M^m = \bigcap_{n=0}^{\infty} M_n^m.$$

So, for example, M^2 would be the "middle 3/4ths" set

$$M^2 = \left[\frac{1}{8}, \frac{7}{8}\right] \cap \left(\left[\frac{1}{16}, \frac{7}{16}\right] \cup \left[\frac{9}{16}, \frac{15}{16}\right]\right) \cap \cdots .$$

A useful way to characterize the sets M^m is to observe that if $x \in [0,1]$ has dyadic expansion $x = .a_1a_2a_3\ldots$, then

$$x \notin M^m \text{ if and only if } a_n = a_{n+1} = \cdots = a_{n+m} \text{ for some } n \in \mathbb{N}.$$

Lemma 13. *If x is a slow point for a function $f \in \Lambda_d^*$ and $x \in M^m$ for some m, then x is a Lipschitz point.*

Proof. Let $x \in S_K(f)$, fix $y \neq x$, and let $I_{n,k}$ be the smallest dyadic interval that contains both x and y. By Lemma 4,

$$|f(y) - f(x)| \leq (K + 4\|f\|_d^*)\, 2^{-n}.$$

Now if c is the midpoint of $I_{n,k}$, then $x \in M^m$ implies

$$|y - x| \geq |c - x| \geq \epsilon_m 2^{-(n+1)}$$

where $\epsilon_m > 0$ can be chosen independently of x and y. Finally,

$$\frac{|f(y) - f(x)|}{|y - x|} \leq \frac{(K + 4\|f\|_d^*)\, 2^{-n}}{\epsilon_m 2^{-(n+1)}} = \frac{2(K + 4\|f\|_d^*)}{\epsilon_m},$$

and x satisfies a Lipschitz condition. $\square$

Set

$$M = \bigcup_{m=1}^{\infty} M^m.$$

Also, recall that $L_P = L_P(f)$ is the set of points that satisfy a Lipschitz condition with constant $P > 0$ for a function f, and let

$$L = L(f) = \bigcup_{P>0} L_P(f)$$

be the set of all Lipschitz points for f. The next proposition should be compared to Zygmund's result, cited in the opening paragraph, that every Λ^* function admits a dense set of Lipschitz points.

Theorem 14. *For every $f \in \Lambda_{d,1}^*$, we have $\dim(S_K \cap L) = \dim(S_K)$.*

Proof. By Lemma 13,

$$S_K \cap M \subseteq S_K \cap L \subseteq S_K,$$

so it suffices to prove $\dim(S_K \cap M) = \dim(S_K)$.

Let's take K to be odd (the proof for even K is similar) and let $m \geq 4$ be even. The first step is to define a set $S_K^m \subseteq L$ as the intersection of a decreasing sequence of sets $S_{K,n}^m$, similar to the sequence $S_{K,n}$ in the proof of Theorem 5, but defined inductively on n. For $n = 1$, set $S_{K,1}^m = [0, 1]$. For the inductive step, we consider several cases.

If $n \neq -2$ or $-1 \pmod{m}$ we set

$$S^m_{K,n+1} = \{x \in S^m_{K,n} : |D_{n+1}f(x)| \leq K\}.$$

If $n = -2 \pmod{m}$, we define $S^m_{K,n+1}$ in the following way. Take each dyadic interval $I_{n,k} \subseteq S^m_{K,n}$ and bisect it. Because K is odd and $D_nf(x)$ is even, $|D_{n+1}f(x)| \leq K$ on both halves of $I_{n,k}$. However, we shall just keep a half that ensures we eventually wind up in the middle set M^m. To be precise, if $I_{n,k}$ was the right half of its "parent" interval $I_{n-1,k'}$ at the $(n-1)$st dyadic stage, then we include only the left half of $I_{n,k}$ in the composition of $S^m_{K,n+1}$. If $I_{n,k}$ was the left half of $I_{n-1,k'}$, then we keep only the right half. The punch-line of this part of the construction (once the rest is finished) will be that

$$\bigcap_{n=1}^{\infty} S^m_{K,n} \subseteq M^m.$$

If $n = -1 \pmod{m}$, then we have just carried out the "middle set adjustment," and we want to follow this with a step that returns the random walk $D_nf(x)$ to its previous distribution (scaled by 1/4.) To do this, set

$$S^m_{K,n+1} = \{x \in S^m_{K,n} : D_{n+1}f(x) = D_{n-1}f(x)\}.$$

In effect, this again means looking at each dyadic interval $I_{n,k}$ in $S^m_{K,n}$ and keeping exactly half. It follows that when $n = -1 \pmod{m}$,

$$m(S^m_{K,n+1}) = \frac{1}{2}m(S^m_{K,n}) = \frac{1}{4}m(S^m_{K,n-1}).$$

We now define S^m_K by

$$S^m_K = \bigcap_{n=1}^{\infty} S^m_{K,n}.$$

To compute the Hausdorff dimension, we proceed just as in Theorem 5. The key step in this argument is estimating the probabilities $p^{(m)}_{K,n} = m(S^m_{K,n})$. Let Q_K be the same sub-stochastic matrix as before with dominant eigenvalue $r > 0$, and let $D_{1/4} = (1/4)I$ where I is the $2K+1$ dimensional identity matrix. Then $p^{(m)}_{K,mn}$ can be computed by summing the entries of $[D_{1/4}Q_K^{m-2}]^n\mathbf{v_0}$, and this time we may assert that there are strictly positive constants d_0 and d_1 such that $d_0r_m^n \leq p^{(m)}_{K,mn} \leq d_1r_m^n$, where $r_m = (1/4)r^{m-2}$. Following the argument in Theorem 5 through, we are led to the conclusion that

$$\dim(S^m_K) = 1 + (1/m)\log_2 r_m.$$

By its construction, $S_K^m \subseteq S_K \cap M^m$, and the proof is then completed by observing

$$\lim_{m\to\infty} [\dim(S_K^m)] = \lim_{m\to\infty} [1 + (1/m)\log_2 r_m] = 1 + \log_2 r = \dim(S_K). \quad \square$$

6 Random Coefficients.

The middle set M, crucial to the proof of Theorem 14, is universal in that it does not depend on a particular function f. What is especially curious here is that M has measure zero and, for each $f \in \Lambda_{d,1}^*$, the set $S_K(f)$ also has measure zero. Yet still these sets intersect, and the overlap is substantial enough to yield Theorem 14.

For the van der Waerden–Takagi function $w(x)$, it is the case that $S_K(w)$ is contained in M and hence $L(w)$. This observation offers an extremely short proof of Theorem 7, albeit without the estimates on the Lipschitz constants. The alternating van der Waerden function $v(x)$ represents the other extreme where $S_K(v) \cap M$ is quite small. For this latter example, we were again able to prove that $S_K(v) \subseteq L(v)$, but the argument here rests on the nested cancellations innate to alternating series. Examples where all slow points satisfy a Lipschitz condition are, in fact, the rare exceptions. For a "random" function in $\Lambda_{d,1}^*$, we should not anticipate being so fortunate.

Theorem 15. *Let $f(x,\omega) \in \Lambda_{d,1}^*$ be the random function*

$$f = f(x,\omega) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} b_{n,k}(\omega) t_{n,k}(x)$$

where $b_{n,k}(\omega)$ are i.i.d. Bernoulli random variables on a probability space Ω taking values ± 1 with probability $1/2$. For each K, let H^α be α–dimensional Hausdorff measure where $\alpha = 1 + \log_2 r$ as in Theorem 5. Then

$$H^\alpha(S_K(f) \cap L(f)) = 0 \quad \omega\text{–almost surely.}$$

For the following discussion fix a constant $P > 0$ and natural number K. Before beginning the proof, we establish the following geometric fact.

Lemma 16. *Let $f \in \Lambda_{d,1}^*$ be given, and consider two adjacent dyadic intervals $I_{m,k}$ and $I_{m,k+1}$. If $|D_m f(x)| \le K$ for $x \in I_{m,k}$ and $|D_m f(x)| \ge 2P + K + 1$ for $x \in I_{m,k+1}$, then no point in $I_{m,k}$ can satisfy a Lipschitz condition with constant P.*

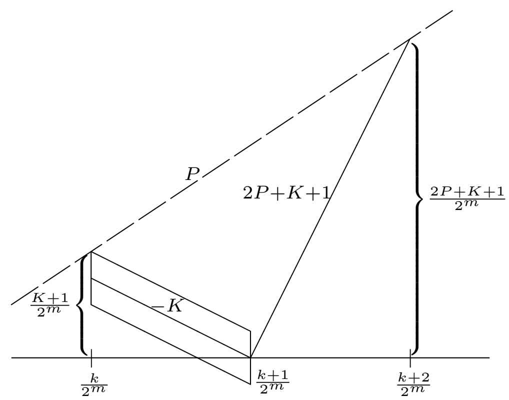

Figure 4: CRITICAL CASE WHERE $D_m f = -K$ ON $I_{m,k}$ AND $D_m f = 2P+K+1$ ON $I_{m,k+1}$.

PROOF. Set $y = (k+2)/2^m$ and let $x \in I_{m,k}$ be arbitrary. Then (see Figure 4),

$$\left|\frac{f(y)-f(x)}{y-x}\right| > \frac{[(2P+K+1)-(K+1)]2^{-m}}{2 \cdot 2^{-m}} = P. \qquad \square$$

PROOF OF THEOREM 15. The crux of the argument is to use the divergent quality of the modulus of a random walk to show that the situation described in Lemma 16 occurs with sufficient frequency. In particular, if $\{Y_l\}_{l=0}^{\infty}$ is a classical random walk starting from 0, we can choose l' so that

$$\operatorname{prob}\{|Y_l| > 2(P+K+1)\} \geq \frac{1}{2} \text{ for all } l \geq l'.$$

Now fix $l_0 > l'$. For each $f(x,\omega) \in \Lambda_{d,1}^*$, we define a sequence of sets $\{S_{K,nl_0}^P\}_{n=0}^{\infty}$ inductively on n. The reference point for this sequence is (again) the sequence $S_{K,n}$ from the proof of Theorem 5.[2] Although the sets S_{K,nl_0} depend on the function f, it is significant that the measures $m(S_{K,nl_0})$ do not.

[2]Although it is similar, this notation should not be associated or confused with that used in the proof of Theorem 14.

For $n = 0$, set $S^P_{K,0} = S_{K,0} = [0,1]$. Given S^P_{K,nl_0}, let $S^P_{K,(n+1)l_0} \subseteq S_{K,(n+1)l_0} \cap S^P_{K,nl_0}$ consist of those dyadic intervals in $S_{K,(n+1)l_0}$ of the $(n+1)l_0$ generation that are not eliminated after applying the criterion in Lemma 16. That is,

$$S^P_{K,(n+1)l_0} = [S_{K,(n+1)l_0} \cap S^P_{K,nl_0}] \setminus \bigcup_{k'} I_{(n+1)l_0,k'},$$

where $I_{(n+1)l_0,k'}$ is part of the union to be removed if we can determine that a Lipschitz condition with constant P is impossible because $|D_{(n+1)l_0} f(x)| \geq 2P + K + 1$ for $x \in I_{(n+1)l_0,k'\pm 1}$.

It follows that L_P, the set of all Lipschitz points of f with constant P, satisfies

$$S_K \cap L_P \subseteq \bigcap_{n=0}^{\infty} S^P_{K,nl_0}. \tag{6.1}$$

In an effort to derive a statement about the size of S^P_{K,nl_0}, let's fix our attention on an arbitrary interval $I_{nl_0,k} \subseteq S^P_{K,nl_0}$ and look ahead l_0 steps to the $(n+1)l_0$ generation. For each interval $I_{(n+1)l_0,k'}$ in $S_{K,(n+1)l_0} \cap I_{nl_0,k}$, one of the endpoints is necessarily a dyadic point of the $(n+1)l_0$ generation, but the other is from some earlier generation. Let $p_0 > 0$ be the probability, which depends only on l_0 and l', that the other endpoint is of the $(n+1)l_0 - l'$ generation or earlier. If $I_{(n+1)l_0,k'}$ is such an interval, then our choice of l' implies that with probability at least $1/2$

$$\begin{aligned} |D_{(n+1)l_0} f(x)| &\geq ||D_{nl_0+1} f(x)| - 2(P + K + 1)| \\ &\geq |(K+1) - 2(P + K + 1)| = 2P + K + 1 \end{aligned}$$

for all x in an interval adjacent to $I_{(n+1)l_0,k'}$, and we may apply Lemma 16. Thus, to construct the sequence $\{S^P_{K,nl_0}\}$, we begin with the intervals in the corresponding sequence $\{S_{K,nl_0}\}$, but at each stage remove them at a rate sufficient to conclude that

$$E[m(S^P_{K,nl_0})] \leq \left(1 - \frac{p_0}{2}\right)^n m(S_{K,nl_0}). \tag{6.2}$$

To finish the proof, it suffices to show that

$$E\left[H^\alpha\left(S_K \cap L_P\right)\right] = 0 \text{ for all } P > 0,$$

and this will follow from (6.1) if we can establish

$$E\left[H^\alpha\Big(\bigcap_{n=0}^{\infty} S^P_{K,nl_0}\Big)\right] = 0 \text{ for all } P > 0.$$

Returning to the technique from the proof of Theorem 5, let N_{K,nl_0}^P be the number of intervals from the nl_0 generation that make up S_{K,nl_0}^P so that

$$\begin{aligned} H^\alpha\left(\bigcap_{n=0}^{\infty} S_{K,nl_0}^P\right) &\leq \liminf_{n\to\infty}\ N_{K,nl_0}^P \left(2^{-nl_0}\right)^\alpha \\ &= \liminf_{n\to\infty}\ m(S_{K,nl_0}^P)2^{(1-\alpha)nl_0} \\ &= \liminf_{n\to\infty} m(S_{K,nl_0}^P)r^{-nl_0}. \end{aligned}$$

Taking expectations and combining (6.2) with the estimate on $m(S_{K,nl_0})$ from (3.2) yields

$$E\left[H^\alpha\left(\bigcap_{n=0}^{\infty} S_{K,nl_0}^P\right)\right] \leq \liminf_{n\to\infty}\ \left(1-\frac{p_0}{2}\right)^n c_1 = 0,$$

and the proof is complete. $\square$

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