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VERTICALLY RIGID FUNCTIONS

Abstract

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be vertically rigid provided its graph $G(f) = \{\langle x, f(x)\rangle \colon x \in \mathbb{R}\}$ is isometric with the graph of the function kf for every non-zero $k \in \mathbb{R}$. We show that the group homomorphisms f from $\langle \mathbb{R}, +\rangle$ into $\langle \mathbb{R}^+, \cdot\rangle$ is vertically rigid if and only if it is an epimorphism. Some other examples of vertically rigid functions will also be given. A problem of characterizing all vertically rigid functions remains open.

1 Vertically Rigid Sets.

A subset A of the plane $\mathbb{R}^2$ is *vertically rigid* if for every $k > 0$ its vertical stretching $A(k) = \{\langle x, ky\rangle \colon \langle x, y\rangle \in A\}$ is isometric with A. Notice that if $A \subseteq \mathbb{R}^2$ is vertically rigid, then A is isometric with $A(k)$ for every non-zero $k \in \mathbb{R}$, since $A(-k)$ is isometric with $A(k)$ via reflection with respect to the x-axis. A function $f\colon \mathbb{R} \to \mathbb{R}$ is vertically rigid provided so is its graph.

Note that if $A \subseteq \mathbb{R}^2$ is vertically rigid, then neither its inner Lebesgue measure $m_*(A)$ nor its outer Lebesgue measure $m^*(A)$ can be finite and positive, since isometries preserve Lebesgue measure, while $m_*(A(k)) = km_*(A)$ and

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$m^*(A(k)) = km^*(A)$ for every $k > 0$. (See e.g. [6, thm 1, p. 57].) It is also clear that the complement of a vertically rigid set is also vertically rigid. The simplest vertically rigid sets are of the form $S \times I$, where I is any unbounded interval in $\mathbb{R}$ and S is an arbitrary subset of $\mathbb{R}$. Also, any line and any half plane, open or closed, are vertically rigid.

For this paper the most interesting example of a vertically rigid set is the graph of the exponential function $f(x) = a^x$. In fact, it is vertically rigid only via horizontal translation, since for every $k > 0$ we have $kf(x) = a^{\log_a k} a^x = a^{x+\log_a k} = f(x+\log_a k)$. Other examples of vertically rigid sets via horizontal translation include $\{\langle x, y\rangle : 0 \le y \le a^x\}$ and $\{\langle x, y\rangle : -a^x \le y \le a^x\}$.

2 Vertically Rigid Functions.

First notice the following simple fact.

Fact 1. *If $f\colon \mathbb{R} \to \mathbb{R}$ is vertically rigid, then so are bf and $f + b$ for every $b \in \mathbb{R}$.*

PROOF. Clearly mf is vertically rigid. We need to show that so is $f + b$. Let $k > 0$. Then $G(k(f + b)) = G(kf + kb) = G(kf) + \langle 0, kb\rangle = (T_{kb} \circ i)[G(f)]$, where i is an isometry with $i[G(f)] = G(kf)$ and T_{kb} is a vertical translation by kb. $\square$

Since all exponential functions $f(x) = a^x$ are vertically rigid, so are also the functions $h(x) = b + a^x$. In addition, all linear functions $f(x) = mx + b$ are vertically rigid. Dragan Janković of Cameron University has conjectured, in private communication, that these are the only types of continuous vertically rigid functions. So far, we were not able to prove this claim. However, the next theorem provides a multitude of discontinuous vertically rigid functions.

Theorem 2. *Let f be a non-constant group homomorphism from the additive group $\langle\mathbb{R}, +\rangle$ to the multiplicative group $\langle\mathbb{R}^+, \cdot\rangle$. Then f is vertically rigid if and only if f is an epimorphism, that is, when $f[\mathbb{R}] = \mathbb{R}^+$.*

PROOF. If f is surjective and $k > 0$, then $k = f(a)$ for some $a \in \mathbb{R}$. Thus $kf(x) = f(a)f(x) = f(x + a)$ for all $x \in \mathbb{R}$. So, f is vertically rigid via horizontal translation.

Conversely, assume that $f[\mathbb{R}] \ne \mathbb{R}^+$. We will show that f is not vertically rigid. For this, let $d \in \mathbb{R}^+ \setminus f[\mathbb{R}]$ and let $k > 0$ be such that $d \in kf[\mathbb{R}]$. (Take $k = d/z$ for any $z \in f[\mathbb{R}]$.) We will show that $G(f)$ is not isometric with $G(kf)$. First notice, that there exists an additive function g (i.e., a group homomorphism from $\langle \mathbb{R}, + \rangle$ into itself) such that $f = \exp \circ g$. This easy fact can be found in Kuczma [6, p. 308]. Since g is not of the form $g(x) = ax$ (as then we would have $f[\mathbb{R}] = \mathbb{R}^+$), the graph of g is dense in $\mathbb{R}^2$. (See [6, p. 277] or [3].) Thus, $G(f)$ and $G(kf)$ are dense in $\mathbb{R} \times \mathbb{R}^+$.

Now, by way of contradiction, assume that there is an isometry i of the plane with $i[G(f)] = G(kf)$. The closure of each set $G(f)$ and $G(kf)$ is $\mathbb{R} \times [0, \infty)$ and so, being a homeomorphism, i must map the upper half plane into itself. In particular, i maps the x-axis $L = \mathbb{R} \times \{0\}$ onto itself. Now, since i preserves the distance, for every $x, y \in \mathbb{R}$ with $i(\langle x, f(x) \rangle) = \langle y, kf(y) \rangle$ we have

$$f(x) = \mathrm{dist}(\langle x, f(x) \rangle, L) = \mathrm{dist}(i(\langle x, f(x) \rangle), L) = kf(y).$$

Using this equation with y for which $kf(y) = d$ we obtain that $d = kf(y) = f(x) \in f[\mathbb{R}]$, contradicting the choice of d. $\square$

As mentioned in the proof, f is a homomorphism from $\langle \mathbb{R}, + \rangle$ into $\langle \mathbb{R}^+, \cdot \rangle$ if and only if $f = \exp \circ g$ for an additive function g. This representation and Theorem 2 allow an easy transformation of additive functions into the vertically rigid functions, which is very useful due to the extensive literature on the additive functions. (See e.g. [1, 2, 4, 5, 6, 7, 8].) For example, there exists a vertically rigid homomorphism f from $\langle \mathbb{R}, + \rangle$ to $\langle \mathbb{R}^+, \cdot \rangle$ with $\ker(f) = \mathbb{Q}$, since there exists an additive function g onto $\mathbb{R}$ with $\ker(g) = \mathbb{Q}$. There exists a vertically rigid $f \colon \mathbb{R} \to \mathbb{R}^+$ with $\mathfrak{c}$-dense graph, since there is an appropriate additive function. For the same reason, there exists a vertically rigid homomorphism f from $\langle \mathbb{R}, + \rangle$ to $\langle \mathbb{R}^+, \cdot \rangle$ whose kernel is uncountable and contains no perfect set, so it is not Borel. This last example can be constructed using [3, thm. 7.3.4], which states that a Hamel basis H and a function $h : H \to \mathbb{R}$ exist such that $h^{-1}(r)$ is a Bernstein set for every $r \in \mathbb{R}$.

3 Final Remarks and Questions.

Notice that the set of vertically rigid functions is not closed under addition.

Example 3. *Since e^x and e^{-x} are vertically rigid functions, so also are $\frac{1}{2}e^x$ and $\frac{1}{2}e^{-x}$. However, the sum of these functions, $f(x) = \cosh x$ is not vertically rigid. For the point $\langle 0, 1 \rangle$ on the graph is a unique point of maximum curvature 1. But, for each $k > 1$, $G(kf)$ has a unique point of maximum curvature k. So $G(kf)$ cannot be isometric to $G(f)$, since the isometry preserves curvature.*

We will finish with some questions.

Question 1. Do vertically rigid functions exist which are neither in form $b + \exp \circ g$ for some additive g nor have a line graph?

A negative answer to Question 1 would prove the conjecture of Janković.

Question 2. Can non-linear additive functions be vertically rigid via rotation?

Question 3. Are functions with line graphs the only ones which are both vertically rigid and horizontally rigid?

References

[1] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, 1966.

[2] A. L. Cauchy, *Cours d'Analyse de l'Ecole Polytechnique*, 1. *Analyse Algébrique*, V., Paris, 1821, (*Oeuvres* (2) **3**, Paris, 1897).

[3] K. Ciesielski, *Set Theory for the Working Mathematician*, Cambridge University Press, (1997).

[4] C. F. Gauss, *Theoria motus corporum coelestium*, Hamburg, 1809, (Werke VII, Leipzig, 1906; pp. 240–244).

[5] G. Hamel, *Einer basis aller zahlen und die unstetigen l ösungen der funktionalgleichung* $f(x+y) = f(x) + f(y)$, Math. Ann., **60** (1905), 459–462.

[6] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities,* Uniwersytet Śląski, (1985).

[7] M. Laczkovich, *Conjecture and Proof*, The Mathematical Association of America, (2001).

[8] A. M. Legendre, *Eléments de géometrie*, Paris, 1791, Note II.