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ON SUBCONTINUITY

Abstract

A new characterization of subcontinuity of functions is given. Relations among subcontinuity, local boundedness and local compactness are studied. The set of points of subcontinuity of functions is investigated.

1 Introduction.

Fuller in his paper [FU] introduced and studied the notion of subcontinuous functions. Mimna and Wingler in their paper [MW] investigated some relations among subcontinuity, local boundedness and local compactness of functions. Our paper provides some generalizations and improvements of results from [MW] and also [FU]. We also study the sets of points of continuity of subcontinuous functions, as well as the sets of points of subcontinuity of functions.

For a topological space X and a subset $A \subset X$ we denote by $\overline{A}$, $int(A)$, ∂A the closure, the interior and the boundary of A, respectively.

If (X, d) is a metric space we denote by $G(x, r)$ ($B(x, r)$) the open (closed) ball with the center x and the radius r.

For all basic notions we refer to Engelking's General Topology [EN].

2 Relations among Subcontinuity, Local Boundedness and Local Compactness.

Definition 2.1. [FU] Let X and Y be topological spaces. We say that a function $f : X \to Y$ is subcontinuous at $x \in X$ if for every net $\{x_a : a \in \mathcal{A}\}$ in X converging to x, there is a convergent subnet of $\{f(x_a) : a \in \mathcal{A}\}$.
A function f is subcontinuous if it is subcontinuous at every point of X.

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Definition 2.2. [MW] Let X and Y be topological spaces. A function $f : X \to Y$ is locally compact at $x \in X$ if there is a compact subset K of Y such that $x \in \operatorname{int}(f^{-1}(K))$.

Definition 2.3. [MW] Let X be a topological space and Y be a metric space. Then function $f : X \to Y$ is locally bounded at $x \in X$ if there is a neighborhood V of x such that $f(V)$ is bounded.

Definition 2.4. Let X be a topological space and Y be a metric space. Then function $f : X \to Y$ is locally totally bounded (or simply l.t.b.) at $x \in X$ if for every $r \in \mathbb{R}^+$ there is a finite subset F of Y and a neighborhood V of x such that $f(V) \subset \cup\{G(y,r) : y \in F\}$.

It is clear, that function f, which is l.t.b. at x is also locally bounded at x, for there is a finite subset F of Y and a neighborhood V of x such that $f(V) \subset \cup\{G(y,1) : y \in F\}$; i.e., $f(V)$ is bounded.

Theorem 2.1. *Let X and Y be topological spaces. $f : X \to Y$ is subcontinuous at x if and only if for every open cover $\mathcal{U}$ of Y there is a finite subset $\mathcal{F}$ of $\mathcal{U}$ and a neighborhood V of x such that $f(V) \subset \cup\mathcal{F}$.*

PROOF. Suppose that f is subcontinuous at x and there is an open cover $\mathcal{U}$ of Y such that for every neighborhood V of x and every finite subset $\mathcal{F}$ of $\mathcal{U}$ there is $x_{(V,\mathcal{F})} \in V$ such that $f(x_{(V,\mathcal{F})}) \notin \cup\mathcal{F}$. Put

$$\mathcal{A} = \{(\mathcal{V},\mathcal{F}) : \mathcal{V} \text{ is a neighborhood of } \mathcal{X} \text{ and } \mathcal{F} \text{ is a finite subset of } \mathcal{U}\}.$$

Define the following direction $\geq$ on $\mathcal{A}$: $(V_1,\mathcal{F}_1) \geq (V_2,\mathcal{F}_2)$ iff $V_1 \subset V_2$ and $\mathcal{F}_1 \supset \mathcal{F}_2$. It is easy to verify that the net $\{x_{(V,\mathcal{F})} : (V,\mathcal{F}) \in \mathcal{A}\}$ converges to x. The subcontinuity of f at x implies that $\{f(x_{(V,\mathcal{F})}) : (V,\mathcal{F}) \in \mathcal{A}\}$ has a cluster point $y \in Y$. Let $U \in \mathcal{U}$ be such that $y \in U$. There is $(V,\mathcal{F}) \in \mathcal{A}$ such that $(V,\mathcal{F}) \geq (X,\{U\})$ with $f(x_{(V,\mathcal{F})}) \in U$. By the assumption $f(x_{(V,\mathcal{F})}) \notin \cup\mathcal{F}$ and $\mathcal{F} \supset \{U\}$; i.e., $f(x_{(V,\mathcal{F})}) \notin U$ contrary to supposition.

For the second implication suppose that for every open cover $\mathcal{U}$ of Y there is a finite subset $\mathcal{F}$ of $\mathcal{U}$ and a neighborhood V of x such that $f(V) \subset \cup\mathcal{F}$ and f is not subcontinuous at x. Thus there is a net $\{x_a : a \in \mathcal{A}\}$ converging to x such that $\{f(x_a) : a \in \mathcal{A}\}$ has no cluster point; i.e., for every $y \in Y$ there is an open neighborhood U_y of y and a_y such that for every $a \geq a_y$ is $f(x_a) \notin U_y$. $\mathcal{U} = \{U_y : y \in Y\}$ is an open cover of Y and therefore there exists $\mathcal{F} = \{U_{y_1},\dots,U_{y_n}\}$ and a neighborhood V of x such that $f(V) \subset \cup\mathcal{F}$. Since $\{x_a\}$ is converging to x there is a_0 such that for every $a \geq a_0$ is $x_a \in V$ and thus $f(x_a) \in \cup\mathcal{F}$. Also there is a_1 such that $a_1 \geq a_0$ and $a_1 \geq a_{y_i}$ for $i = 1,\dots,n$. Hence for every $a \geq a_1$ both of the following are true $f(x_a) \in \cup\mathcal{F}$ and $f(x_a) \notin U_{y_i}$ for every $i = 1,\dots,n$ contrary to supposition. □

From Theorem 2.1 we have the following corollary, which is an improvement of the first part of Theorem 2 in [MW].

Corollary 2.1. *Let X be a topological space and Y be a metric space. If $f : X \to Y$ is subcontinuous at x, then it is also l.t.b. at x and hence locally bounded at x.*

Proof. Let $r \in \mathbb{R}^+$. $\mathcal{U}= \{G(y, r) : y \in Y\}$ is an open cover of Y. According to Theorem 2.1, there is a neighborhood V of x and $\mathcal{F}= \{G(y_1, r), \ldots, G(y_k, r)\}$ such that $f(V) \subset \cup\mathcal{F}$ and hence f is l.t.b. at x. □

Corollary 2.2. *Let X be a topological space and Y be a locally compact topological space. If $f : X \to Y$ is subcontinuous at x then it is also locally compact at x.*

Proof. Since Y is locally compact, for every $y \in Y$ there is an open neighborhood U_y of y and a compact subset K_y of Y such that $U_y \subset K_y$. $\mathcal{U}= \{U_y : y \in Y\}$ is an open cover of Y and according to Theorem 2.1 there is a neighborhood V of x and $\mathcal{F}= \{U_{y_1}, \ldots, U_{y_n}\}$ such that $f(V) \subset \cup\mathcal{F}= \cup_{i=1}^{n} U_{y_i} \subset \cup_{i=1}^{n} K_{y_i}$ which is compact and hence f is locally compact at x. □

We can now prove an improvement of the first part of Theorem 3 in [MW].

Theorem 2.2. *Let X and Y be topological spaces. If $f : X \to Y$ is locally compact at x, then it is subcontinuous at x.*

Proof. Since f is locally compact at x, there is a neighborhood V of x and a compact subset K of Y such that $f(V) \subset K$. Since K is compact, for every open cover $\mathcal{U}$ of Y, there is a finite subset $\mathcal{F}$ of $\mathcal{U}$ such that $\cup\mathcal{F}\supset K \supset f(V)$ and hence f is subcontinuous at x. □

Theorem 2.3. *Let X be a topological space and Y be a complete metric space. If $f : X \to Y$ is l.t.b. at x, it is subcontinuous at x.*

Proof. Suppose that $\{x_a : a \in \mathcal{A}\}$ is a net in X, converging to $x \in X$. For every $n \in \mathbb{N}$ there is a finite set F and a neighborhood V of x such that $f(V) \subset \cup\{G(y, 1/n) : y \in F\}$. Since $\{x_a : a \in \mathcal{A}\}$ converges to x, it is eventually in V and since $f(V) \subset \cup\{G(y, 1/n) : y \in F\}$, there are $\mathcal{A}_n \subset \mathcal{A}$ and $y_n \in Y$ such that $\{f(x_a) : a \in \mathcal{A}_n\}\subset G(y_n, 1/n)$ and $\{x_a : a \in \mathcal{A}_n\}$ is subnet of $\{x_a : a \in \mathcal{A}\}$. Since $\{x_a : a \in \mathcal{A}_n\}$ converges to x, we may suppose that $\mathcal{A}_{n+1} \subset \mathcal{A}_n$. ($\mathcal{A}_n$ and y_n are constructed by the induction where $\mathcal{A}_0 = \mathcal{A}$.) Put $\mathcal{B}= \{(a,n) : a \in \mathcal{A}_n, n \in \mathbb{N} \cup \{0\}\}$. Define the direction $\succeq$ by $\mathcal{B}$: $(a_1,n_1) \succeq (a_2,n_2)$ iff $a_1 \geq a_2$ and $n_1 \geq n_2$. For $b = (a,n) \in \mathcal{B}$ put $x_b = x_a$. $\{f(x_b) : b \in \mathcal{B}\}$ is subnet of $\{f(x_a) : a \in \mathcal{A}\}$ and it is a Cauchy net, because for every $b \succeq (a,n)$, $f(x_b) \in G(y_n, 1/n)$. Since Y is complete, $\{f(x_b) : b \in \mathcal{B}\}$ is convergent and hence f is subcontinuous at x. □

Theorem 2.4. *Let X and Y be topological spaces and $f : X \to Y$ be a subcontinuous function. If K is a compact subset of X, then every open cover $\mathcal{U}_0$ of $\overline{f(K)}$ contains a finite subfamily $\mathcal{F}_0$ such that $\overline{f(K)} \subset \cup\{\overline{W} : W \in \mathcal{F}_0\}$.*

PROOF. For an arbitrary open cover $\mathcal{U}_0$ of $\overline{f(K)}$ there is a $\mathcal{U}= \mathcal{U}_0 \cup \{Y \setminus \overline{f(K)}\}$ an open cover of Y. For every $x \in K$ there is an open neighborhood V_x of x and a finite subset $\mathcal{F}_x$ of $\mathcal{U}$ such that $f(V_x) \subset \cup \mathcal{F}_x$. $\mathcal{V}= \{V_x : x \in K\}$ is an open cover of K and therefore there are $V_{x_1}, \dots, V_{x_n}$ such that $K \subset \cup_{i=1}^n V_{x_i}$; i.e., $f(K) \subset f(\cup_{i=1}^n V_{x_i}) \subset \cup_{i=1}^n f(V_{x_i}) \subset \cup_{i=1}^n (\cup \mathcal{F}_{x_i}) = \cup(\cup_{i=1}^n \mathcal{F}_{x_i})$. Put $\mathcal{F}_0 = \{U : U \in \mathcal{F}_{x_i}, i = 1, 2, \dots n\} \setminus \{Y \setminus \overline{f(K)}\}$. Then $\mathcal{F}_0$ is a finite subset of $\mathcal{U}_0$. Since $f(K) \subset \cup \mathcal{F}_0$, $\overline{f(K)} \subset \cup\{\overline{W} : W \in \mathcal{F}_0\}$. □

Corollary 2.3. *Let X and Y be topological spaces, Y Hausdorff and $f : X \to Y$ be a subcontinuous function. If K is a compact subset of X, then $\overline{f(K)}$ is H-closed in Y.*

The following corollary is an improvement of Theorem 2.1 in [FU].

Corollary 2.4. *Let X and Y be topological spaces, Y regular and $f : X \to Y$ be a subcontinuous function. If K is a compact subset of X, then $\overline{f(K)}$ is compact in Y.*

Corollary 2.5. *Let X be a locally compact topological space and let Y be a regular topological space. If $f : X \to Y$ is subcontinuous, then it is locally compact.*

PROOF. Let K be a compact neighborhood of x. By Corollary 2.4 $\overline{f(K)}$ is compact and thus f is locally compact. □

Definition 2.5. [HO] Let Y be a metric space. We say that Y is b-compact if every closed ball in Y is compact.

The following theorem completes Remark 1 in [MW].

Theorem 2.5. *Let Y be a metric space. The following are equivalent:*

1. *Y is b-compact,*
2. *For every topological space X and every function $f : X \to Y$, f is subcontinuous if and only if it is locally bounded.*

Proof. According to Corollary 2.1 and Theorem 2.2, it is sufficient to prove $2 \Rightarrow 1$. We prove that an arbitrary closed ball $B(y,r)$ in Y is compact. Since Y is a metric space, it is sufficient to show, that every sequence $\{y_n : n \in \mathbb{N}\}$ in $B(y,r)$ has a cluster point in $B(y,r)$. So let $\{y_n : n \in \mathbb{N}\}$ be a sequence in $B(y,r)$. Let $X = \mathbb{R}$ be the space of reals with the usual topology. Define $f : X \to Y$ by

$$f(x) = \begin{cases} y_n & x = 1/n \; n = 1, 2, \ldots \\ y & \text{otherwise.} \end{cases}$$

Then of course f is bounded. Thus by the assumption f is subcontinuous. Since $\{1/n\} \to 0$, $\{y_n : n \in \mathbb{N}\}$ has to have a convergent subnet; i.e., $\{y_n : n \in \mathbb{N}\}$ has a cluster point z, which is of course in $B(y,r)$. Thus $B(y,r)$ is compact. □

We can also prove analogous characterizations for locally compact and l.t.b. functions.

Theorem 2.6. *Let Y be a topological space. The following are equivalent:*

1. *Y is locally compact,*
2. *For every topological space X and every function $f : X \to Y$, f is subcontinuous if and only if it is locally compact.*

Proof. We will only prove that $2 \Rightarrow 1$. Let f be an identity on Y. Hence f is continuous. Thus by assumption f is locally compact. Let y be a point of Y. Then there is a neighborhood U of y and a compact subset K of Y such that $K \supset f(U) = U$. Hence Y is locally compact. □

Theorem 2.7. *Let Y be a metric space. The following are equivalent:*

1. *Y is complete,*
2. *For every topological space X and every function $f : X \to Y$, f is subcontinuous if and only if it is l.t.b..*

PROOF. It suffices to prove $2 \Rightarrow 1$. Let $\{y_n : n \in \mathbb{N}\}$ be an arbitrary Cauchy sequence in Y. Let $X = \mathbb{R}$ be the space of reals with the usual topology. Define $f : X \to Y$ by

$$f(x) = \begin{cases} y_n & x = 1/n \; n = 1, 2, \dots \\ y_1 & \text{otherwise.} \end{cases}$$

Since $f(X)$ is totally bounded, f is l.t.b.. By assumption f is subcontinuous and since $1/n \to 0$, $\{y_n : n \in \mathbb{N}\}$ must have a cluster point and consequently it is convergent. □

3 The Set of Points of Continuity of a Subcontinuous Function.

Holá and Piotrowski [HP] showed that the set of points of continuity of a function with values in a Hausdorff topological space with the following property:

there is a sequence $\{\mathcal{G}_n : n \in \mathbb{N}\}$ of open covers of Y such that if $y \in G_n \in \mathcal{G}_n$ for each n, and if W is an open set in Y which contains y, then $\cap\{G_j : 1 \leq j \leq n\} \subset W$ for some n,

is a G_δ-set. Notice that all developable spaces fulfil this condition. Bolstein [BO] showed that every G_δ-set in an almost-resolvable space coincides with the set of points of continuity of a real valued function. Now the question is whether every G_δ-set in an almost-resolvable space is the set of points of continuity of a subcontinuous function.

Definition 3.1. [BO] A topological space is almost-resolvable if it is a countable union of sets with void interiors.

Theorem 3.1. *Let X be an almost-resolvable topological space. Let Y be a first countable Hausdorff topological space, which contains a non isolated point y_0. Let H be a G_δ-set in X. There is a subcontinuous function $f : X \to Y$ such that $C(f) = H$.*

PROOF. The following proof uses ideas from [BO]. Let $\{y_n : n \in \mathbb{N}\}$ be a sequence of points of Y, such that $y_n \to y_0$ and let $\{O(y_n) : n \in \mathbb{N}\}$ be a sequence of open sets such that $y_n \in O(y_n)$ and $y_0 \notin O(y_n)$ for $n \in \mathbb{N}$ and $O(y_n) \cap O(y_m) = \emptyset$ for $m, n \in \mathbb{N}$ and $n \neq m$. Let $F = X \setminus H$. Since F is an F_σ set, $F = \cup_{n \in \mathbb{N}} F_n$ where $F_n \subset F_{n+1}$ are closed sets. Let $E_n = F_n \setminus F_{n-1}$ where $F_0 = \emptyset$. Since X is almost-resolvable, $X = \cup_{m \in \mathbb{N}} A_m$ where A_m are pairwise disjoint and have void interiors. Let $E_{mn} = int(E_n) \cap A_m$ so $int(E_n) = \cup_{m\in\mathbb{N}} E_{mn}$. Let us define $f : X \to Y$ by

$$f(x) = \begin{cases} y_0 & x \notin F \\ y_n & x \in E_n \setminus \text{int}(E_n)\ n = 1, 2, \dots \\ y_{n+m} & x \in E_{mn}\ m, n = 1, 2, \dots . \end{cases}$$

The function f satisfies $f(X \setminus F_n) \subset \{y_{n+1}, y_{n+2}, \dots, y_0\}$, $f(F) \subset \{y_1, y_2, \dots\}$ and $f(X) \subset \{y_0, y_1, y_2, \dots\}$. Since for every $n \in \mathbb{N}$, $X \setminus F_n$ is an open neighborhood of the set H and since $y_n \to y_0$, f is continuous at each point of H. Since all points of $f(F)$ are isolated in $f(X)$, to show that f is discontinuous at every point of F it suffices to show f is not constant on every open V which meets F. Suppose V meets F. Then V meets E_n for some n. If V meets int(E_n), then since each E_{mn} has void interior, V meets E_{mn} at least for two values m. Hence f is not constant on V. If V and int(E_n) are disjoint, then V contains $x \in E_n \setminus \text{int}(E_n)$ so $x \in V \cap E_n \subset V \cap (X \setminus F_{n-1})$ which is an open neighborhood of x and it has to meet $(X \setminus E_n)$. So $\emptyset \neq V \cap (X \setminus F_{n-1}) \cap (X \setminus E_n) = V \cap (X \setminus F_n)$. Since $f(X \setminus F_n) \subset \{y_{n+1}, y_{n+2}, \dots, y_0\}$ and $f(x) = y_n$, f is not constant at V. We have the subcontinuity of f from the compactness of $f(X)$. □

4 The Set of Points of Subcontinuity of a Function.

In this section we will denote the set of points of subcontinuity of a function f by $SC(f)$. The following propositions are evident.

Proposition 4.1. *Let X be a topological space and Y be a metric space and $f : X \to Y$ be a function. Then the set of points of local boundedness of f is open.*

Proposition 4.2. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the set of points of local compactness of f is open.*

Corollary 4.1. *Let X and Y be topological spaces and Y be locally compact. Let $f : X \to Y$ be a function. Then $SC(f)$ is open.*

Theorem 4.1. *Let X be a topological space and Y be a metric space and $f : X \to Y$ be a function. Then the set of points of l.t.b. of f is G_δ.*

Proof. For every $n \in \mathbb{N}$ put

$$W_n = \{x \in X : \text{there is a neighborhood } V \text{ of } x \text{ and finite subset } F \text{of } Y \\ \text{such that } f(V) \subset \cup\{G(y, 1/n) : y \in F\}\}.$$

Then of course W_n is open. The set of points of l.t.b. of f is $\cap_{n\in\mathbb{N}} W_n$ and thus it is G_δ. $\square$

Corollary 4.2. *Let X be a topological space and Y be a complete metric space. Let $f : X \to Y$ be a function. Then $SC(f)$ is a G_δ-set.*

Definition 4.1. A topological space is σ-resolvable if it is a union of infinitely many pairwise disjoint dense sets.

Lemma 4.1. *Let X and Y be topological spaces and $f : X \to Y$ be a subcontinuous function at x. Every net in $W = \cap\{f(V) : V$ is a neighborhood of $x\}$ has a cluster point (not necessarily in W).*

PROOF. Let $\{y_a : a \in \mathcal{A}\}$ be a net in W. For every neighborhood V of x and for every $a \in \mathcal{A}$ there is $x_{(a,V)} \in V$ such that $f(x_{(a,V)}) = y_a$. Consider the natural direction on $\mathcal{B} = \{(a, V) : a \in \mathcal{A}, V$ is a neighborhood of $x\}$. Therefore $\{x_b : b \in \mathcal{B}\}$ is a net converging to x. Since $\{f(x_b) : b \in \mathcal{B}\}$ is a subnet of $\{y_a : a \in \mathcal{A}\}$ and f is subcontinuous at x, $\{y_a : a \in \mathcal{A}\}$ has a cluster point. $\square$

Theorem 4.2. *Let X be a σ-resolvable topological space. Let Y be a non countably compact topological space. Let H be an open domain; i.e., $H = \operatorname{int}(\overline{H})$. Then there is a function $f : X \to Y$ such that $SC(f) = H$.*

PROOF. Since Y is not countably compact, there is a sequence $\{y_n : n \in \mathbb{N}\}$ without a cluster point. Since X is σ-resolvable, $X = \cup_{n\in\mathbb{N}} A_n$, where A_n are pairwise disjoint and dense in X. Define $f : X \to Y$ by

$$f(x) = \begin{cases} y_n & x \in A_n \cap (X \setminus \overline{H})\ n = 1, 2, \dots \\ y_0 & \text{otherwise.} \end{cases}$$

1. If $x \in H$, then H is an open neighborhood of x and $f(H) = \{y_0\}$. Hence f is continuous at x.
2. If $x \in (X \setminus \overline{H})$, then for every open neighborhood V of x is $V \cap (X \setminus \overline{H})$ open and not empty. Hence for every $n \in \mathbb{N}$ is $V \cap (X \setminus \overline{H}) \cap A_n \neq \emptyset$ and $f(V) \supset \{y_n : n \in \mathbb{N}\}$. According to Lemma 4.1, f is not subcontinuous at x.

3. If $x \in \partial H$. Suppose that there is a neighborhood V of x such that $V \subset \overline{H}$. Then $x \in int(\overline{H}) = H$. But $x \in \partial H \subset X \setminus H$ contrary to supposition. This implies that for every open neighborhood V of x, $V \cap (X \setminus \overline{H})$ is an open and nonempty. Hence for every $n \in \mathbb{N}$ is $A_n \cap V \cap (X \setminus \overline{H})$ not void. Hence $f(V) \supset \{y_n : n \in \mathbb{N}\}$ and f is not subcontinuous at x.

Finally $SC(f) = H$. □

Corollary 4.3. *Let X be a σ-resolvable topological space and Y be a non countably compact topological space. Then there is $f : X \to Y$ with $SC(f) = \emptyset$.*

Theorem 4.3. *Let X be a σ-resolvable metric space and Y be a non countably compact topological space. Then for every open set H in X, there is a function $f : X \to Y$ with $SC(f) = H$.*

Proof. W.l.o.g. we may suppose that $\partial H \neq \emptyset$, otherwise we can use the previous theorem. Put $H_n = \{x \in H : d(x, \partial H) > 1/n\}$. H_n is open, $\overline{H}_n \subset H_{n+1}$ and $H = \cup_{n\in\mathbb{N}} H_n$. Let $G_n = H_{n+1} \setminus \overline{H}_n$. Let y_0 be a point of Y. Since Y is not countably compact, there is a sequence $\{y_n : n \in \mathbb{N}\}$ without a cluster point. Since X is σ-resolvable, $X = \cup_{m\in\mathbb{N}} A_m$, where A_m are pairwise disjoint and dense in X. Define $f : X \to Y$ by

$$f(x) = \begin{cases} y_m & x \in A_m \cap (X \setminus \overline{H}) \; m = 1, 2, \dots \\ y_m & x \in G_n \cap A_m \; n = 1, 2, \dots ; m = 1, 2, \dots , n \\ y_0 & \text{otherwise.} \end{cases}$$

1. If $x \in H$, then there is $n \in \mathbb{N}$, such that H_n is an open neighborhood of x and $f(H_n) \subset \{y_0, y_1, \dots , y_n\}$, which is compact. Hence f is subcontinuous at x.

2. If $x \in (X \setminus \overline{H})$, then for every open neighborhood V of x, $V \cap (X \setminus \overline{H})$ is open and nonempty. Hence for every $m \in \mathbb{N}$ is $V \cap (X \setminus \overline{H}) \cap A_m \neq \emptyset$ and $f(V) \supset \{y_m : m \in \mathbb{N}\}$. According to Lemma 4.1, f is not subcontinuous at x.

3. If $x \in \partial H$, then for every open neighborhood V of x meets G_n for infinitely many $n \in \mathbb{N}$. Since $V \cap G_n$ is open, for every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}, n > m$, such that $A_m \cap V \cap G_n$ is not void. Hence $f(V) \supset \{y_m : m \in \mathbb{N}\}$ and f is not subcontinuous at x.

Finally $SC(f) = H$. $\square$

Theorem 4.4. *Let X be a σ-resolvable topological space and Y be a complete metric space with a point y_0 with no compact neighborhood. Let $H = \cap_{n\in\mathbb{N}} H_n$ be a subset of X, where $\{H_n : n \in \mathbb{N}\}$ is a sequence of open domains. Then there is a function $f : X \to Y$, such that $SC(f) = H$.*

Proof. Let $n \in \mathbb{N}$. Since $B(y_0, 1/n)$ is not compact, it contains a sequence $\{y^n_m : m \in \mathbb{N}\}$ without a cluster point. Since X is $\sigma-$resolvable, $X = \cup_{n\in\mathbb{N}} A_n$, where A_n are pairwise disjoint and dense in X. We may also suppose that $H_{n+1} \subset H_n$. Let $H_0 = X$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} y^n_m & x \in A_m \cap (H_{n-1} \setminus \overline{H}_n)\ n, m = 1, 2, \ldots \\ y_0 & \text{otherwise.} \end{cases}$$

1. If $x \in H$, then for every $n \in \mathbb{N}$ is H_{n+1} an open neighborhood of x and $f(H_{n+1}) \subset B(y_0, 1/(n + 1)) \subset G(y_0, 1/n)$; i.e., f is l.t.b. at x and hence subcontinuous at x.
2. If $x \in (H_{n-1} \setminus \overline{H}_n)$, then for every open neighborhood V of x, $V \cap (H_{n-1} \setminus \overline{H}_n)$ is open and nonempty. Hence for every $m \in \mathbb{N}$, $V \cap (H_{n-1} \setminus \overline{H}_n) \cap A_m \neq \emptyset$ and $f(V) \supset \{y^n_m : m \in \mathbb{N}\}$. According to Lemma 4.1, f is not subcontinuous at x.
3. If $x \in \partial H_n$, where $n = \min\{i \in \mathbb{N} : x \in \partial H_i\}$, then $x \in H_{n-1}$. Suppose that there is a neighborhood V of x such that $V \subset \overline{H}_n$. Then $x \in \operatorname{int}(\overline{H}_n) = H_n$. But $x \in \partial H_n \subset X \setminus H_n$ contrary to supposition. This implies that for every open neighborhood V of x, $V \cap (H_{n-1} \setminus \overline{H}_n)$ is open and non-void. Hence for every $m \in \mathbb{N}$, $A_m \cap V \cap (H_{n-1} \setminus \overline{H}_n)$ is non-void. Hence $f(V) \supset \{y^n_m : m \in \mathbb{N}\}$ and f is not subcontinuous at x.

Finally $SC(f) = H$. $\square$

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