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WAVELETS AND BESOV SPACES ON MAULDIN-WILLIAMS FRACTALS

Abstract

A. Jonsson has constructed wavelets of higher order on self-similar sets, and characterized Besov spaces on totally disconnected self-similar sets, by means of the magnitude of the coefficients in the wavelet expansion of the function. For a class of self-similar sets, W. Jin shows that such wavelets can be constructed by recursively calculating moments. We extend their results to a class of graph-directed self-similar sets, introduced by R. D. Mauldin and S. C. Williams.

1 Introduction.

Wavelet bases and multiresolution analysis on fractals has been studied in several papers (see e.g. [14, 11, 15, 3, 9]). R. S. Strichartz [9] defines continuous piecewise linear wavelets, and constructs a multiresolution analysis on several fractals.

A. Jonsson introduces Haar type wavelets of higher order on self-similar sets in [15]; i.e., piecewise polynomials of degree $\leq m$, which are continuous on totally disconnected self-similar sets, and constructs wavelet bases using multiple Haar type mother wavelets of higher order. Jonsson then characterizes Besov spaces on a class of totally disconnected self-similar sets, by means of the magnitude of the coefficients in the wavelet expansion of a function. Following his method, we generalize this in Theorem 5.2, and Theorem 5.4, to graph-directed self-similar sets, introduced by R. D. Mauldin and S. C. Williams in [10].

Jonsson's construction of the wavelet bases involves the Gram-Schmidt procedure, which in general is difficult to apply, because the inner product in

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$L^2(\mu)$ is not easily calculated on fractals. However, for Haar type polynomials, the Gram-Schmidt procedure can be reduced to calculating moments. W. Jin [14] shows that, for a class of self-similar sets in $\mathbb{R}^n$, the moments can be calculated recursively. We extend the result by Jin to a class of strongly connected Mauldin-Williams fractals in Theorem 4.3.

2 Mauldin-Williams Fractals.

A *digraph* is a finite directed graph (V, E), in which every vertex has at least one edge leaving it, and there is one edge with two vertices leaving it. We allow several edges between vertices and edges from a vertex to itself, and enumerate the vertices from 1 to q; i.e., $V = \{1, 2, \dots, q\}$.

Let E_{ij} be the set of edges from vertex i to vertex j, and let E_i be the set of edges leaving the vertex i.

For $i, j \in V$ and positive integers k, let $\mathcal{E}_{ij}^k$ denote the set of paths of length k from i to j. When we leave out an index in $\mathcal{E}_{ij}^k$ and write $\mathcal{E}_i^k$, $\mathcal{E}^k$, $\mathcal{E}_{ij}$, or $\mathcal{E}_i$, we mean that the index left out can take on any admissible value. For notational purposes, we let the set of vertices be included in $\mathcal{E}$. If $e = e_1 e_2 \dots e_n$ and $\tilde{e} = \tilde{e}_1 \tilde{e}_2 \dots \tilde{e}_m$ are paths, we write $e\tilde{e}$ for the path $e_1 e_2 \dots e_n \tilde{e}_1 \tilde{e}_2 \dots \tilde{e}_m$.

By an *infinite path*, we mean a sequence $e^* = e_1 e_2 \dots$, such that the restriction $e^*|n = e_1 e_2 \dots e_n$ of e^* to the first n characters, is a path. Let $\mathcal{E}^*$ be the set of all infinite paths, and let $\mathcal{E}_i^*$ be the set of infinite paths with initial vertex i.

Define $t(e) = j$ for a path e that terminates at the vertex j, and let $t(i) = i$ for a vertex i.

A *similitude with contraction factor* r is a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, such that $|T(x) - T(y)| = r|x - y|$ for all $x, y \in \mathbb{R}^n$, for some fix $0 < r < 1$.

Definition 2.1. The ordered pair $((V, E), \{T_e\}_{e \in E})$, is a *Mauldin-Williams graph* (MW-graph), if (V, E) is a digraph, and T_e is a similitude with contraction factor $0 < r_e < 1$ for each e in E.

We use the notation $T_e = T_{e_1} \circ T_{e_2} \circ \dots \circ T_{e_m}$ and $r_e = r_{e_1} r_{e_2} \cdots r_{e_m}$, for $e = e_1 e_2 \dots e_m \in \mathcal{E}^m$.

Given a MW-graph, it is shown in [10] that there exist a unique collection $\{K_i\}_{i \in V}$, of non-empty compact sets, which we will refer to as *Mauldin-Williams sets* (MW-sets), such that

$$K_i = \bigcup_{i=1}^{q} \bigcup_{e \in E_{ij}} T_e(K_j). \tag{1}$$

Iterating (1) we get that $K_i = \cup_{e\in\mathcal{E}_i^m} K_e$, where $K_e = T_e(K_{t(e)})$.

We call $K = \cup_{i\in V} K_i$ a *Mauldin-Williams fractal* (MW-fractal), which is called the graph-directed construction object in [10]. For a more on Mauldin-Williams graphs, see for example [13, 16, 10].

To a MW-graph we associate a matrix $A(t)$, for $t \geq 0$, by defining the (i, j)-th entry of $A(t)$ to be $a_{ij}(t) = \sum_{e\in E_{ij}} r_e^t$, with $a_{ij} = 0$ if $E_{ij} = \emptyset$.

If A is a square matrix, then the spectral radius $\rho(A)$ of A, is the largest, in absolute value, eigenvalue of A. It can be shown that there exists a unique $d \geq 0$, such that $\rho(A(d)) = 1$. This d is called the *dimension of the MW-graph* and we call $A(d)$ the *construction matrix*. Let H^d denote the d-dimensional Hausdorff measure, and $H^d\ |F$ the restriction of H^d to the set F.

A MW-graph is *strongly connected* if for every pair of vertices i and j in V, there is a directed path from i to j.

Theorem 2.2. [10] *If a strongly connected MW-graph has dimension d, then $H^d(K_i) < \infty$ for all $i \in V$.*

It is not necessary that the MW-graph is strongly connected for the Hausdorff measure to be finite. It does however depend on the structure of the graph; see [10] for details.

A MW-graph satisfies the *open set condition* (OSC) if there exist non-empty open sets $\{U_i\}_{i\in V}$ such that for each $i \in V$ $\cup_{e\in E_{ij}} T_e(U_j) \subset U_i$, with disjoint union.

Theorem 2.3. [8] *If a strongly connected MW-graph has dimension d, then*

$$OSC \iff H^d(K_i) > 0 \text{ for all } i \in V \iff H^d(K) > 0.$$

The proof of the implication $\implies$ of the left $\iff$ can be found in [10], while the converse is proven in [8], as is the right implication $\iff$.

We say that two sets E and F are *essentially disjoint* (with respect to the d-dimensional Hausdorff measure) if $H^d(E \cap F) = 0$.

Proposition 2.4. [8] *If a MW-graph is strongly connected, then the sets $\{K_e : e \in E_i\}$ are pairwise essentially disjoint for all $i \in V$.*

Corollary 2.5. *If a MW-graph is strongly connected, the sets $\{K_e\}_{e\in\mathcal{E}_i^k}$ are pairwise essentially disjoint for all $k \geq 1$ and $i \in V$.*

Assume that the MW-sets $\{K_i\}$ are pairwise essentially disjoint, and let $\mu_i = H^d|K_i$. Then $\mu = \sum_{i\in V} \mu_i$ has support K, and $\mu|K_i = \mu_i$. Each measure μ_i is invariant in the sense that

$$\mu_i(A) = \sum_{j=1}^{q} \sum_{e\in E_{ij}} r_e^d \mu_j(T_e^{-1}(A)) \tag{2}$$

for all Borel sets $A \subseteq \mathbb{R}^n$. By (2) it follows that

$$\int_{K_i} f(x)\,\mathrm{d}\mu_i(x) = \sum_{j=1}^{q} \sum_{e \in E_{ij}} r_e^d \int_{K_j} f(T_e(x))\,\mathrm{d}\mu_j(x), \tag{3}$$

for all Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}$. Furthermore, we have that

$$\int_{K_e} f(x)\,\mathrm{d}\mu_i(x) = r_e^d \int_{K_j} f(T_e(x))\,\mathrm{d}\mu_j(x) \text{ for all } e \in \mathcal{E}_{ij}, \tag{4}$$

and especially $\mu_i(K_e) = r_e^d \mu_j(K_j)$. Since $\operatorname{diam} K_e = r_e \operatorname{diam} K_j$, we also have that

$$\mu_i(K_e) = (\operatorname{diam} K_e)^d \mu_j(K_j)(\operatorname{diam} K_j)^{-d}. \tag{5}$$

Definition 2.6. Let $0 < d \leq n$ and let μ be a non-negative Borel measure on $\mathbb{R}^n$ with $\operatorname{supp}(\mu) = F$. Then μ is a d-measure on F if there exists constants $c_1, c_2 > 0$ such that $c_1 r^d \leq \mu(F \cap B(x,r)) \leq c_2 r^d$ for all closed balls $B(x,r)$, with $x \in F$ and $0 < r \leq 1$. If there exists a d-measure on a closed set F we say that F is a d-set.

Remark. We can replace $0 < r \leq 1$ with $0 < r \leq r_0$, where $r_0 > 0$, in Definition 2.6 without altering the meaning. The restriction of the d-dimensional Hausdorff measure to a d-set F will act as a canonical d-measure on F (see [17]).

Proposition 2.7. *If a strongly connected MW-graph has dimension d, then the MW-graph satisfies the OSC iff the MW-fractal K is a d-set.*

PROOF. If K is a d-set, then, by Theorem 2.3, the OSC is satisfied, since the Hausdorff measure acts as a canonical d-measure on any d-set. Let $\mu = \sum_{i=1}^q \mu_i$, where $\mu_i = H^d|K_i$, and put $M = \max_j \mu_j(K_j)$, $m = \min_j \mu_j(K_j)$, $D = \max_j \operatorname{diam} K_j$, $r_0 = \min_j \operatorname{diam} K_j$ and $r_{\min} = \min_{e \in E} r_e$. We will use r_0 in Definition 5 according with the remark above.

Let $i \in V$, $x \in K_i$ and $0 < r \leq r_0$. First we show that $\mu_i(B(x,r)) \geq c_0 r^d$ for some $c_0 > 0$. We can find $e \in \mathcal{E}_i^p$, for some integer $p \geq 1$, such that $r r_{\min} \leq \operatorname{diam} K_e < r$, and with $x \in K_e$. Then $K_e \subseteq B(x,r)$, so by (5) we have that

$$\mu_i(B(x,r)) \geq \mu_i(K_e) \geq r^d \frac{r_{min}^d m}{D^d}.$$

Next we will show that $\mu_i(B(x,r)) \leq c_1 r^d$ for some $c_1 > 0$. If $e^* = e_1 e_2 \ldots \in \mathcal{E}^*$ is an infinite path, then $K_{e^*} = \cap_{m \geq 1} K_{e^*|m}$ is a singleton. If $z \in K_i$, there is at least one infinite path $e^* \in \mathcal{E}^*$ such that $z = K_{e^*}$. Choose exactly one such infinite path e_y to each $y \in B(x,r) \cap K_i$ and let p_y be the smallest positive integer such that

$$r_{\min} r \leq \operatorname{diam} K_{e_y|p_y} = r_{e_1} \cdot \ldots \cdot r_{e_{p_y}} \operatorname{diam} K_{t(e_{p_y})} < r. \tag{6}$$

Let I be the restrictions of all such infinite paths with initial vertex i, that is

$$I = \bigcup_{y \in B(x,r) \cap K_i} \{e_y | p_y\},$$

where we have chosen e_y and p_y, as explained above.

Note that, if $e_z|p_1, e_w|p_2 \in I$, and $e_w|p_1 = e_z|p_1$, then $p_1 = p_2$ because otherwise p_2 would not be the smallest possible integer satisfying (6). Therefore, by Corollary 2.5, $\{K_e\}_{e \in I}$ is a collection of pairwise essentially disjoint sets.

The number of elements in I is bounded by a constant $c > 0$, where c does not depend on r. To see this, let $\{U_j\}$ be the sets in the OSC and assume each U_j contains a ball with radius R. If $U_e = T_{e_1} \circ \ldots \circ T_{e_p}(U_{t(e)})$, then $\{U_e\}_{e \in I}$ is a family of pairwise disjoint sets, where each U_e contains a ball with radius $r_{e_1} \ldots r_{e_p} R \geq R r_{\min} r_0 r$. Then there must be a constant $c > 0$ so that the number of elements in I is less then c.

It now follows, since $B(x,r) \cap K_i \subseteq \cup_{e \in I} K_e$, that

$$\begin{aligned} \mu_i(B(x,r)) &\leq \sum_{e \in I} \mu_i(K_e) = \sum_{e \in I} r_e^d H^d(K_{t(e)}) \\ &\leq \sum_{e \in I} r_{e_1}^d \ldots r_{e_p}^d (\operatorname{diam} K_{t(e)})^d \frac{M}{r_0^d} \leq \frac{cM}{r_0^d} r^d = c_1 r^d \end{aligned}$$

Hence each μ_i is a d-measure on K_i. It is easy to see that μ is a d-measure on $K = \cup_{i \in V} K_i$. □

3 Sets Preserving Markov's Inequality.

We use the notation $\mathbb{N} = \{0, 1, 2, \ldots\}$, and write $z^m = z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$ for $z \in \mathbb{R}^n$ and $m \in \mathbb{N}^n$. Let $\mathcal{P}_m$ denote the set of real polynomials in $\mathbb{R}^n$ of total degree at most m.

Definition 3.1. A closed set $F \subseteq \mathbb{R}^n$ preserves Markov's inequality if for every fixed positive integer m there exist a constant $c > 0$, such that for all polynomials $P \in \mathcal{P}_m$ and closed balls $B = B(x,r)$, $x \in F$, $0 < r \leq 1$, we have that

$$\max_{F \cap B} |\nabla P| \leq \frac{c}{r} \max_{F \cap B} |P|. \tag{7}$$

Remark. We can replace $0 < r \leq 1$ with $0 < r \leq r_0$, where $r_0 > 0$, without altering the meaning of Definition 3.1.

The space $\mathcal{P}_m$ has dimension $D_0 = \binom{n+m}{n}$ as a vector space, and if F preserves Markov's inequality and μ is a d-measure on F, then $\mathcal{P}_m$ will have the same dimension D_0 as a subspace of $L^2(\mu)$ (see [4]).

We let $\|f\|_p$ denote the standard L^p-norm with respect to μ, and $\|f\|_{p,F}$ the L^p-norm with respect to $\mu|F$.

If each MW-set K_i preserves Markov's inequality and $1 \leq p \leq \infty$ there exists constants $c_1, c_2 > 0$ such that

$$c_1\|P\|_{\infty,K_e} \leq [\mu_i(K_e)]^{-p}\|P\|_{p,K_e} \leq c_2\|P\|_{\infty,K_e}, \tag{8}$$

for all $e \in \mathcal{E}_i$ and $P \in \mathcal{P}_m$. To show (8) we will use that, if a set F preserves Markov's inequality, then there exists a constant $c > 0$ such that $\|P\|_{\infty,F} \leq c\|P\|_{p,F}$, for all $P \in \mathcal{P}_m$ (see [17]). If $e \in \mathcal{E}_{ij}$, (4) gives us that

$$\begin{aligned}
\|P\|_{\infty,K_e} &= \|P \circ T_e\|_{\infty,K_j} \\
&\leq c\Big(\int_{K_j} |P \circ T_e|^p \,\mathrm{d}\mu_j\Big)^{1/p} = c\Big(r_e^{-d}\int_{K_e} |P|^p \,\mathrm{d}\mu_i\Big)^{1/p} \\
&= c\Big(\frac{\mu_j(K_j)}{\mu_i(K_e)}\int_{K_e} |P|^p \,\mathrm{d}\mu_i\Big)^{1/p} \leq \frac{1}{c_1}\Big(\frac{1}{\mu_i(K_e)}\int_{K_e} |P|^p \,\mathrm{d}\mu_i\Big)^{1/p}
\end{aligned}$$

The right inequality in (8) is trivial.

Proposition 3.2. *Let $\{K_i\}$ be the MW-sets associated with a MW-graph. If K_i is not a subset of any $n-1$ dimensional subspace of $\mathbb{R}^n$ for any $i \in V$, then each K_i preserves Markov's inequality.*

Remark. The MW-graph in Proposition 3.2 does not need to satisfy the OSC, nor be strongly connected.

Theorem 3.3. [7] *$F \subseteq \mathbb{R}^n$ preserves Markov's inequality if there exists a constant $c > 0$ so that for every closed ball $B = B(x,r)$, where $x \in F$ and $0 < r \leq 1$, there are $n+1$ affinely independent points $a_i \in F \cap B$, $i = 1, \ldots, n+1$, such that the n-dimensional ball inscribed in the convex hull of $a_1, \ldots, a_{n+1}$ has radius no less then cr.*

We will use Theorem 3.3 (see [7] for a proof) to prove Proposition 3.2. Proposition 3.2 is known for IFS, cf. [6].

PROOF OF PROPOSITION 3.2. Let $r_{\min} = \min_{e\in E} r_e$, $D = \max_j \operatorname{diam} K_j$ and $r_0 = \min_j \operatorname{diam} K_j$. Suppose $x \in K_i$ and that $0 < r \leq r_0$. Since $x \in K_i$ there exists $e \in \mathcal{E}_i^*$ such that $x = \cap_{m=1}^{\infty} K_{e|m}$. Let p be the smallest positive integer such that $r_{min} r \leq \operatorname{diam} K_{e|p} < r$. Since K_m is not a subset of any $n-1$-dimensional subspace of $\mathbb{R}^n$, there exists $n+1$ affinely independent points $y_l^m \in K_m$, $l = 1, \ldots, n+1$. Assume we can inscribe a ball with radius c_m in the simplex spanned by $\{y_1^m, \ldots, y_{n+1}^m\}$ and let $c_0 = \min_m c_m$. Suppose that $t(e|p) = j$ and define $a_l = T_{e|p}(y_l^j)$ for $l = 1, \ldots, n+1$. Then $a_l \in K_{e|p} \subseteq B(x,r) \cap K_i$ and we can inscribe a ball with radius $r^* \geq r_{e_1} r_{e_2} \cdot \ldots \cdot r_{e_p} c_j$ in the simplex spanned by $\{a_1, \ldots, a_{n+1}\}$, since $T_{e|p}$ is a similitude. Therefore we can inscribe a ball in the convex hull of $a_1, \ldots, a_{n+1}$, with radius $r^* \geq c_0 r_{e_1} r_{e_2} \cdot \ldots \cdot r_{e_p} = c_0 \operatorname{diam} K_{e|p} / \operatorname{diam} K_j \geq c_0 r_{\min} r / D = cr$. By Theorem 3.3, K_i preserves Markov's inequality. □

4 Moments and Wavelets.

In this section we will describe one way of constructing a wavelet basis for $L^2(\mu)$, introduced in [15], and show that moments can be calculated recursively for a class of strongly connected MW-fractals.

The key assumptions in the construction of the wavelets are: (i) the MW-sets are d-sets, (ii) they preserves Markov's inequality, and (iii) $\mu(K) = \sum_{e \in E} \mu(K_e)$, where μ is a d-measure on K. A strongly connected MW-graph that satisfies the OSC and has essentially disjoint MW-sets fulfils (i) and (iii), while Proposition 3.2 helps us determine that (ii) is fulfilled.

Example 4.1. An example of a strongly connected MW-fractal is the Hany fractal, introduced in [2] and further studied in [1]. All twelve similitudes in the MW-graph describing the Hany fractal (Figure 2) have contraction factor 1/3.

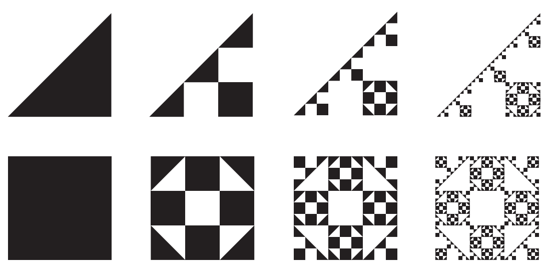

Figure 1: The first four iterations in the construction of the Hany fractal.

In Example 4.5 we give another example of a MW-fractal that is given by a strongly connected MW-graph. An example of a MW-fractal that is not

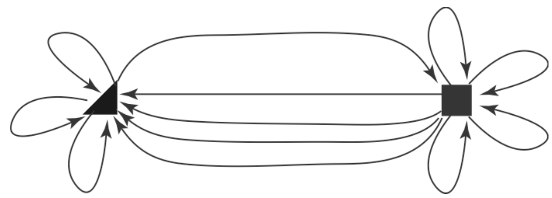

Figure 2: The digraph for the Hany fractal.

strongly connected but still fulfils (i)–(iii) is the von Koch snowflake domain. However, the boundary of the snowflake; i.e., the closed von Koch curve, is a strongly connected MW-fractal, as are all fractals that are an essentially disjoint union of n copies of a self-similar fractal.

For $i \in V$ let $S_0^i = \mathcal{P}_m$ and S_k^i be the space of functions f, as a subspace of $L^2(\mu_i)$, such that f is a polynomial in $\mathcal{P}_m$ on each K_e, for $e \in \mathcal{E}_i^k$, except perhaps in points belonging to several different K_e. Note that the set of all such points has zero μ-measure. We then get a nested sequence $S_0^i \subset S_1^i \subset S_2^i \dots$ of subspaces of $L^2(\mu_i)$. Let $W_0^i = S_0^i$ and $W_{k+1}^i = S_{k+1}^i \ominus S_k^i$ for $k \geq 0$, where $\ominus$ denotes the orthogonal complement. Then W_1^i will have dimension $D_1^i = D_0|E_i| - D_0$. Suppose that we have an orthonormal basis $\psi^{i,1}, \dots, \psi^{i,D_1^i}$ in W_1^i each with support in K_i and define

$$\psi_e^\sigma(x) = \Big[\frac{\mu_i(K_e)}{\mu_j(K_j)}\Big]^{-1/2} (\psi^{t(e),\sigma} \circ T_e^{-1})(x)$$

for $e \in \mathcal{E}_i$ and $\sigma = 1, \dots, D_e$, where $D_e = D_1^{t(e)}$. Then $\{\psi_e^\sigma\}_{e \in \mathcal{E}_i^k}$ will form an orthonormal basis in W_{k+1}^i for $k \geq 1$. Let $\phi_1^i, \dots, \phi_{D_0}^i$ be an orthonormal basis in $W_0^i = S_0^i$. To simplify the notation, we let $\mathcal{E}^0 = V$ with $\mathcal{E}_i^0 = \{i\}$, and $\psi_i^\sigma = \psi^{i,\sigma}$ for $i \in V$. Then $\{\psi_e^\sigma : k \geq 0, e \in \mathcal{E}_i^k, 1 \leq \sigma \leq D_e\}$ together with $\{\phi_l^i : 1 \leq l \leq D_0\}$ will form a orthonormal basis in $L^2(\mu_i)$ since $L^2(\mu_i) = \bigoplus_{k \geq 0} W_k^i$. Since the MW-sets K_i are assumed to be pairwise essentially disjoint, we have that $L^2(\mu) = \bigoplus_{k \geq 0} \bigoplus_{i=1}^q W_k^i$. Therefore

$$f = \sum_{i=1}^{q} \sum_{l=1}^{D_0} \alpha_l^i \phi_l^i + \sum_{k=0}^{\infty} \sum_{e \in \mathcal{E}^k} \sum_{\sigma=1}^{D_e} \beta_e^\sigma \psi_e^\sigma, \tag{9}$$

is a valid representation for f in $L^2(\mu)$, where $\beta_e^\sigma = \int f \psi_e^\sigma \, \mathrm{d}\mu$ and $\alpha_l^i = \int f \phi_l^i \, \mathrm{d}\mu$. Furthermore, this representation also holds in $L^p(\mu)$ for $1 \leq p \leq \infty$, see [15] for a proof of this in the case of an IFS.

Lemma 4.2. *With the notation above, there exists a constant $c > 0$, not depending on the wavelet basis, such that*

$$\|\psi_e^\sigma\|_p \le c\mu(K_e)^{(1/p-1/2)} \text{ for all } e \in \mathcal{E}. \tag{10}$$

Remark. By (5), Lemma 4.2 remains true if we replace $\mu(K_e)$ in (10) with $\operatorname{diam}(K_e)^d$.

PROOF OF LEMMA 4.2. Assume that $e \in \mathcal{E}_{ij}$. Since $\psi^{j,\sigma}$ is a polynomial on each $K_{\tilde{e}}$ for $\tilde{e} \in E_j$ we can use (8) twice to show that there is a constant $c_0 > 0$ not depending on the wavelet basis such that $\|\psi^{j,\sigma}\|_p \le c_0$.

$$\begin{aligned}
\|\psi^{j,\sigma}\|_p &\le \sum_{\tilde{e}\in E_j} \|\psi^{j,\sigma}\|_{p,K_{\tilde{e}}} \le c_2 \sum_{\tilde{e}\in E_j} \mu(K_{\tilde{e}})^p \|\psi^{j,\sigma}\|_{\infty,K_{\tilde{e}}} \\
&\le c_3 \sum_{\tilde{e}\in E_j} \mu(K_{\tilde{e}})^p \mu(K_{\tilde{e}})^{-2} \|\psi^{j,\sigma}\|_{2,K_{\tilde{e}}} \\
&\le c_0 \sum_{\tilde{e}\in E_j} \|\psi^{j,\sigma}\|_{\infty,K_{\tilde{e}}} = c_0 \|\psi^{j,\sigma}\|_2 = c_0.
\end{aligned}$$

Then, by using (5) and (4), we get that

$$\begin{aligned}
\|\psi_e^\sigma\|_p^p &= \int_{K_e} \Big|\Big(\frac{\mu(K_e)}{\mu(K_j)}\Big)^{-1/2} (\psi^{j,\sigma} \circ T_e^{-1})\Big|^p \, \mathrm{d}\mu \\
&= \Big(\frac{\mu(K_e)}{\mu(K_j)}\Big)^{-p/2} r_e^d \|\psi^{j,\sigma}\|_p^p \le c_0^p \mu(K_e)^{-p/2} r_e^d \\
&\le c\mu(K_e)^{d(1/p-1/2)p}.
\end{aligned}$$

□

For $F \subset \mathbb{R}^n$, and multi-indices $\mathbf{m} \in \mathbb{N}^n$ and $\mathbf{z} \in \mathbb{R}^n$, we define the *moments of μ over F* by

$$M(F, \mathbf{m}) := \int_F \mathbf{z}^{\mathbf{m}} \, \mathrm{d}\mu = \int_F z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} \, \mathrm{d}\mu,$$

and call $|\mathbf{m}| = m_1 + m_2 + \ldots + m_n$ the order of the moment. Recall that the D_0 is the dimension of $\mathcal{P}_m$ seen as a subspace of $L^2(\mu)$. Let $P_1, \ldots, P_{D_0}$ be the monomials of degree $\le m$ and define $g_k^i = P_k \chi_i$, $k = 1, 2, \ldots, D_0$, where χ_i denotes the characteristic function on K_i.

Enumerate all $e \in E_i$ so that $E_i = \{e_{i0}, \ldots, e_{ik_i}\}$, where $k_i = |E_i| - 1$ and let $g_{jk}^i = P_k$ on $K_{e_{ij}}$ and 0 elsewhere, for $j = 1, 2, \ldots, k_i$, and $k = 1, 2, \ldots, D_0$. Then $\{g_k^i\}_k$ together with $\{g_{jk}^i\}_{j,k}$ form a linearly independent set in S_1^i which we will orthogonalize using the Gram-Schmidt procedure and obtain orthonormal basis for S_0^i and W_1^i. We use the standard inner product $< f, g >= \int_K fg\,\mathrm{d}\mu$ and L^2-norm $\|f\|_2 =< f, f >^{1/2}$. Let

$$\phi_1^i = \frac{g_1^i}{\|g_1^i\|_2} \text{ and } h_k^i = g_k^i - \sum_{l=1}^{k-1} < g_l^i, \phi_l^i > \phi_l^i \text{ where } \phi_k^i = \frac{h_k^i}{\|h_k^i\|_2}$$

for $k = 2, 3, \ldots, D_0$. Then $\{\phi_k^i\}_{k=1}^{D_0}$ will be an orthonormal basis in S_0^i. Continuing the Gram-Schmidt procedure on the remaining functions g_{jk}^i, we obtain an orthonormal basis $\{\psi_{jk}^i : j = 1, 2, \ldots, k_i \text{ and } k = 1, \ldots, D_0\}$ for W_1^i. In this construction we need to calculate all moments of order $\leq 2m$ over the MW-sets K_i, and over the sets K_e, for $e \in E$.

If $B = [b_{ij}]$ is a $n \times n$ matrix we define the matrix norm by

$$\|B\| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |b_{ij}|.$$

The similitudes $T_e : \mathbb{R}^n \to \mathbb{R}^n$ can be written as $T_e(\mathbf{z}) = A_e\mathbf{z} + \mathbf{b}_e$, where $A_e = [a_{eij}]$ is an $n \times n$ matrix, and $\mathbf{b}_e \in \mathbb{R}^n$.

Theorem 4.3. *Suppose a strongly connected MW-graph, that satisfies the OSC, has construction matrix $A = A(d)$, essentially disjoint MW-sets, and similitudes $T_e(\mathbf{z}) = A_e\mathbf{z} + \boldsymbol{b}_e$. If*

$$\|A\| \max_{e \in E} \|A_e\| < 1, \tag{11}$$

then the moments of all orders over K_i can be calculated recursively.

If we know the moments over all K_i, then we can calculate the moments over K_e for all $e \in E$ by using (4). Note that the condition (11) implies that $\|A_e\| < 1$ for all $e \in E$ since $\|A\| \geq \rho(A) = 1$.

Example 4.4. The dimension of the Hany fractal is $d = \ln((7+\sqrt{17})/2)/\ln 3$. If the similitudes are given by $T_e(\mathbf{z}) = A_e(\mathbf{z}) + b_e$, then $\|A_e\| = 1/3$ for all edges e. The construction matrix is

$$A = \begin{bmatrix} 3(\frac{1}{3})^d & (\frac{1}{3})^d \\ 4(\frac{1}{3})^d & 4(\frac{1}{3})^d \end{bmatrix} = \begin{bmatrix} \frac{6}{7+\sqrt{17}} & \frac{2}{7+\sqrt{17}} \\ \frac{8}{7+\sqrt{17}} & \frac{8}{7+\sqrt{17}} \end{bmatrix}$$

Hence it follows by Theorem 4.3 that the moments can be calculated recursively.

Example 4.5. In this example we will illustrate the method described in this section. Let $K = K_1 \cup K_2$, where K_1 is the Sierpinski gasket with vertices $(0,0)$, $(1,0)$ and $(1/2,1/2)$, and K_2 is K_1 reflected in the x-axis. We consider K as a MW-fractal given by the digraph (V,E) in Figure 4 together with the similitudes

$$\begin{array}{ll} T_a(z) = z/2 + (1/4,1/4) & T_b(z) = z/2 + (1/2,0) \\ T_c(z) = -z/2 + (1/2,0) & T_d(z) = -z/2 + (1/2,0) \\ T_e(z) = z/2 + (1/4,-1/4) & T_f(z) = z/2 + (1/2,0), \end{array}$$

where $z = (x,y)$. Let us begin the construction of an orthonormal basis for $L^2(K)$. We do this with polynomials of at most degree one, which means that (18) in Theorem 5.2 below, will be valid for $0 < \alpha < 2$.

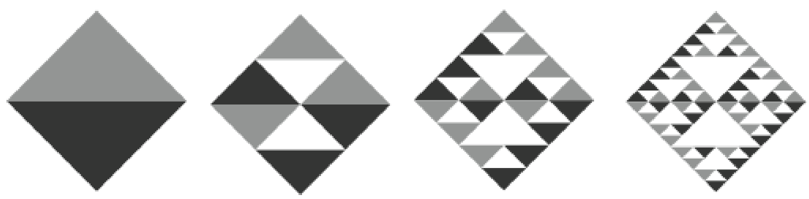

Figure 3: The first four iterates in the construction of K.

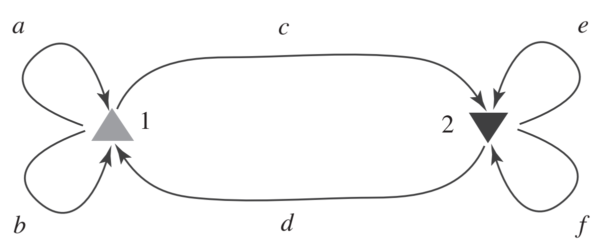

Figure 4: The directed graph generating K.

It is easy to see that (11) is satisfied, so we can calculate the moments recursively. We need to calculate the moments of order ≤ 2 over K_i. Let μ be the restriction of the d-dimensional Hausdorff measure to K, where $d = \ln 3/\ln 2$, such that $\mu(K_i) = 1$. Let $M_i(k,l)$ be the moment

$$M_i(k,l) = \int_{K_i} x^k y^l \, \mathrm{d}\mu$$

and let the $\mathbf{M}_1 = (M_1(1,0), M_1(0,1), M_2(1,0), M_2(0,1))$ be the moments of order 1. Note that $M_i(0,0) = \mu(K_i) = 1$. Using (12) we have that

$$\begin{aligned} M_1(1,0) &= \int_{K_1} x \,\mathrm{d}\mu = \frac{1}{3}\int_{K_2} x \circ T_c \,\mathrm{d}\mu + \frac{1}{3}\int_{K_1} x \circ T_a \,\mathrm{d}\mu + \frac{1}{3}\int_{K_1} x \circ T_b \,\mathrm{d}\mu \\ &= \frac{1}{3}\int_{K_2} (-x/2 + 1/2) \,\mathrm{d}\mu + \frac{1}{3}\int_{K_1} (x/2 + 1/4) \,\mathrm{d}\mu + \frac{1}{3}\int_{K_1} (x/2 + 1/2) \,\mathrm{d}\mu \\ &= -\frac{1}{6} M_2(1,0) + \frac{1}{3} M_1(1,0) + \frac{5}{12}. \end{aligned}$$

Doing this for every moment of order 1, we arrive at the equation system $(I - \Gamma_1)\mathbf{M}_1 = \mathbf{R}_1$, where

$$\Gamma_1 = \begin{bmatrix} \frac{1}{3} & 0 & -\frac{1}{6} & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{1}{6} \\ -\frac{1}{6} & 0 & \frac{1}{3} & 0 \\ 0 & -\frac{1}{6} & 0 & \frac{1}{3} \end{bmatrix} \text{ and } \mathbf{R}_1 = \begin{bmatrix} \frac{5}{12} \\ \frac{1}{12} \\ \frac{5}{12} \\ -\frac{1}{12} \end{bmatrix}.$$

Solving this equation system we get that $\mathbf{M}_1 = (1/2, 1/6, 1/2, -1/6)$. In a similar way, the moments of order 2 are

$$\begin{aligned} \mathbf{M}_2 &= (M_1(2,0), M_1(1,1), M_1(0,2), M_2(2,0), M_2(1,1), M_2(0,2)) \\ &= (11/36, 1/12, 5/108, 11/36, -1/12, 5/108). \end{aligned}$$

Let χ_i be the characteristic functions on K_i. Put $g_1 = \chi_1$, $g_2 = x\chi_1$ and define $h_1 = g_1$ and let the first function in the Gram-Schmidt procedure be $\phi_1^1 = h_1/\|h_1\|_2 = \chi_1$. Continuing the orthonormalization procedure, we let

$$\begin{aligned} h_2 &= g_2 - < g_2, \phi_1^1 > \phi_1^1 = x\chi_1 - \chi_1 \int_{K_1} x \,\mathrm{d}\mu \\ &= x\chi_1 - \chi_1 M(1,0) = (x - \frac{1}{2})\chi_1, \end{aligned}$$

and since

$$\begin{aligned} \|h_2\|_2^2 &= \int_{K_1} (x - \frac{1}{2})^2 \,\mathrm{d}\mu = \int_{K_1} (x^2 - x + \frac{1}{4}) \,\mathrm{d}\mu \\ &= M_1(2,0) - M_1(1,0) + \frac{1}{4} M_1(0,0) = \frac{1}{18}, \end{aligned}$$

we let $\phi_2^1 = h_2/\|h_2\|_2 = 3\sqrt{2}(x - 1/2)\chi_1$. Continuing, we get the functions

$$\phi_1^i = \chi_i, \quad \phi_2^i = \frac{3}{\sqrt{2}}\,(2x - 1)\,\chi_i, \quad \phi_3^i = \frac{3}{\sqrt{6}}\left(6y + (-1)^i\right)\chi_i$$

Then $\{\phi_j^i\}_j$ will be an ON-basis for S_0^i, so that $\{\phi_j^i\}_{i,j}$ is the required basis for $S_0 = S_0^1 \oplus S_0^2$. In a similar way we can produce an ON-basis $\{\psi_j^i\}_{i,j}$ for $W_1 = W_1^1 \oplus W_1^2$, where $W_1^i = S_1^i \setminus S_0^i$.

To prove Theorem 4.3, we need the following lemma; see e.g. [14] for a proof.

Lemma 4.6. *If $D = [d_{ij}]$ is an $n \times n$ matrix such that $d_{ii} > 0$ and $d_{ii} > \sum_{i \neq j} |d_{ij}|$, $i = 1, 2, \dots, n$, then D is non-singular.*

PROOF OF THEOREM 4.3. Observe that the moment of order 0 over K_i is $M(K_i, \mathbf{0}) = \mu(K_i)$. Assume that $\mathbf{m} = (m_1, m_2, \dots, m_n) \neq \mathbf{0}$ and that all moments of order less then $|\mathbf{m}|$ are known. By (3), we get that

$$\begin{aligned} M(K_i, \mathbf{m}) &= \int_{K_i} \mathbf{z}^{\mathbf{m}} \, \mathrm{d}\mu = \sum_{j=1}^{q} \sum_{e \in E_{ij}} r_e^d \int_{K_j} \mathbf{z}^{\mathbf{m}} \circ T_e \, \mathrm{d}\mu \\ &= \sum_{j=1}^{q} \sum_{e \in E_{ij}} r_e^d \int_{K_j} \prod_{k=1}^{n} \Big(\sum_{l=1}^{n} a_{ekl} z_l + b_{ek} \Big)^{m_k} \mathrm{d}\mu. \end{aligned} \tag{12}$$

By the multinomial theorem, we have that

$$\Big(\sum_{l=1}^{n} a_{ekl} z_l \Big)^{m_k} = \sum \frac{m_k!}{p_1! p_2! \cdots p_n!} a_{ek1}^{p_1} a_{ek2}^{p_2} \cdots a_{ekn}^{p_n} z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n},$$

where the sum is taken over $p_1 + p_2 + \ldots + p_n = m_k$ and $p_l \geq 0$. Thus, expanding $\prod_{k=1}^{n} (\sum_{l=1}^{n} a_{ekl} z_l)^{m_k}$ yields a polynomial of degree equal to $|\mathbf{m}| = m_1 + m_2 + \ldots + m_n$. Using that $(a+b)^{m_k} = \sum_{l=0}^{m_k} \binom{m_k}{l} a^{m_k - l} b^l = a^{m_k} + \sum_{l=1}^{m_k} \binom{m_k}{l} a^{m_k - l} b^l$, letting $a = \sum_{l=1}^{n} a_{ekl} z_l$ and $b = b_{ek}$, it follows that

$$\prod_{k=1}^{n} \Big(\sum_{l=1}^{n} a_{ekl} z_l + b_{ek} \Big)^{m_k} = \prod_{k=1}^{n} \Big(\sum_{l=1}^{n} a_{ekl} z_l \Big)^{m_k} + P(e, \mathbf{m})$$

where $P(e, \mathbf{m})$ is a polynomial of degree at most $|\mathbf{m}| - 1$.

There are p moments of order equal to $|\mathbf{m}|$, where p is the number of combinations of $m_1, m_2, \dots, m_n$ such that $m_1 + m_2 + \ldots + m_n = |\mathbf{m}|$. Enumerate the moments over K_i of order $|\mathbf{m}|$ from 1 to p, denoting them M_{is} for $1 \leq s \leq p$. Let m be the enumeration of the moment $\mathbf{m}$; i.e., $M_{im} = M(K_i, \mathbf{m})$. Then, by (12), we get that

$$M_{im} = \sum_{j=1}^{q} \sum_{e \in E_{ij}} r_e^d \int_{K_j} \prod_{k=1}^{n} \Big(\sum_{l=1}^{n} a_{ekl} z_l \Big)^{m_k} \mathrm{d}\mu + R(i, m),$$

where $R(i,m)$ is a sum of moments of order less then or equal to $|\mathbf{m}|-1$.

Now, consider the product

$$\prod_{k=1}^{n}\Big(\sum_{l=1}^{n} d_{kl}z_l\Big)^{m_k}, \tag{13}$$

where $\mathbf{m}=(m_1,m_2,\dots,m_n)$, and $[d_{kl}]$ is an $n\times n$ matrix. Let s be the number of the moment over K_i, given by $\mathbf{s}=(s_1,s_2,\dots,s_n)$; i.e., $M_{is}=M(K_i,\mathbf{s})$. If $\Lambda_m(s,[d_{kl}])$ is the sum of all coefficients of terms in the expansion of (13) with polynomial part $z_1^{s_1}z_2^{s_2}\cdots z_n^{s_n}$, then $|\Lambda_m(s,[d_{kl}])|\le\Lambda_m(s,[|d_{kl}|])$, and

$$\sum_{s=1}^{p}\Lambda_m(s,[d_{kl}])=\prod_{k=1}^{n}\Big(\sum_{k=1}^{n} d_{kl}\Big)^{m_k}.$$

Hence it follows that

$$\sum_{s=1}^{p}|\Lambda_m(s,[a_{ekl}])|\le\sum_{s=1}^{p}\Lambda_m(s,[|a_{ekl}|])=\prod_{k=1}^{n}\Big(\sum_{l=1}^{n}|a_{ekl}|\Big)^{m_k}.$$

Then

$$\sum_{s=1}^{p}|\alpha_{ems}|\le\prod_{k=1}^{n}\Big(\sum_{l=1}^{n}|a_{ekl}|\Big)^{m_k}\le\prod_{k=1}^{n}\|A_e\|^{m_k}=\|A_e\|^{|\mathbf{m}|}, \tag{14}$$

where $\alpha_{ems}=\Lambda_m(s,[a_{ekl}])$. We now get that

$$M_{im}=\sum_{j=1}^{q}\sum_{e\in E_{ij}} r_e^d\sum_{k=1}^{p}\alpha_{emk}M_{jk}+R(i,m)=\sum_{k=1}^{p}\sum_{j=1}^{q}\gamma_{ijmk}M_{jk}+R(i,m),$$

where $\gamma_{ijmk}=\sum_{e\in E_{ij}} r_e^d\alpha_{emk}$. Put

$$\mathbf{M}=(M_{11},M_{12},\dots,M_{1p},M_{21},M_{22},\dots,M_{qp}),$$

$$\mathbf{R}(|\mathbf{m}|)=(R_{11},R_{12},\dots,R_{1p},R_{21},R_{22}\dots,R_{qp}),$$

where $R_{kl} = R(k,l)$, and

$$\Gamma_{|\mathbf{m}|} = \begin{bmatrix} \gamma_{1111} & \gamma_{1112} & \cdots & \gamma_{111p} & \gamma_{1211} & \gamma_{1212} & \cdots & \gamma_{1q1p} \\ \gamma_{1121} & \gamma_{1122} & \cdots & \cdots & \cdots & \cdots & \cdots & \gamma_{1q2p} \\ \vdots & \vdots & \ddots & & & & & \vdots \\ \gamma_{11p1} & \gamma_{11p2} & \cdots & \cdots & \gamma_{11pp} & & & \gamma_{1qpp} \\ \gamma_{2111} & \gamma_{2112} & \cdots & \cdots & \gamma_{211p} & \gamma_{2211} & \cdots & \gamma_{2q1p} \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & \vdots \\ \vdots & & & & & & \ddots & \vdots \\ \gamma_{q1p1} & \gamma_{q1p2} & \cdots & \gamma_{q1pp} & \gamma_{q2p1} & \cdots & \cdots & \gamma_{qqpp} \end{bmatrix}.$$

Note that $\operatorname{diag}(\Gamma_{|\mathbf{m}|}) = (\gamma_{1111}, \gamma_{1122}, \ldots, \gamma_{11pp}, \gamma_{2211}, \ldots, \gamma_{qqpp})$. We are then left to solve the equation system

$$(I - \Gamma_{|\mathbf{m}|})\mathbf{M} = \mathbf{R}(|\mathbf{m}|),$$

where I is the identity matrix. The vector $\mathbf{R}(|\mathbf{m}|)$ is known by assumption, so we need to show that $(I - \Gamma_{|\mathbf{m}|})$ is non-singular, which we will do using Lemma 4.6. If $x_i = H^d(K_i)$, we have that

$$x_i \sum_{e \in E_{ii}} r_e^d = \sum_{e \in E_{ii}} r_e^d x_i \le \sum_{j=1}^{q} \sum_{e \in E_{ij}} r_e^d x_j = x_i,$$

and since $x_i > 0$ we have that $\sum_{e \in E_{ii}} r_e^d \le 1$. First we show that the diagonal elements of $(I - \Gamma_{|\mathbf{m}|})$ are greater then 0. Using that $\|A_e\| < 1$ and $|\mathbf{m}| \ge 1$, we have that

$$\begin{aligned} |\gamma_{iimm}| = \left| \sum_{e \in E_{ii}} r_e^d \alpha_{emm} \right| &\le \sum_{e \in E_{ii}} r_e^d |\alpha_{emm}| \\ &\le \sum_{e \in E_{ii}} r_e^d \|A_e\|^{|\mathbf{m}|} < \sum_{e \in E_{ii}} r_e^d \le 1, \end{aligned}$$

which proves that the diagonal elements $(1 - \gamma_{iimm}) > 0$. Next we investigate the second condition in Lemma 4.6. We need to show that

$$\sum_{(j,k) \ne (i,m)} |-\gamma_{ijmk}| < (1 - \gamma_{iimm}).$$

We get, by (14), that

$$\begin{aligned}\gamma_{iimm} + \sum_{(j,k)\neq(i,m)} |-\gamma_{ijmk}| &\leq \sum_{k=1}^{p}\sum_{j=1}^{q} |\sum_{e\in E_{ij}} r_e^d \alpha_{emk}| \\ &\leq \sum_{j=1}^{q}\sum_{e\in E_{ij}} r_e^d \Big(\sum_{k=1}^{p} |\alpha_{emk}|\Big) \leq \sum_{j=1}^{q}\sum_{e\in E_{ij}} r_e^d \|A_e\|^{|\mathbf{m}|} \\ &\leq \sum_{j=1}^{q}\sum_{e\in E_{ij}} r_e^d \|A_e\| \leq \max_{e\in E}\|A_e\| \sum_{j=1}^{q}\sum_{e\in E_{ij}} r_e^d \\ &\leq \|A\| \max_{e\in E}\|A_e\| < 1.\end{aligned}$$

So by Lemma 4.6, $(I - \Gamma_{|\mathbf{m}|})$ is non-singular and the proof is complete. □

5 Besov Spaces.

The Besov spaces $B_\alpha^{p,q}(F)$ by Jonsson and Wallin, are defined on d-sets $F \subseteq \mathbb{R}^n$, see [17] for a thorough treatment.

A net of mesh r is a subdivision of $\mathbb{R}^n$ into equally sized half open cubes Q with side length r; i.e., cubes of the form $Q = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i < a_i + r\}$. Let $\mathcal{N}_\nu$ be the net with mesh $2^{-\nu}$ with one cube in the net having a corner at the origin and define $\mathcal{N}_\nu(F) = \{Q \in \mathcal{N}_\nu : Q \cap F \neq \emptyset\}$. Suppose μ is a d-measure on $F \subseteq \mathbb{R}^n$, with F preserving Markov's inequality, $1 \leq p, q \leq \infty$, $\alpha > 0$, and $[\alpha]$ denotes the integer part of α. For $Q \in \mathcal{N}_\nu(F)$ let $P_Q(f)$ be the orthogonal projection of $L^1(\mu, 2Q)$ onto the subspace $\mathcal{P}_{[\alpha]}$ of $L^2(\mu, 2Q)$, that is $P_Q(f) = \sum_{|j|\leq[\alpha]} \pi_j \int_{2Q} f\pi_j \,\mathrm{d}\mu$, where $\{\pi_j\}_j$ is an orthonormal basis in the subspace $\mathcal{P}_{[\alpha]}$ of $L^2(\mu, 2Q)$. Here $2Q$ denotes the cube with the same center as Q but with sides two times that of Q.

Definition 5.1. Let ν_0 be an integer and suppose $f : F \to \mathbb{R}$ is given. Define the sequence $\{A_\nu\}_{\nu=\nu_0}^\infty$ by

$$\Big(\sum_{Q\in\mathcal{N}_\nu(F)} \int_{2Q} |f - P_Q(f)|^p \,\mathrm{d}\mu\Big)^{1/p} = 2^{-\nu\alpha} A_\nu. \tag{15}$$

Then a function $f \in L^p(\mu)$ belongs to $B_\alpha^{p,q}(F)$ if

$$\|f\|_{B_\alpha^{p,q}(F)} := \|f\|_p + \Big(\sum_{\nu\geq\nu_0} A_\nu^q\Big)^{1/q} < \infty. \tag{16}$$

If p or q equals infinity, we interpret the expressions in Definition 5.1 in the natural limiting way.

Let $J_\nu = \{e \in \mathcal{E} : 2^{-\nu} \leq \operatorname{diam} K_e < 2^{-\nu+1}\}$ and let ν_1 be an integer. Define $\|\{\beta_e^\sigma\}\|$ for a sequence $\{\beta_e^\sigma\}_{e\in J_\nu,\nu\geq\nu_1}$ by

$$\|\{\beta_e^\sigma\}\| = \Big(\sum_{\nu\geq\nu_1}\Big(2^{\nu\alpha p}2^{\nu d(p/2-1)}\sum_{e\in J_\nu}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|^p\Big)^{q/p}\Big)^{1/q}. \tag{17}$$

Theorem 5.2. *Let $1 \leq p, q \leq \infty$, $\alpha > 0$, $f \in B_\alpha^{p,q}(K)$, $m \geq [\alpha]$ and f has the representation (9). Then*

$$\Big(\sum_{i\in V}\sum_{l=1}^{D_0}|\alpha_l^i|^p\Big)^{1/p} + \|\{\beta_e^\sigma\}\| \leq c\|f\|_{B_\alpha^{p,q}(K)} \tag{18}$$

where c does not depend on f or the wavelet basis.

Remark. Note that the m in Theorem 5.2 refers to $\mathcal{P}_m$ in the wavelet construction in Section 4.

Lemma 5.3. *If $Q \in \mathcal{N}_{\nu-2}$ there exists a constant $c_1 > 0$, independent of ν and Q, such that there are at most c_1 of the $e \in J_\nu$ with $K_e \cap Q \neq \emptyset$.*

PROOF. Let $r_0 = \max_{e\in E} r_e$ and define $I_Q = \{e \in J_\nu : K_e \cap Q \neq \emptyset\}$. To each $e \in I_Q$ define e^* to be the shortest path $e^* \in I_Q$ with $K_e \subseteq K_{e^*}$ and let $M = \{e^* : e \in I_Q\}$. Then $\{K_e\}_{e\in M}$ is an collection of pairwise essentially disjoint sets. If $e \in I_Q$ then $K_e \subseteq 2Q$, and since μ is a doubling measure, we have that

$$\sum_{e\in M}\mu(K_e) \leq \mu(2Q) \leq \mu(B(x, 8\sqrt{n}2^{-\nu})) \leq c_2 2^{-\nu d}$$

for any $x \in Q \cap K$. By (5) it follows that there is a constant c_3, such that $\mu(K_e) \geq c_3 2^{-\nu d}$ so the number of elements in M is bounded by c_2/c_3. If $e \in M$ and there is $e_1 \in I_Q$ such that $K_{e_1} \subseteq K_e$, then $e_1 = e\tilde{e}$ for some path $\tilde{e}$. Since $2^{-\nu} \leq \operatorname{diam} K_{e_1} \leq r_0^{|\tilde{e}|}\operatorname{diam} K_e \leq r_0^{|\tilde{e}|}2^{-\nu+1}$, there is a constant k such that $|\tilde{e}| \leq k$. Therefore the number of elements in I_Q is less then $c_1 = c_4c_2/c_3$ if c_4 is the number of elements in $\mathcal{E}^k$. □

PROOF OF THEOREM 5.2. We give the proof for $1 \leq p, q < \infty$, since only minor modifications are needed for the other cases. Let c denote constant that can differ from line to line. Let ν_1 be an integer such that $\max_i \operatorname{diam} K_i < 2^{-\nu_1+1}$. To each $e \in J_\nu$, $\nu \geq \nu_1$, choose exactly one $Q_e \in \mathcal{N}_{\nu-2}$ such that

$K_e \cap Q_e \neq \emptyset$, and let $P_{Q_e} = P_{Q_e}(f)$. Then $K_e \subseteq 2Q_e$ and since ψ_e^σ is orthogonal to P_{Q_e} whenever $m \geq [\alpha]$, we have that

$$\begin{aligned}|\beta_e^\sigma| &= |\int f\psi_e^\sigma \,\mathrm{d}\mu| = |\int (f - P_{Q_e})\psi_e^\sigma \,\mathrm{d}\mu| \\ &\leq \Big(\int_{K_e} |f - P_{Q_e}|^p \,\mathrm{d}\mu\Big)^{1/p}\Big(\int |\psi_e^\sigma|^{p'} \,\mathrm{d}\mu\Big)^{1/p'} \\ &\leq c\Big(\int_{2Q_e} |f - P_{Q_e}|^p \,\mathrm{d}\mu\Big)^{1/p}\mu(K_e)^{(1/p'-1/2)}.\end{aligned}$$

Then, by the remark after Lemma 4.2, we have that

$$|\beta_e^\sigma| \leq c2^{-\nu d(1/p'-1/2)}\Big(\int_{2Q_e} |f - P_{Q_e}|^p \,\mathrm{d}\mu\Big)^{1/p}$$

The right side is independent of σ, and p' is the dual index to p so it follows that

$$\sum_{\sigma=1}^{D_e} |\beta_e^\sigma|^p \leq c2^{-\nu d(p/2-1)} \int_{2Q_e} |f - P_{Q_e}|^p \,\mathrm{d}\mu.$$

By Lemma 5.3 a cube $Q \in \mathcal{N}_{\nu-2}(K)$ can intersect only a finite number c_1 of the K_e for $e \in J_\nu$, where c_1 is independent of ν and Q. By this we get that

$$\begin{aligned}\sum_{e\in J_\nu}\sum_{\sigma=1}^{D_e} |\beta_e^\sigma|^p &\leq c2^{-\nu d(p/2-1)} \sum_{e\in J_\nu}\int_{2Q_e} |f - P_{Q_e}|^p \,\mathrm{d}\mu \\ &\leq cc_1 2^{-\nu d(p/2-1)} \sum_{Q\in\mathcal{N}_{\nu-2}(K)} \int_{2Q} |f - P_Q|^p \,\mathrm{d}\mu \\ &\leq c2^{-\nu d(p/2-1)}2^{-(\nu-2)\alpha p}A_{\nu-2}^p\end{aligned}$$

where A_ν is given by (15). Then it follows that

$$\|\{\beta_e^\sigma\}\| = \Big(\sum_{\nu\geq\nu_1}\Big(2^{\nu\alpha p}2^{\nu d(p/2-1)}\sum_{e\in J_\nu}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|^p\Big)^{q/p}\Big)^{1/q} \leq c\Big(\sum_{\nu\geq\nu_1-2} A_\nu^q\Big)^{1/q}.$$

It is clear that $|\alpha_l^i| \leq c\|f\|_p$ and then (18) follows if we let $\nu_0 = \nu_1 - 2$ and $F = K$ in Definition 5.1. □

Next we will prove a partial converse of Theorem 5.2. We can not expect a complete converse to be true, since the functions in the wavelet basis do not need to be in $B_\alpha^{p,q}(K)$, see [15].

Theorem 5.4. *Let $\alpha > 0$ and $1 \le p, q \le \infty$. If the sets $\{K_e\}_{e\in E}$ are pairwise disjoint and $f \in L^1(\mu)$, then*

$$\|f\|_{B_\alpha^{p,q}(K)} \le c\Big(\Big(\sum_{i\in V}\sum_{l=1}^{D_0}|\alpha_l^i|^p\Big)^{1/p} + \|\{\beta_e^\sigma\}\|\Big), \tag{19}$$

where c does not depend on f or the wavelet basis.

We will prove this using a characterization of $B_\alpha^{p,q}(K)$ using atoms; see e.g. [15], and [5] (see [12] for details). We write π_ν instead of $\mathcal{N}_\nu$ when we consider the elements as closed cubes. Suppose that F is a d-set with d-measure μ, $\alpha > 0$ and $1 \le p, q \le \infty$. Let k be the integer such that $k < \alpha \le k+1$. A function $a \in C^k(\mathbb{R}^n)$ is an (α, p)-atom if there exist a closed cube Q in $\mathbb{R}^n$ with $\operatorname{supp}(a) \subset 3Q$ and that

$$|D^j a(x)| \le s(Q)^{\alpha-|j|-d/p},\ x \in \mathbb{R}^n,\ |j| \le k,$$

where $s(Q)$ denotes the side length of Q. We write a_Q for an atom associated to Q.

Definition 5.5. Let ν_0 be an integer. Then $f \in B_\alpha^{p,q}(F)$ if there are (α, p)-atoms a_Q and $s_Q \in \mathbb{R}$ such that

$$f = \sum_{\nu=\nu_0}^{\infty}\sum_{Q\in\pi_\nu} s_Q a_Q, \tag{20}$$

with convergence in $L^p(\mu)$ and that

$$\Big(\sum_{\nu=\nu_0}^{\infty}\Big(\sum_{Q\in\pi_\nu}|s_Q|^p\Big)^{q/p}\Big)^{1/q} < \infty. \tag{21}$$

The *norm* of f is the infimum of (21) taken over all possible representations of f on the form in (20).

Suppose $g \in C^\infty(\mathbb{R}^n)$, $e = e_1e_2\ldots e_k \in \mathcal{E}_{ij}^k$, and $g_e(x) = g \circ T_e^{-1}(x)$. Let $x^* = T_e^{-1}(x)$ so that $g_e(x) = g(x^*)$ and let $\mathbf{u} \in \mathbb{R}^n$ be a unit vector. Then for some unit vector $\mathbf{v} \in \mathbb{R}^n$, we have that

$$\begin{aligned}(D_{\mathbf{u}}g_e)(x) &= \lim_{h\to0}\frac{g_e(x+h\mathbf{u})-g_e(x)}{h} = \lim_{h\to0}\frac{g((x+h\mathbf{u})^*)-g(x^*)}{h}\\ &= \lim_{h\to0}\frac{g_e(x^*+r_{e_1}^{-1}r_{e_2}^{-1}\cdot\ldots\cdot r_{e_k}^{-1}h\mathbf{v})-g(x^*)}{h}\\ &= r_{e_1}^{-1}r_{e_2}^{-1}\cdot\ldots\cdot r_{e_k}^{-1}D_{\mathbf{v}}g(x^*) = r_{e_1}^{-1}r_{e_2}^{-1}\cdot\ldots\cdot r_{e_k}^{-1}(D_{\mathbf{v}}g)_e(x)\\ &= \frac{\operatorname{diam}K_j}{\operatorname{diam}K_e}(D_{\mathbf{v}}g)_e(x)\end{aligned} \tag{22}$$

Let $\mathbf{l} = (l_1, \ldots, l_n)$ be a multi-index. Then, by iterating (22), there is a sequence of unit vectors $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_{|\mathbf{l}|}$ such that

$$\begin{aligned}(D^{\mathbf{l}} g_e)(x) &= \Big(\frac{\operatorname{diam} K_j}{\operatorname{diam} K_e}\Big)^{|\mathbf{l}|} (D_{\mathbf{v}_1} D_{\mathbf{v}_2} \ldots D_{\mathbf{v}_{|\mathbf{l}|}} g)_e(x) \\ &= \Big(\frac{\operatorname{diam} K_j}{\operatorname{diam} K_e}\Big)^{|\mathbf{l}|} D_{\mathbf{v}_1} D_{\mathbf{v}_2} \ldots D_{\mathbf{v}_{|\mathbf{l}|}} g(x^*)\end{aligned}$$

We then get that

$$|(D^{\mathbf{l}} g_e)(x)| \leq c(\operatorname{diam} K_e)^{-|\mathbf{l}|} \max_{|\mathbf{m}|=|\mathbf{l}|} |D^{\mathbf{m}} g(x^*)| \tag{23}$$

Let $d(A,B) = \inf\{|x-y| : x \in A, x \in B\}$ be the distance between two sets.

Lemma 5.6. *Suppose that the sets $\{K_e\}_{e\in E}$ are pairwise disjoint and that δ is the minimum distance between any two of these sets. If $e = e_1 e_2 \ldots e_{k+1} \in \mathcal{E}^{k+1}$, where $k \geq 1$, then*

$$d(K_e, K \setminus K_e) \geq r_{e_1} r_{e_2} \cdot \ldots \cdot r_{e_k} \delta.$$

PROOF. Since $K = \cup_{\tilde{e} \in \mathcal{E}^{k+1}} K_{\tilde{e}}$, with a disjoint union, we have that $d(K_e, K \setminus K_e) = \min\{d(K_e, K_{\tilde{e}}) : \tilde{e} \in \mathcal{E}^{k+1}, \tilde{e} \neq e\}$. Suppose that $\tilde{e} = \tilde{e}_1 \tilde{e}_2 \ldots \tilde{e}_{k+1} \in \mathcal{E}^{k+1}$ and let l be the smallest integer such that $e_{l+1} \neq \tilde{e}_{l+1}$ so that $\tilde{e} = e_1 e_2 \ldots e_l \tilde{e}_{l+1} \ldots \tilde{e}_{k+1}$. If $l = 1$, then $d(K_e, K_{\tilde{e}}) \geq \delta$. If $l > 1$ we have that

$$\begin{aligned}d(K_e, K_{\tilde{e}}) &= d(T_{e_1 e_2 \ldots e_l}(K_{e_{l+1} \ldots e_{k+1}}), T_{e_1 e_2 \ldots e_l}(K_{\tilde{e}_{l+1} \ldots \tilde{e}_{k+1}})) \\ &\geq r_{e_1} r_{e_2} \cdot \ldots \cdot r_{e_l} d(K_{e_{l+1}}, K_{\tilde{e}_{l+1}}) \geq r_{e_1} r_{e_2} \cdot \ldots \cdot r_{e_l} \delta\end{aligned}$$

□

Assume that the sets $\{K_e\}_{e\in E}$ are pairwise disjoint and let δ be as in Lemma 5.6. If $P \in S_1^j$, meaning that P is a polynomial of degree $\leq m$ on each K_e with $e \in E_j$. We let P be defined for $x \in \mathbb{R}^n$ with $d(x, K_j) < \delta/2$ by extending P to such x by letting P coincide with the polynomial defined by P on K_e whenever $d(x, K_e) < \delta/2$, and $P(x) = 0$ if $d(x, K_j) \geq \delta/2$. For $e = e_1 e_2 \ldots e_{k+1} \in \mathcal{E}_{ij}$, let $P_e(x) = P \circ T_e^{-1}(x)$, so that $P_e(x) = 0$ if $d(x, K_e) \geq r_{e_1} \cdot \ldots \cdot r_{e_k} \delta/2$. Now, choose $\Phi^j \in C_0^\infty(\mathbb{R}^n)$ such that $\Phi^j(x) \equiv 1$ on K_j and $\Phi^j(x) \equiv 0$ if $d(x, K_j) \geq \delta/2$ and define $\Phi_e(x) = \Phi^j \circ T_e^{-1}(x)$ and $\Phi_j = \Phi^j$ for $j \in V$. Then $\Phi_e(x) P_e(x) = (\Phi^j P)_e(x) \in C_0^\infty(\mathbb{R}^n)$ and $(\Phi^j P)_e \neq 0$ only if $d(x, K_e) \leq (\delta/2) r_{e_1} \cdot \ldots \cdot r_{e_k} = (\delta \operatorname{diam} K_e)/(2 \operatorname{diam} K_j)$. Then, since $\delta \leq \operatorname{diam} K_j$, we have that $\operatorname{diam}(\operatorname{supp}(\Phi_e)) \leq 2 \operatorname{diam}(K_e)$.

Lemma 5.7. *If* $\mathbf{l}$ *is a multi-index,* $e \in \mathcal{E}$ *and* $j \in V$*, there exists a constant* $c > 0$ *such that*

$$\|D^{\mathbf{l}}(\Phi_e \psi_e^\sigma)\|_\infty \leq c(\mathrm{diam} K_e)^{-|\mathbf{l}|}(\mu(K_e))^{-1/2} \tag{24}$$

and

$$\|D^{\mathbf{l}}(\Phi^j \phi_l^j)\|_\infty \leq c(\mathrm{diam} K_j)^{-|\mathbf{l}|}(\mu(K_j))^{-1/2}, \tag{25}$$

where c *depends on* K_j*,* Φ^j*,* n*,* m *and* $\mathbf{l}$.

Remark. We can of course replace the right side of (25) with just a constant c, but choose to express us this way in order to simplify the notation later in this section.

PROOF OF LEMMA 5.7. If $P \in S_1^j$, $d(x, K_e) < \delta/2$, $x_0 \in K_e$ and $x \in B = B(x_0, \delta/2)$ we have, by the remark after Definition 3.1, that

$$\begin{aligned} |D^{\mathbf{l}}P(x)| &\leq \max_{x \in B} |D^{\mathbf{l}}P(x)| \leq c \max_{x \in K_j \cap B} |D^{\mathbf{l}}P(x)| \\ &\leq c \max_{x \in K_j \cap B} |P(x)| = c\|P\|_{\infty, K_e} \end{aligned}$$

Therefore, by (8),

$$|D^{\mathbf{l}}\psi^{j,\sigma}(x)| \leq c\|\psi^{j,\sigma}\|_{\infty,K_e} \leq c\|\psi^{j,\sigma}\|_{2,K_e} \leq c\|\psi^{j,\sigma}\|_{2,K_j} = c, \tag{26}$$

which gives us (24) for $e \in V$. Similarly we have that $|D^{\mathbf{l}}\phi_l^j(x)| \leq c$ which implies (25). Inequality (24) follows from (23) and (26), since

$$D^{\mathbf{l}}(\Phi_e \psi_e^\sigma)(x) = D^{\mathbf{l}}(\Phi^j \psi^{j,\sigma})_e(x) \Big(\frac{\mu(K_e)}{\mu(K_j)}\Big)^{-1/2},$$

for $e \in \mathcal{E}_{ij}$. □

PROOF OF THEOREM 5.4. Assume that the right side of (19) is finite and let ν_1 be an integer such that $\max_i \mathrm{diam}\, K_i < 2^{-\nu_1+1}$. To each K_e, $e \in J_\nu$ and $\nu \geq \nu_1$, we associate exactly one $Q_e \in \pi_{\nu-2}$ with $Q_e \cap K_e \neq \emptyset$. For $Q \in \pi_{\nu-2}$, we define $I_Q = \{e \in J_\nu : Q \text{ is associated to } K_e\}$. Lemma 5.3 holds if we replace $\mathcal{N}_{\nu-2}$ with $\pi_{\nu-2}$, so there is a constant c_1 not depending on ν or Q such that I_Q contains no more then c_1 elements. Define the partial sum f_N as

$$f_N = \sum_{i \in V} \sum_{l=1}^{D_0} \alpha_l^i \phi_l^i + \sum_{\nu=\nu_1}^{N-1} \sum_{e \in J_\nu} \sum_{\sigma=1}^{D_e} \beta_e^\sigma \psi_e^\sigma.$$

Combining Lemma 5.3 with the fact that, for a fixed k, the functions ψ_e^σ have disjoint support for different $e \in \mathcal{E}^k$, and using the inequality

$$(\sum_{m=1}^{n} x_m)^p \leq n^{p-1} \sum_{m=1}^{n} x_m^p, \tag{27}$$

we get that

$$\begin{aligned} |\sum_{e\in J_\nu} \sum_{\sigma=1}^{D_e} \beta_e^\sigma \psi_e^\sigma(x)|^p &\leq \Big(\sum_{e\in J_\nu} \sum_{\sigma=1}^{D_e} |\beta_e^\sigma||\psi_e^\sigma(x)| \Big)^p \\ &\leq c_1^{p-1} D^{p-1} \sum_{e\in J_\nu} \sum_{\sigma=1}^{D_e} |\beta_e^\sigma|^p |\psi_e^\sigma(x)|^p, \end{aligned}$$

where $D = \max_{e\in E} D_e$. By the remark after Lemma 4.2, $\|\psi_e^\sigma\|_p^p \leq 2^{vd(p/2-1)}$. If q' is the dual index to q and $M > N > 1$, we have that

$$\begin{aligned} \|f_M - f_N\|_p &= \|\sum_{\nu=N}^{M-1} \sum_{e\in J_\nu} \sum_{\sigma=1}^{D_e} \beta_e^\sigma \psi_e^\sigma \|_p \leq \sum_{\nu=N}^{M-1} \Big(\int_K |\sum_{e\in J_\nu} \sum_{\sigma=1}^{D_e} \beta_e^\sigma \psi_e^\sigma(x)|^p \, \mathrm{d}\mu \Big)^{1/p} \\ &\leq c \sum_{\nu=N}^{M-1} \Big(\sum_{e\in J_\nu} \sum_{\sigma=1}^{D_e} |\beta_e^\sigma|^p \|\psi_e^\sigma\|_p^p \Big)^{1/p} \\ &\leq c \sum_{\nu=N}^{M-1} 2^{-v\alpha} \Big(2^{\nu d(p/2-1)} 2^{\nu\alpha p} \sum_{e\in J_\nu} \sum_{\sigma=1}^{D_e} |\beta_e^\sigma|^p \Big)^{1/p} \\ &\leq c \Big(\sum_{\nu=N}^{M-1} 2^{-\nu\alpha q'} \Big)^{1/q'} \Big(\sum_{\nu=N}^{M-1} \Big(2^{\nu d(p/2-1)} 2^{\nu\alpha p} \sum_{e\in J_\nu} \sum_{\sigma=1}^{D_e} |\beta_e^\sigma|^p \Big)^{q/p} \Big)^{1/q} \\ &\leq c 2^{-\alpha N} \|\{\beta_e^\sigma\}\| \end{aligned}$$

Thus $\{f_N\}$ is a Cauchy sequence in $L^p(\mu)$, which implies that the wavelet series (9) of f converges to f in $L^p(\mu)$. Therefore, by defining ψ_e^σ, ϕ^j and Φ^j as discussed earlier, we can represent f as

$$f = \sum_{i\in V} \sum_{l=1}^{D_0} \alpha_l^i \Phi^i \phi_l^i + \sum_{k=0}^{\infty} \sum_{e\in\mathcal{E}^k} \sum_{\sigma=1}^{D_e} \beta_e^\sigma \Phi_e \psi_e^\sigma.$$

For $Q \in \pi_\nu$ define

$$f_Q = \sum_{i\in V} \Big[\sum_{l=1}^{D_0} \alpha_l^i \Phi^i \phi_l^i \Big]_Q + \sum_{e\in I_Q} \sum_{\sigma=1}^{D_e} \beta_e^\sigma \Phi_e \psi_e^\sigma,$$

where $[\cdot]_Q$ means that it is present only if Q is associated to K_i, and put $f_Q = 0$ if $I_Q = \emptyset$.

Let $k_Q = \max_{|\mathbf{l}|\le[\alpha]+1} 2^{-(\nu-2)|\mathbf{l}|}\|D^{\mathbf{l}} f_Q\|_\infty$ and

$$a_Q = \begin{cases} f_Q 2^{-(\nu-2)(\alpha-d/p)}/k_Q & \text{if } k_Q \neq 0 \\ 0 & \text{if } k_Q = 0. \end{cases}$$

Then a_Q is an (α,p)-atom with $k = [\alpha]+1$. If we let $s_Q = k_Q 2^{(\nu-2)(\alpha-d/p)}$ we have that $f_Q = s_Q a_Q$ and that $f = \sum_{\nu=\nu_1}^{\infty}\sum_{Q\in\pi_{\nu-2}} s_Q a_Q$. By Lemma 5.7, we get that

$$\begin{aligned} |D^{\mathbf{l}} f_Q(x)| &\le \sum_{i\in V}\Big[\sum_{l=1}^{D_0}|\alpha_l^i||D^{\mathbf{l}}(\Phi^i\phi_l^i)|\Big]_Q + \sum_{e\in I_Q}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|\|D^{\mathbf{l}}\Phi_e\psi_e^\sigma| \\ &\le c\sum_{i\in V}\Big[\sum_{l=1}^{D_0}|\alpha_l^i|(\operatorname{diam} K_i)^{-|\mathbf{l}|}\mu(K_i)^{-1/2}\Big]_Q \\ &\quad + c\sum_{e\in I_Q}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|(\operatorname{diam} K_e)^{-|\mathbf{l}|}\mu(K_e)^{-1/2} \\ &\le c\Big(\sum_{i\in V}\Big[\sum_{l=1}^{D_0}|\alpha_l^i|2^{\nu|\mathbf{l}|}\Big]_Q + \sum_{e\in I_Q}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|2^{\nu|\mathbf{l}|}2^{\nu d/2}\Big) \end{aligned}$$

Since the number of elements in I_Q is bounded by a constant independent of Q and ν, we can use (27) and get that

$$\|D^{\mathbf{l}} f_Q\|_\infty \le c2^{\nu|\mathbf{l}|}\Big(\sum_{i\in V}\Big[\sum_{l=1}^{D_0}|\alpha_l^i|^p\Big]_Q + 2^{\nu dp/2}\sum_{e\in I_Q}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|^p\Big)^{1/p}.$$

By this we can estimate k_Q, using (27), with

$$k_Q \le c\Big(\sum_{i\in V}\Big[\sum_{l=1}^{D_0}|\alpha_l^i|^p\Big]_Q + 2^{\nu dp/2}\sum_{e\in I_Q}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|^p\Big)^{1/p},$$

so that

$$\begin{aligned}\sum_{Q\in\pi_{\nu-2}} |s_Q|^p &= \sum_{Q\in\pi_{\nu-2}} k_Q^p 2^{(\nu-2)(\alpha p-d)}\\ &\le c\Big(\sum_{Q\in\pi_{\nu-2}}\sum_{i\in V}\Big[\sum_{l=1}^{D_0}|\alpha_l^i|^p\Big]_Q + \sum_{Q\in\pi_{\nu-2}} 2^{\nu\alpha p}2^{\nu d(p/2-1)}\sum_{e\in I_Q}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|^p\Big)\\ &= c\Big(\sum_{Q\in\pi_{\nu-2}}\sum_{i\in V}\Big[\sum_{l=1}^{D_0}|\alpha_l^i|^p\Big]_Q + 2^{\nu\alpha p}2^{\nu d(p/2-1)}\sum_{e\in J_\nu}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|^p\Big).\end{aligned}$$

Using (27) or that $(a+b)^r \le a^r + b^r$ for $0 < r \le 1$, we get that

$$\begin{aligned}\Big(\sum_{Q\in\pi_{\nu-2}} |s_Q|^p\Big)^{q/p} \le c \sum_{Q\in\pi_{\nu-2}} &\Big(\sum_{i\in V}\Big[\sum_{l=1}^{D_0}|\alpha_l^i|^p\Big]_Q\Big)^{q/p}\\ &+ c\Big(2^{\nu\alpha p}2^{\nu d(p/2-1)}\sum_{e\in J_\nu}\sum_{\sigma=1}^{D_e}|\beta_e^\sigma|^p\Big)^{q/p}.\end{aligned}$$

Hence, we get that

$$\sum_{\nu=\nu_1}^{\infty}\Big(\sum_{Q\in\pi_{\nu-2}} |s_Q|^p\Big)^{q/p} \le c\Big(\Big(\sum_{i\in V}\sum_{l=1}^{D_0}|\alpha_l^i|^p\Big)^{q/p} + \|\{\beta_e^\sigma\}\|^q\Big)$$

and (19) follows. □

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