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USING DETERMINACY TO INSCRIBE COMPACT NON-σ-POROUS SETS INTO NON-σ-POROUS PROJECTIVE SETS

Abstract

Under the assumption of projective determinacy, we show that one can obtain non-σ-porous compact subsets for given projective non-σ-porous sets, with respect to the regular and strong porosities for dimension zero compact metric spaces.

1 Introduction.

In Zajíček's 1987 survey on porosity and σ-porosity, the following question was posted (Zajíček [3], 4.20).

(Q) Let B be a Borel non-σ-porous subset of a metric space X. Does there exists a closed non-σ-porous set $F \subset B$?

This question was answered positively by J. Pelant for complete metric spaces, and (independently) by M. Zelený for compact metric spaces. They wrote a joint paper (see Zelený-Pelant [7]) presenting a more accessible version of Pelant's proof, which in fact, proves the result for analytic sets. Later, Zelený and Zajíček [6] present Zelený's non-constructive proof, in this paper they make use of some concepts (introduced in Zajíček-Zelený [4]) which allows the theorem to be proved not just for regular porosity, but for other types of porosities, though, not including strong porosity. In the words of Zajíček, for the strong porosity "the question remains open" (Zajíček [6]). In the case of strong porosity I do not know of any other partial answer (to this question) than the one presented here.

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In this paper we present a different, direct method for proving Zelený's and Pelant's result for dimension zero compact metric spaces (Proposition 3.1). The method presented is based on a determinacy argument. It is because of the nature of this argument that the proofs can be extended to obtain compact non-σ-porous subsets for any projective non-σ-porous set (in dimension zero compact metric spaces), under the assumption of projective determinacy of course (Theorem 3.6). Also, by making subtle changes to the proofs of these results (stated for the regular porosity) one can obtain the same type of results, but for the strong porosity (Theorem 4.5). This way we give a positive partial answer to the question just mentioned above.

2 Definitions and Lemmas.

We start this section by introducing the notions of *metric porosity* (This is the regular porosity mentioned in the introduction.) and the more general notion of *abstract porosity.* Then we give a characterization for compact zero dimensional metric spaces. And finally, we will define a particular abstract porosity called *det-porosity* and prove that the *metric* and *det*-porosities are equivalent.

Definition 2.1. Given a metric space (X, d), $A \subset X$, $x \in X$ and $\delta > 0$. Let $\Lambda(A, x, \delta) = \sup\{r > 0 : (\exists B_r(z) \subset B_\delta(x))(B_r(z) \cap A = \emptyset)\}$, and let

$$p(x, A) = \limsup_{\delta \to 0^+} \frac{\Lambda(A, x, \delta)}{\delta}$$

where $B_r(z)$ denotes the open ball with center z and radius r.

Say that A is *porous at* x, if $p(x, A) > 0$. The set A is said to be *porous* if it is porous at x for every $x \in A$. The ideal I_m given by *metric porosity* is the ideal σ-generated by the porous subsets of X. Finally, we say that a set $B \subset X$ is *σ-porous* if $B \in I_m$.

Farah and Zapletal [1] generalized (for Polish spaces) the notion of porosity to that of "abstract porosity." Given a Polish space X, an *abstract porosity* is an order preserving map $\text{por}_U : \mathcal{P}(U) \to \mathcal{B}(X)$ $(a \subset b \Rightarrow \text{por}_V(a) \subset \text{por}_V(b))$, where $\mathcal{P}(U)$ is the power set of U, $\mathcal{B}(X)$ is the set of all Borel subsets of X, and U is a countable collection of $\mathcal{B}(X)$.

If por_U is an abstract porosity, the U-porous sets are the sets A of the form $A \subset \text{por}_V(a) \setminus \bigcup a$ (for $a \in U$). The *porosity ideal* I_U associated with por_U is the ideal σ-generated by the U-porous sets. As before, the σ-U-porous sets are the elements of I_U. As Farah and Zapletal pointed out, it is easy to see that if $\langle X, d\rangle$ is a Polish space, then the metric porosity is an abstract porosity.

From the following set of propositions we will be giving a characterization of the compact metric spaces of *dimension zero*. This characterization will be used in order to define a particular abstract porosity needed to establish the results. The first of the propositions states a property that holds for any compact metric space, it is given after the following basic definition.

Definition 2.2. Let (X, d) be a metric space, let $\epsilon > 0$ and let $x, y \in X$. An $\epsilon(x, y)$-*path* is a finite subset of X of the form $\{x = x_0, x_1, \ldots, x_r = y\}$ such that for each $i \in \{0, \ldots, r-1\}$, $d(x_i, x_{i+1}) < \epsilon$.

Proposition 2.3. *Let (X, d) be a compact metric space. For each $\epsilon > 0$, define the equivalence $\sim_\epsilon$ (in X) by: $x \sim_\epsilon y$ if and only if there is an $\epsilon(x, y)$-path. Then, for each $x \in X$, $[x]_\epsilon$ is clopen, and $X/\sim_\epsilon$ is finite.*

Proof. If $y \in [x]_\epsilon$, then $B_\epsilon(y) \subset [x]_\epsilon$. On the other hand, if y is an accumulation point of $[x]_\epsilon$, take $z \in [x]_\epsilon$ such that $d(z, y) < \epsilon$. Then $y \in [z]_\epsilon = [x]_\epsilon$. So $X/\sim_\epsilon$ is an open covering of X of disjoint sets, but X is compact, therefore $X/\sim_\epsilon$ must be finite. □

Definition 2.4. Let (X, d) be a compact metric space. For each $n \in \omega$ define the set W_n as the set $X/\sim_{1/n}$. That is to say, $W_n = \{[x]_{1/n} : x \in X\}$ where $[x]_{1/n}$ is the equivalence class of x with respect to the relation $\sim_{1/n}$ defined in Proposition 2.3. So, for each $n \in \omega$, each class in W_n is partitioned into finitely many classes of W_{n+1}.

By a topological space of *dimension zero* we mean a topological space (X, τ) that has a clopen basis. This way, a metric space is *dimension zero* if it has dimension zero as a topological space.

Lemma 2.5. *If (X, d) is a compact dimension zero metric space, then for all $x, y \in X$ $(x \neq y)$ there is a positive real ϵ such that there are no $\epsilon(x, y)$-paths.*

Proof. Assume the contrary. For each $n \in \omega$ let $p_n = \{x = x_0^n, \ldots, x_{i_n}^n = y\}$ be a $(1/n)(x, y)$-path. Since X is dimension zero, there is a clopen set A such that $y \in A$ and $x \notin A$. Define $P = (\bigcup_{n \in \omega} p_n) \cap A^c$ and $\hat{P} = \{z \in P : z = x_{j_n}^n \Rightarrow x_{j_n+1}^n \in A\}$. We claim that there must be a point $\hat{z} \in A^c$ which is an accumulation point of $\hat{P}$. Otherwise, for each $z \in A^c$, take O_z open such that $z \in O^z$ and $O_z \cap \hat{P}$ is finite. Since A^c is compact, we can extract a finite sub-covering of A^c, which gives a contradiction. But also, $\hat{z}$ should be an accumulation point of the set $\{x_{j_n+1}^n : x_{j_n}^n \in \hat{P}\} \subset A$ because one can always find $m \in \omega$ large enough such that the pairs $x_{j_m+1}^m, x_{j_m}^m$ and $x_{j_m}^m, \hat{z}$ are close enough. Since A is also closed, we reach the contradiction, $\hat{z} \in A \cap A^c$. □

For each $n \in \omega$, and for W_n as defined in Definition 2.4, let r_n be defined as $r_n = |W_n|$, and let T_i^n (for $i \leq r_n$) denote the i-th element of W_n. (Observe that for each $n \in \omega$, $r_n \leq r_{n+1}$.)

Remark 2.6. If a compact space (X, d) is dimension zero, then for each $x \in X$ there is a unique $\hat{x} \in {}^{\omega}\omega$ such that $x \in T^n_{\hat{x}(n)}$ (for all $n \in \omega$). For each $a \in {}^{\omega}\omega$ there is at most one $x \in X$ such that $\hat{x} = a$. For each $x \in X$ call $\hat{x}$ the *description of* x. Let $\hat{X}$ be the subspace of the Baire space $\mathcal{N}$ formed by these descriptions. Consider the 1-1 map $x \mapsto \hat{x}$ from X onto $\hat{X}$. Observe that the map is an homeomorphism. First, the map is continuous. A basic open ball $\mathcal{O}_t := \{\hat{x} \in \hat{X} : \hat{x} \restriction m = t\}$ (where $t \in {}^{<\omega}\omega$ and $m = \mathrm{dom}(t)$) in $\hat{X}$, comes from $[x]_{1/m}$ (with x such that $\hat{x} \in \mathcal{O}_t$) under this map. Then, since X is compact, the 1-1 continuous map $x \mapsto \hat{x}$ must be an homeomorphism. The existence of this homeomorphism will often be exploited.

The next task is to define a particular ideal I_d (for dimension zero compact metric spaces). This will be the ideal σ-generated by the *det-porous* sets given by the *det-porosity.* To define this abstract porosity for compact dimension zero spaces (X, d), consider the countable basis V of X defined as follows.

For each $k \in \omega$, let T_i^k be the i-th element of W_k (i.e., the i-th $\sim_{1/k}$ class). Let $\{B_{1/k^2}(z_1), \dots B_{1/k^2}(z_s)\}$ be a finite sub-covering of $\{B_{1/k^2}(z) : z \in T_i^k\}$ and let $R_i^k = \{B_{1/k}(z_1), \dots, B_{1/k}(z_s)\}$. This way, each $z \in T_i^k$ is within a distance of $1/k^2$ from one of the centers z_j. Finally, let $V_k = \bigcup\{R_i^k : i \leq r_k\}$ (where $r_k = |W_k|$) and let $V = \bigcup_k V_k$.

Define $\mathrm{por}_V : P(V) \to \mathcal{B}(X)$, the *det-porosity* of X, by $x \in \mathrm{por}_V(a)$ if and only if there is $m \in \omega$, and a decreasing sequence of positive reals $\{\delta_n\}$ converging to 0 such that for all $n \in \omega$ there is a ball t_n in V such that:

1. $t_n \in a$,
2. $\delta_n < m \cdot \mathrm{rd}(t_n)$ ($\mathrm{rd}(t)$ denotes the radius of the ball t),
3. $t_n \subset B_{\delta_n}(x)$.

Let I_d be the σ-ideal given by this type of porosity. We will refer to the V-porous sets related to this type of porosity by the term *det-porous sets.*

The next and last property of this section states the equivalence between the metric porosity and the det-porosity. In the proof of the results included in the next sections we use this equivalence.

Proposition 2.7. *Let (X, d) be a dimension zero compact metric space, let I_m be the ideal given by the metric porosity and let I_d be the one given by the det-porosity. Then, for all $A \subset X$, A is a porous set if and only if A is a det-porous set. Therefore, $I_m = I_d$.*

PROOF. Let A be a porous set. We show that $A \subset \text{por}_V(a) \setminus \bigcup a$, where $a = \{t \in V : t \cap A = \emptyset\}$. Fix $x \in A$, by definition of the metric porosity, there is $m \in \omega$ and a sequence $\{\delta_n\}$ of positive reals that converges to 0 such that $\lim_{\delta_n \to 0} \frac{\Lambda(A,x,\delta_n)}{\delta_n} > 1/m$. For all but finitely many n, there is an open ball $t_n := B_{r_n}(z_n)$ (not necessarily in V), such that t_n is a subset of $B_{\delta_n}(x)$, $t_n \cap A = \emptyset$ and such that $\delta_n/m < \text{rd}(t_n) \leq \Lambda(A, x, \delta_n)$. Re-enumerate the sequence $\{\delta_n, t_n\}$ in order to make sure that for all $n \in \omega$ the properties just mentioned hold.

We show next that there is a sequence of balls $\hat{t}_n \in V$ such that for each n, $\hat{t}_n \subset t_n$ and $\lim_{n\to\infty}(\hat{r}_n/r_n) = 1$ (where $\hat{r}_n = rd(\hat{t}_n)$ and $r_n = rd(t_n)$). For each n, let $k_n \in \omega$ be the least $k \geq 2$ such that $1/(k-1) \leq r_n$. Let $\hat{z}_n \in X$ be such that $d(z_n, \hat{z}_n) \leq 1/k_n^2$ and such that $B_{1/k_n}(\hat{z}_n) \in V_{k_n}$. (As we pointed out during the construction of the V_k's, this point $\hat{z}_n$ exists.) Observe that if we take $\hat{t}_n = B_{1/k_n}(\hat{z}_n)$, then $\hat{t}_n \subset t_n$. If $d(\hat{z}_n, y) < 1/k_n$, then $d(z_n, y) \leq d(z_n, \hat{z}_n) + d(\hat{z}_n, y) < 1/k_n^2 + 1/k_n < 1/(k_n - 1)$. (This last inequality is true for all $k \geq 2$.) Now, since $\hat{r}_n = 1/k_n$, and for $k_n \geq 3$, $r_n < 1/(k_n - 2)$, it follows that $\dfrac{k_n - 2}{k_n} \leq \dfrac{\hat{r}_n}{r_n} < 1$ and so it is clear that $\lim_{n\to\infty}(\hat{r}_n/r_n) = 1$.

The fact that $\lim_{n\to\infty}(\hat{r}_n/r_n) = 1$ together with $\lim_{n\to\infty}(r_n/\delta_n) \geq 1/m$, implies that $\lim_{n\to\infty}(\hat{r}_n/\delta_n) \geq 1/m$. So, for all but finitely many n, $(\hat{r}_n/\delta_n) > 1/2m$; that is to say, for all but finitely many n, $\hat{r}_n > \delta_n/2m$. Hence, this tail of the sequence $\{\hat{t}_n\}$ together with the corresponding tail of $\{\delta_n\}$ and together with $\hat{m} = 2m$, verify that $x \in \text{por}_V(a) \setminus \bigcup a$.

For the other direction, let $A \subset \text{por}_V(b) \setminus \bigcup b$ be a det-porous set. For $x \in A$, let $m, \{\delta_n\}$ and $\{t_n\}$ be the witnesses of the fact that $x \in \text{por}_V(b)$. Observe that $t_n \cap A = \emptyset$ for all n, since $A \subset \text{por}_V(b) \setminus \bigcup b$ and $t_n \in b$. Now, for $\delta > 0$ let n_0 be such that $0 < \delta_{n_0} < \delta$. Then, it clear that $\dfrac{\Lambda(A, x, \delta_{n_0})}{\delta_{n_0}} > 1/m$ (because $\dfrac{\Lambda(A, x, \delta_{n_0})}{\delta_{n_0}} \geq \frac{\text{rd}(t_n)}{\delta_{n_0}}$). Therefore A is porous. □

3 Inscribing Compact Non-σ-Porous Sets.

In this section we present the "determinacy argument" that will allow us to obtain the non-σ-porous compact subsets for given non-σ-porous projective sets, for the two different types of porosities mentioned in section 1. This argument is inspired by an example given in Farah-Zapletal [1].

Proposition 3.1. *If (X, d) is a dimension zero compact metric space and let I_m be the ideal given by the metric porosity. Then every I_m-positive Borel set has an I_m- positive compact subset.*

PROOF. Considering the equivalence stated in Proposition 2.7, we prove the proposition in terms of the ideal I_d instead. For $B \subset X$, consider the two player game (Eve and Adam) G_B defined as follows. For her n-th move, Eve plays the n-th step on the description of a point y in X; that is to say, she plays the integer $\hat{y}(n)$ such that $y \in T^n_{\hat{y}(n)}$. Adam responds by playing pairs $\langle t, k\rangle$ such that:

- $k \in n$,
- $t \in \hat{V}_{n^2} \equiv_{def} V_{n^2} \cup V_{n^2+1} \cup \cdots \cup V_{(n+1)^2-1}$,
- $t \subset T^n_{\hat{y}(n)}$.

Eve is building the description $\hat{y} \in {}^{\omega}\omega$ of an element y in X. Adam is building a sequence $\langle a_k \in P(V) : k \in \omega\rangle$ at his n-th play, Adam is adding a finite number of elements of a_k for each $k \in n$. So for each $k \in \omega$,

$$a_k = \{t \in V : \exists n(k \in n) \text{ and } \langle t, k\rangle \text{ was played at level } n\}.$$

Let A be the σ-det-porous set $A =_{def} \bigcup_k(\mathrm{por}_V(a_k) \setminus \bigcup a_k))$. Then, Eve wins if and only if $y \in B \setminus A$. So the payoff set of the game G_B is the set $\{\langle \hat{y}, \langle a_k : k \in \omega\rangle\rangle : y \in B \setminus \bigcup_k(\mathrm{por}_V(a_k) \setminus \bigcup a_k)\}$.

The key fact of our proof is Lemma 3.3 below. In the proof of this lemma we use the following observation.

Remark 3.2. Let $a \in P(V)$, let $y \in \mathrm{por}_V(a) \setminus \bigcup a$ and let m, $\{\delta_n\}$, $\{t_n\}$ witness this. For each n, let $N_n \in \omega$ be the unique N such that $t_n \in \hat{V}_{N^2}$. For all but finitely many n, $\delta_n < 1/N_n$ take n large enough such that $N_n > m$. Then $\mathrm{rd}(t_n) \leq 1/(N_n)^2$ and $\delta_n < m \cdot \mathrm{rd}(t_n) < N_n \cdot \mathrm{rd}(t_n)$, so $\delta_n < N_n \cdot 1/(N_n)^2 = 1/N_n$.

Lemma 3.3. *Let $B \subset X$. Then $B \in I_d$ if and only if Adam has a winning strategy in the game G_B.*

PROOF. $\Leftarrow$). Let σ be a winning strategy for Adam. For each $k \in \omega$ let B_k be the set of those $y \in X$ such that if Eve plays $\hat{y}$ against σ. Then Adam (following σ) will produce $a_k(y)$ with the property $y \in \mathrm{por}_V(a_k(y)) \setminus \bigcup a_k(y)$. (Observe that in this case $B = \bigcup_k B_k$.)

We claim that for each $k \in \omega$, $B_k \in I_d$ (and so $B \in I_d$). Let $b_k = \{t \in V : t \cap B_k = \emptyset\}$. We show next that $B_k \subseteq \mathrm{por}_V(b_k) \setminus \bigcup b_k$. Let $y_0 \in B_k$. Then $y_0 \in \mathrm{por}_V(a_k(y_0)) \setminus \bigcup a_k(y_0)$, let m, $\{\delta_n\}$ and $\{t_n\}$ be witnesses of this fact. Observe that for all but finitely many n, the sets t_n belongs to b_k by Remark 3.2, for all but finitely many n, $\delta_n < 1/N_n$, so if $y \in t_n$, then $d(y_0, y) < 1/N_n$ (since $t_n \subset B_{\delta_n}(y_0) \subset B_{1/N_n}(y_0)$). This implies that for all $s \leq N_n$, $\hat{y}_0(s) = \hat{y}(s)$. So, at level N_n of the game, when Eve has played $\hat{y} \restriction (N_n + 1)$, σ will tell Adam to play $t_n \in a_k(y)$. Now, the fact that $y \in t_n \in a_k(y)$ makes it impossible for y to be in B_k. Therefore $t_n \cap B_k = \emptyset$, and so $t_n \in b_k$. This implies that the same witnesses of $y_0 \in \mathrm{por}_V(a_k(y_0)) \setminus \bigcup a_k(y_0)$ also witness the fact that $y_0 \in \mathrm{por}_V(b_k) \setminus \bigcup b_k$.

$\Rightarrow$) Say that $B \subset \bigcup_k \mathrm{por}_V(a_k) \setminus \bigcup a_k$. The strategy for Adam will say that at his Nth move, Adam will play all those $\langle t, k\rangle$ such that $k \in N$, $t \in \hat{V}_{N^2}$, $t \subset T^N_{\hat{y}(N)}$ and $t \in a_k$. That is to say, let Adam play all the legal $\langle t, k\rangle$ such that $t \in a_k$. This process produces a sequence $\langle \hat{a}_k : k \in \omega\rangle$ ($\hat{a}_k \subset a_k$) and a set $A = \bigcup_k \mathrm{por}_V(\hat{a}_k) \setminus \bigcup \hat{a}_k$. The claim is that $B \subseteq A$; therefore it is not possible for Eve to play in $B \setminus A$. To prove this claim, let $y \in B$, k fixed such that $y \in \mathrm{por}_V(a_k) \setminus \bigcup a_k$, and let m, $\{\delta_n\}$ and $\{t_n\}$ be witnesses of this. By Remark 3.2, for all but finitely many n, $\delta_n < 1/N_n$. So $t_n \subset B_{\delta_n}(y) \subset B_{1/N_n}(y)$. Therefore, since $y \in T^{N_n}_{\hat{y}(N_n)}$, $B_{1/N_n}(y) \subset T^{N_n}_{\hat{y}(N_n)}$, so $t_n \subset T^{N_n}_{\hat{y}(N_n)}$. That is to say, it is legal for Adam to play t_n at level N_n. Therefore, if Adam plays following the strategy described above when Eve plays y, it must be true that $t_n \in \hat{a}_k$. So the same witnesses will give that $y \in \mathrm{por}_V(\hat{a}_k) \setminus \bigcup \hat{a}_k \subset A$. $\square$

To complete the proof of Proposition 3.1, it has to be observed first that if $B \subset X$ is Borel, then the payoff set for G_B is also Borel. Let T be the tree of all legal plays in the game G_B and let

$$\mathcal{P} = \{\langle \hat{y}, \langle a_k : k \in \omega\rangle\rangle \in [T] : y \in B \setminus \bigcup_k (\mathrm{por}_V(a_k) \setminus \bigcup a_k)\}$$

be the payoff set of G_B. To conclude that $\mathcal{P}$ is Borel, it is enough to show that the following sets are Borel:

1. $\mathcal{B} = \{\langle \hat{y}, \langle a_k : k \in \omega\rangle\rangle \in [T] : y \in B\}$,
2. $A_k = \{\langle \hat{y}, \langle a_k : k \in \omega\rangle\rangle \in [T] : y \in \bigcup a_k\}$ (for each $k \in \omega$),
3. $P_k = \{\langle \hat{y}, \langle a_k : k \in \omega\rangle\rangle \in [T] : y \in \mathit{por}(a_k)\}$ (for each $k \in \omega$).

The set $\mathcal{B}$ is the pre-image of the set $B \subset X$ under the map $\pi : [T] \to X$ defined by $\pi(\langle \hat{y}, \langle a_k\rangle_k\rangle) = y$. This map π is the composition of the projection $\langle \hat{y}, \langle a_k\rangle_k\rangle \mapsto \hat{y}$, with the homeomorphism: $\hat{y} \mapsto y$. So π is continuous.

To show that A_k is Borel (for arbitrary $k \in \omega$), simply observe that for each pair of integers n, m (such that $n + k < m$) the set $A^{n,m}_k = \{\langle \hat{y}, \langle a_k\rangle_k\rangle \in [T] : T^m_{y(m)} \subset a_k(n)\}$ (Here, $a_k(n)$ represents the n-th member of a_k, played by Adam at level $n + k$.) is open, if $t = \langle \hat{y}, \langle a_k\rangle_k\rangle \in A^{n,m}_k$, then the set $\mathcal{O}_{t\restriction m+1} := \{t' \in [T] : t' \restriction m = t \restriction m\}$ is a subset of $A^{n,m}_k$. Any $t' \in \mathcal{O}_{t\restriction m+1}$

is such that $T^m_{\hat{y}'(m)} = T^m_{\hat{y}(m)} \subset a_k(n) = a'_k(n)$ $(m > n + k)$. It should be clear also that $A_k = \bigcup_{m>n+k} \bigcup_n A_k^{n,m}$. If $t \in A_k$, then there exists n such that $t \in a_k(n)$. By the discussions in Remark 2.6, there exists $m > n+k$ such that $T^m_{\hat{y}(m)} \subset a_k(n)$.

Finally, the sets P_k are Borel. Simply observe that the sets P_k can be expressed as the set of those $\langle \hat{y}, \langle a_k \rangle_k \rangle$ in $[T]$ such that

$$\exists m \forall n \exists q \exists s (a_k(s) \subset B_q(y) \wedge q < \min\{1/n, m \cdot \mathrm{rd}(a_k(s))\}).$$

Now, let B be a Borel I-positive set. Since the payoff set of the game G_B is also Borel, the game is determined. But $B \notin I$; hence Eve has a winning strategy σ. Let K be the space of all counterplays of Adam, this space is compact as it is a finite branching tree. If $\hat{C} \subset \hat{X}$ is the image of K under σ (That is to say, $\hat{C}$ is the set of all $\hat{y}$ which are runs according to σ.), then $\hat{C}$ is a compact subset of $\hat{X}$ (because is σ is continuous). Finally, let $C \subset X$ be the image of $\hat{C}$ under the homeomorphism $\hat{y} \mapsto y$. So $C \subset X$ is also compact. But also $C \notin I$, as σ is still a winning strategy for Eve in the game G_C. Finally, since σ is winning for Eve in G_C, we conclude $C \subset B$. □

As we said in the introduction, Proposition 3.1 states a known result which holds for any compact space and in fact is true not just for Borel, but for analytic sets. The "determinacy" nature of our proof makes it possible to extend the result not just to analytic sets, but to projective sets (under the assumption of *projective determinacy*). The next result by Farah and Zapletal [1] allows the extension of Proposition 3.1 for analytic sets without losing the "determinacy" nature of the proof.

Lemma 3.4. (Farah-Zapletal [1]) *If (X, d) is a compact metric space and I is the ideal given by the metric porosity, then every I-positive analytic set has an I- positive Borel subset.*

Corollary 3.5. *If (X, d) is a dimension zero compact metric space and I is the ideal given by the metric porosity. Then every I-positive analytic set has an I- positive compact subset.*

The last result of this section states that, under the assumption projective determinacy, Proposition 3.1 holds for projective sets in general.

If P is a projective set, since Lemma 3.3 is stated for any set $B \subset X$, the lemma holds for P. By assumption, the game G_P is determined, and the same argument follows, so we can state the following.

Theorem 3.6. (PD) *If (X, d) is a dimension zero compact metric space and I is the ideal given by the metric porosity. Then every I-positive projective set has an I-positive compact subset.*

4 The Strong Porosity Case.

In this section we define the "accepted" notion of *strong porosity* and an equivalent abstract porosity. Then, we state the corresponding results (to the previous section) for the strong porosity.

These results answer positively the question stated in Zajíček-Zelený [6] (in the case of zero-dimensional compact spaces) mentioned in the introduction. Just as in the case of the regular porosity, under projective determinacy, the property holds for projective sets.

In Zajíček [3] the definition of strong porosity says that A is strongly porous at x if

$$\limsup_{\delta \to 0^+} \frac{\Lambda(A, x, \delta)}{\delta} \geq 1/2$$

(with $\Lambda(A, x, \delta)$ as in Definition 2.1.) Now, the lim sup defined above could be strictly greater that $1/2$ and this could happen in a way in which the "holes" near x that make this lim sup grater that $1/2$ are not really "close" to x. This observation is made in Mena *et al* [2], they propose and alternative definition of strong porosity, which I think represents better the idea of "having big holes near a point". The notion of strong porosity that they propose is the following.

Definition 4.1. Given a metric space (X, d), $A \subset X$, $x \in X$ and $\delta > 0$. Let $\Lambda\langle A, x, \delta\rangle = \sup\{r > 0 : \exists B_r(z)(B_r(z) \cap A = \emptyset) \wedge d(x, z) + r \leq \delta\}$, and let

$$p\langle x, A\rangle = \limsup_{\delta \to 0^+} \frac{\Lambda\langle A, x, \delta\rangle}{\delta}$$

Say that A is *strongly porous at* x, if $x \notin \overline{A}$ or $p\langle x, A\rangle = 1/2$. The set A is said to be *strongly porous* if it is strongly porous at x for every $x \in A$. The ideal I_s given by the *strong porosity* is the ideal σ-generated by the strongly porous subsets of X. Finally, we say that a set $B \subset X$ is σ-*strongly porous* if $B \in I_S$.

Mena *et al* [2] give an example of a set which is strongly porous at a point according to the old definition of Zajíček [3], but it is not strongly porous at that same point according to Definition 4.1. They also make the following remark about this definition of strong porosity.

Remark 4.2. Let (X, d) be a metric space, $A \subset X$ and $x \in \overline{A}$. Then A is strongly porous at x if and only if there are sequences $\{z_n\} \subset X$ and $\{r_n\} \subset \mathbb{R}$ such that:

1. $\lim_{n\to\infty} z_n = x$

2. $\lim_{n\to\infty} \frac{r_n}{d(x,z_n)} = 1$

3. $B_{r_n}(z_n) \cap A = \emptyset$.

To see that the remark holds, take a sequence $\{B_{r_n}(z_n)\}$ as above and $\delta_n = 2d(x,z_n)$, we get that $d(x,z_n) + r_n < \delta_n$ and $\lim_{n\to\infty} \frac{r_n}{\delta_n} = 1/2$. On the other hand, if A is strongly porous at x and $\{\delta_n\}$, $\{B_{r_n}(z_n)\}$ are sequences such that $B_{r_n}(z_n) \cap A = \emptyset$, $d(x,z_n) + r_n < \delta_n$ and $\lim_{n\to\infty} \dfrac{r_n}{\delta_n} = 1/2$. Then, it must be true that $\lim_{n\to\infty} \dfrac{d(x,z_n)}{\delta_n} = 1/2$ $(2r_n < d(x,z_n) + r_n < \delta_n)$. Therefore, $\lim_{n\to\infty} \dfrac{r_n}{d(x,z_n)} = 1$.

In his 2005 survey on σ-porosity [5], Zajíček proposes this alternative definition of strong porosity (the one just given in the previous remark) as the suitable and natural definition of strong porosity for general metric spaces.

We use this definition of Remark 4.2 as an inspiration for the definition of an equivalent abstract porosity which is defined on subsets of V (where V is the countable basis defined in the previous sections). The reason for using this alternative definition should be obvious at this point. In fact we can take the balls $B_{r_n}(z_n)$ to be inside V (as we show in Proposition 4.3).

Define $\hat{por}_V : P(V) \to \mathcal{B}(X)$, to be the *$\hat{det}$-porosity* of X, defined by $x \in \hat{por}_V(a)$ if and only if there is a sequence of balls $\{B_{r_n}(z_n)\} \subset V$ such that:

1. for each $n \in \omega$, $B_{r_n}(z_n) \in a$

2. $\lim_{n\to\infty} z_n = x$

3. $\lim_{n\to\infty} \frac{r_n}{d(x,z_n)} = 1$.

Proposition 4.3. *Let (X,d) be a dimension zero compact metric space, let I_s be the ideal given by the strong porosity and let $\hat{I}$ be the one given by the $\hat{det}$-porosity. Then, for all $A \subset X$, A is a strongly porous set if and only if A is a $\hat{det}$-porous set. Therefore, $I_s = \hat{I}$.*

PROOF. It is clear that for any $a \in \mathcal{P}(V)$, the set $A := \hat{por}_V(a) \setminus \bigcup a$ is strongly porous: if $x \in A$ and $\{t_n\} \subset V$ is a witness for this, we just have to check that $t_n \cap A = \emptyset$ (for each n), but as each $t_n \in a$ and $A \cap (\bigcup a) = \emptyset$, this holds.

For the other direction, let $A \subset X$ be a strongly porous set and let $a = \{t \in V : t \cap A = \emptyset\}$, we show next that $A \subset \hat{por}_V(a) \setminus \bigcup a$. Let $x \in A$ and let $\{t_n = B_{r_n}(z_n)\}$ be a witness for the strong porosity of A at x. As in the proof of Proposition 2.7, let $\{\hat{t}_n = B_{\hat{r}_n}(\hat{z}_n)\} \subset V$ be such that $\hat{t}_n \subset t_n$ and $\lim_{n\to\infty} \hat{r}_n/r_n = 1$.

We claim that $\{\hat{t}_n\}$ witnesses the fact that $x \in p\hat{o}r_V(a) \setminus \bigcup a$. Since $\hat{t}_n \subset t_n$, it is clear that for each n, $\hat{t}_n \cap A = \emptyset$ (so $\hat{t}_n \in a$). Now, recall that (as in the proof of Proposition 2.7) the sequence $\{\hat{t}_n\}$ was chosen such that $\hat{r}_n = 1/k_n$ and $d(z_n, \hat{z}_n) \leq 1/k_n^2$ (where k_n is the least k such that $1/(k-1) \leq r_n$), so it is also clear that $\lim_{n\to\infty} \hat{z}_n = x$. Finally, to show that $\lim_{n\to\infty} \dfrac{\hat{r}_n}{d(x,\hat{z}_n)} = 1$ observe that since $d(z_n,\hat{z}_n)/d(x,z_n) \leq (k_n-1)/k_n^2$, it is true that $\lim_{n\to\infty}[d(z_n,\hat{z}_n)/d(x,z_n)] = 0$. This implies that $\lim_{n\to\infty}[d(x,\hat{z}_n)/d(x,z_n)] = 1$. ($d(x,z_n) - d(z_n,\hat{z}_n) \leq d(x,\hat{z}_n) \leq d(x,z_n) + d(z_n,\hat{z}_n)$. Divide by $d(x,z_n)$ and take limits.) Therefore

$$\lim_{n\to\infty}[d(x,\hat{z}_n)/d(x,z_n)] = 1 = \lim_{n\to\infty}[d(x,z_n)/d(x,\hat{z}_n)].$$

From here we may conclude that

$$\lim_{n\to\infty} \frac{\hat{r}_n}{d(x,\hat{z}_n)} = \lim_{n\to\infty} \frac{\hat{r}_n}{r_n} \cdot \frac{r_n}{d(x,z_n)} \cdot \frac{d(x,z_n)}{d(x,\hat{z}_n)} = 1. \qquad \square$$

Remark 4.4. For $y \in p\hat{o}r(a) \setminus \bigcup a$ with $\{B_{r_n}(z_n)\}$ as a witness, if N_n is such that $B_{r_n}(z_n) \in \hat{V}_{N_n^2}$, then for all but finitely many n, $d(y,z_n) < 1/(2N_n)$ for otherwise, there would be infinitely many n such that $\dfrac{r_n}{d(y,z_n)} \leq \frac{2}{N_n}$. This subsequence would converge to zero, which is impossible.

The proof of the next theorem follows the same steps as the proof of Proposition 3.1. The proof of the key lemma (the one corresponding to Lemma 3.3) uses Remark 4.4 above instead of Remark 3.2. We use the remark twice (as before), essentially to show that if $\{t_n = B_{r_n}(z_n)\}$ is a witness for $y \in p\hat{o}r_V(a) \setminus \bigcup a$, then for all but finitely many n, $t_n \subset B_{1/N_n}(y)$. If $x \in t_n$, then $d(x,y) \leq d(x,z_n) + d(z_n,y) \leq 1/N_n^2 + 1/2N_n < 1/N_n$ (for $N_n > 2$).

Theorem 4.5. *If (X,d) is a dimension zero compact metric space and I_s is the ideal given by the strong porosity. Then:*

- *Every I_s-positive analytic set has an I_s-positive compact subset.*
- (PD) *Every I_s-positive Projective set has an I_s-positive compact subset.*

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