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FIRST-RETURN LIMITS FOR FUNCTIONS OF SEVERAL VARIABLES

Abstract

In this paper we begin the search for appropriate definitions of first-return continuity and first-return approachability for functions of several real variables. We do this by first reviewing the one dimensional theory and then considering several candidate definitions in higher dimensions.

1 Introduction.

We shall be considering real-valued functions defined on the n -dimensional cube, $\mathbb{I}^n \equiv [-1, 1] \times [-1, 1] \times \dots \times [-1, 1]$ in Euclidean n -space, \mathbb{R}^n . A continuous function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ is uniquely determined by its values on any countable dense set in \mathbb{I}^n . One of the initial questions in first-return analysis is whether a Baire class one function is also completely determined by its values on a countable dense set, and if so, how to determine that countable dense set and how to compute the value of the function at any point in \mathbb{I}^n . In brief, the answer to this question is that a Baire class one function is uniquely determined by its values on some countable dense set, and the algorithm for computing values everywhere is extremely simple. Of course, unlike the situation for continuous functions, not just any countable dense set will work for a Baire class one function. However, if a function is Baire one, there are certain countable dense sets which carry enough information to permit computation of the value of the function at any point, provided that the countable dense set is carefully enumerated. Before explaining further, we need to specify some notation and terminology.

Key Words: first-return, Baire one, continuity, approachability, recoverability, Darboux
Mathematical Reviews subject classification: Primary 26A21; Secondary 26B05

Received by the editors July 7, 2005

Communicated by: Udayan B. Darji

*This work formed a portion of the author's Undergraduate Honors in Mathematics Thesis at Washington and Lee University, written under the direction of Michael J. Evans.

Underlying all of our subsequent definitions is the notion of what we shall call a *trajectory*. A *trajectory* is any sequence $\bar{x} = \{x_n\}_{n=0}^{\infty}$ of distinct points in \mathbb{I}^n , whose range is dense in \mathbb{I}^n . Any countable dense subset S of \mathbb{I}^n is called a support set and, of course, any enumeration of S produces a trajectory.

We shall use the standard symbol $B(x, \rho) \equiv \{y \in \mathbb{I}^n : \|x - y\| < \rho\}$ to denote the open ball of radius ρ centered at x . Furthermore, for each subset A of \mathbb{R}^n , let $r(A)$ denote the first element of the trajectory \bar{x} that belongs to the set A .

Definition 1.1. Let $x \in \mathbb{I}^n$, and let $\bar{x} = \{x_n\}$ be a fixed trajectory. The *first-return route to x* , $\mathcal{R}_{x, \bar{x}} = \{w_k\}_{k=1}^{\infty}$, is defined recursively via

$$w_1 = x_0,$$

$$w_{k+1} = \begin{cases} r(B(x, \|x - w_k\|)) & \text{if } x \neq w_k \\ w_k & \text{if } x = w_k. \end{cases}$$

When the trajectory \bar{x} is understood, we set $\mathcal{R}_x = \mathcal{R}_{x, \bar{x}}$. We say that a function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ is *first-return recoverable with respect to \bar{x}* at x provided that

$$\lim_{k \rightarrow \infty} f(w_k) = f(x),$$

and if this happens for each $x \in I$, we say that f is *first return recoverable with respect to $\{x_n\}$* . Finally, we say that f is *first-return recoverable* if it is first-return recoverable with respect to some trajectory.

The fundamental result concerning first-return recoverable functions is the following.

Theorem 1.1. *A function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ belongs to Baire class one if and only if f is first-return recoverable.*

This was first proved in the one-variable case by Darji, Evans, and O'Malley in [3] and extended to the several variable case by Darji and Evans in [1].

If the point $(z, f(z))$ is an isolated point in the graph of a Baire one function f , then z will have to be in the range of any trajectory used to recover f . In this case, the *first-return route to z* , \mathcal{R}_z , is eventually constantly z . A natural question is whether or not a function with no isolated points on its graph can be recovered in such a way that the chosen sequence approaching each point is not eventually constant. Here is a definition, making this precise, and a theorem, giving a positive answer to this question in the one-variable case.

Definition 1.2. For each $x \in \mathbb{I}^n$, the *first-return approach to x based on $\bar{x} = \{x_n\}$* , $\mathcal{A}_x = \{u_k\}$, is defined recursively via

$$u_1 = r(\mathbb{I}^n \setminus \{x\}), \text{ and } u_{k+1} = r(B(x, \|x - u_k\|) \setminus \{x\}).$$

We say that a function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ is *first-return approachable at x with respect to the trajectory \bar{x}* provided

$$\lim_{k \rightarrow \infty} f(u_k) = f(x).$$

We say that f is *first-return approachable with respect to \bar{x}* provided it is first-return approachable with respect to \bar{x} at each $x \in \mathbb{I}^n$. Likewise, f is said to be *first-return approachable* provided there exists a trajectory with respect to which f is first-return approachable.

The following characterization was presented by Darji, Evans, and Humke in [2].

Theorem 1.2. *The function $f : \mathbb{I} \rightarrow \mathbb{R}$ is Baire one with no isolated points on its graph if and only if f is first-return approachable.*

Definition 1.3. If \mathcal{O} is a subset of \mathbb{I}^n , is open in \mathbb{R}^n , and has x as a limit point, we define the first-return approach to x based on $\bar{x} = \{x_n\}$ relative to \mathcal{O} as the sequence $\mathcal{A}_{x, \mathcal{O}} = \{y_k\}$ where

$$y_1 = r(\mathcal{O} \setminus \{x\}) \text{ and}$$

$$y_{k+1} = r(\mathcal{O} \cap B(x, \|x - y_k\|) \setminus \{x\})$$

and say that a function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ is first return approachable at x with respect to \bar{x} and relative to the specified open set \mathcal{O} provided

$$\lim_{k \rightarrow \infty} f(y_k) = f(x).$$

2 Identifying the Points of Continuity of a First-Return Recoverable Function.

It is well-known that the set of points of continuity of a Baire one function is residual; i.e., it is the complement of a first category set. Thus, from Theorem 1.1, it follows that a first-return recoverable function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ has residually many points of continuity. If a trajectory \bar{x} recovers such an f , the question arises as to whether or not we can use \bar{x} to identify the points of continuity. We shall show that we can. This will lead to a characterization of continuous

functions on \mathbb{I}^n . Interestingly, we will also encounter a characterization of continuous functions on \mathbb{I}^n ($n \geq 2$) which does not apply to continuous functions on \mathbb{I} . We begin with the following lemma.

Lemma 2.1. *Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$ be first-return recoverable on \mathbb{I}^n with respect to the trajectory \bar{x} . Then f is continuous at a point $z \in \mathbb{I}^n$ if and only if f is first-return approachable at z via \bar{x} relative to every open set having z as a limit point.*

PROOF. If f is continuous at z , then it is obviously first-return approachable at z via \bar{x} relative to every open set having z as a limit point. Conversely, suppose f is first-return recoverable via \bar{x} on \mathbb{I}^n and fails to be continuous at a point z in \mathbb{I}^n . We shall construct an open set G having z as a limit point such that f is not first-return approachable at z via \bar{x} relative to G . Since f is not continuous at z , there exists a sequence $\{z_n\}$ converging to z and an $\epsilon > 0$ such that for each $n \in \mathbb{N}$,

- 1 : $\|z_{n+1} - z\| < \frac{\|z_n - z\|}{4}$.
- 2 : $|f(z_n) - f(z)| > \epsilon$.

We shall build the open set G by placing a small ball $B(z_n, r_n)$ about each z_n . We define the radii r_n inductively as follows:

Step 1: Choose $r_1 < \frac{\|z_1 - z\|}{2}$ so small that $\|f(r(B(z_1, r_1))) - f(z_1)\| < \frac{\epsilon}{2}$. Set $y_1 = r(B(z_1, r_1))$, and let j_1 be such that $y_1 = x_{j_1}$.

Step k : Now, let $k \in \mathbb{N}$ and assume that r_k, y_k , and j_k have all been defined. Choose $r_{k+1} < \frac{\|z_{k+1} - z\|}{2}$ so small that

- i). $(B(z_{k+1}, r_{k+1}) \setminus \{z_{k+1}\}) \cap \{x_j : j \leq \max(j_1, j_2, \dots, j_k)\} = \emptyset$ and
- ii). $|f(r(B(z_{k+1}, r_{k+1}))) - f(z_{k+1})| < \frac{\epsilon}{2}$.

Set $y_{k+1} = r(B(z_{k+1}, r_{k+1}))$, and let j_{k+1} be such that $x_{j_{k+1}} = y_{k+1}$.

Note that the balls $B(z_k, r_k)$ are pairwise disjoint and that the open set $G = \cup_{k=1}^{\infty} B(z_k, r_k)$ has z as a limit point. However, f is not first-return approachable at z via \bar{x} relative to G because, by construction, the approach $\mathcal{A}_{z,G}$ contains a subsequence of the sequence $\{y_n\}$, and for each n , $|f(y_n) - f(z)| > \frac{\epsilon}{2}$. □

As an immediate consequence, we have the following characterization of continuity:

Theorem 2.1. *A function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ is continuous if and only if there is a trajectory \bar{x} such that at each $x \in \mathbb{I}^n$, f is first-return approachable at x via \bar{x} relative to every open set having x as a limit point.*

Next, we shall show that for $n \geq 2$, we can replace open sets in the above lemma and theorem by open connected sets. Neither result is true when $n = 1$, as can be seen by considering the function

$$f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

For this function, it is easy to think of a trajectory \bar{x} with respect to which f will be first-return approachable at 0 relative to every connected set having 0 as a limit point, and an open set \mathcal{O} having 0 as a limit point for which f will not be first-return approachable at 0 via \bar{x} relative to \mathcal{O} .

Lemma 2.2. *Suppose that $f : \mathbb{I}^n \rightarrow \mathbb{R}$ ($n \geq 2$) has the property that there exists a trajectory \bar{x} such that for some $x \in \mathbb{I}^n$, f is first-return approachable at x via \bar{x} relative to every open connected set having x as a limit point. Then f is first-return approachable at x via \bar{x} relative to every open set having x as a limit point.*

PROOF. Let f , \bar{x} , and x be as described. Let \mathcal{O} be a subset of \mathbb{I}^n that is open in \mathbb{R}^n and has x as a limit point; i.e., $x \in \mathcal{O}$. Let $\{y_n\}_{n=1}^\infty$ be the first-return approach to x via \bar{x} relative to \mathcal{O} . We shall construct an open connected set \mathcal{O}^* having x as a limit point and for which

$$\mathcal{A}_{x, \mathcal{O}^*} = \mathcal{A}_{x, \mathcal{O}} = \{y_n\}_{n=1}^\infty,$$

from which it follows that

$$\lim_{n \rightarrow \infty} f(y_n) = f(x),$$

completing our proof.

For each positive integer $k \geq 2$, let T_k be a connected subset of $\mathbb{I}^n \setminus B(x, \|y_k - x\|)$ that is open as a subset of \mathbb{R}^n and that has three more properties:

$$(1) y_{k-1} \in \overline{T_k}; \quad (2) y_k \in \overline{T_k}; \quad \text{and} \quad (3) T_k \cap \{x_j : j \leq s_k\} = \emptyset.$$

Now let \mathcal{O}_1 denote the union of the connected components of \mathcal{O} that contain at least one point y_k . Since \mathbb{R}^n is locally connected, each of these components is open as a subset of \mathbb{R}^n . It follows that \mathcal{O}_1 , a subset of \mathbb{I}^n , is open as a subset of \mathbb{R}^n . Then let $\mathcal{O}^* = \mathcal{O}_1 \cup (\cup_{k=2}^\infty T_k)$. It follows that \mathcal{O}^* , a subset of \mathbb{I}^n , is an open subset of \mathbb{R}^n . Also, a standard argument shows that \mathcal{O}^* is connected. Furthermore, by construction, we have

$$\mathcal{A}_{x, \mathcal{O}^*} = \mathcal{A}_{x, \mathcal{O}}$$

and our proof is complete. □

From this and Theorem 2.1, we immediately obtain the following characterization of continuous functions $f : \mathbb{I}^n \rightarrow \mathbb{R}$ when $n \geq 2$.

Theorem 2.2. *Let $n \geq 2$. A function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ is continuous if and only if there is a trajectory \bar{x} such that at each $x \in \mathbb{I}^n$, f is first-return approachable at x via \bar{x} relative to every open connected set having x as a limit point.*

3 Some One-Variable First-Return Concepts that Carry Over Nicely to Several Variables.

There are a number of theorems concerning the first-return behavior of functions of one variable whose statements make sense in the several variable setting. Often the proof required for the several variable result requires only cosmetic changes to the one variable proof, substituting open balls for open intervals, for example. Other proofs require at least some mild reworking. In this section, we will give examples of both types. These will provide insight into how other results might be obtained.

As an example of a situation where some mild reworking is necessary to take into account the more complex geometry of \mathbb{I}^n , we start with a result of Evans and Humke from [6] which states that a function on \mathbb{I} is first-return recoverable everywhere if and only if it is first-return recoverable except at a scattered set of points, which is defined as follows:

Definition 3.1. A set S in \mathbb{I}^n is called scattered if S contains no dense-in-itself subset; equivalently, S is scattered if and only if it is a countable G_δ set.

Before stating and proving this result in the several variable setting, we look into a little background information. First, consider a function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ that is first-return recoverable via \bar{x} at each $x \in \mathbb{I}^n$ except at a finite set $E = \{e_1, \dots, e_j\}$. Then, clearly it is first-return recoverable everywhere on \mathbb{I}^n via the trajectory $\{y_i\}$ where

$$y_i = \begin{cases} e_i, & \text{if } i = 1, \dots, j; \\ x_{i-j}, & \text{if } i \geq j + 1. \end{cases} \quad (1)$$

On the other hand, if g denotes the characteristic function of the set Q of points in \mathbb{I}^n , all of whose coordinates are rational, then g is easily seen to be

first-return approachable at each point of $\mathbb{I}^n \setminus Q$ via any trajectory lying in $\mathbb{I}^n \setminus Q$. However, g is not everywhere first-return recoverable via any trajectory because g has no points of continuity and hence is not a Baire one function.

Theorem 3.1. *Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$. The following are equivalent:*

1. f is recoverable.
2. f is recoverable except on a scattered set.

PROOF. Since it is obvious that (1) \Rightarrow (2), it only remains to show that (2) \Rightarrow (1). To this end, suppose D is a support set, $E \subset \mathbb{I}^n \setminus D$ is scattered, and $\bar{x} = \{x_j\}$ is an ordering of D which recovers f except on E . We shall produce an ordering \bar{y} of $D \cup E$ which recovers f on \mathbb{I}^n . More specifically, we shall define \bar{y} in such a way that for each $x \in \mathbb{I}^n \setminus E$, the first return route to x based on the trajectory \bar{y} , $\mathcal{R}_{x,\bar{y}}$, and the first return route to x based on the trajectory \bar{x} , $\mathcal{R}_{x,\bar{x}}$, have a common tail sequence. Indeed, we shall arrange things so that for each $x \in \mathbb{I}^n \setminus E$, $\mathcal{R}_{x,\bar{y}}$ contains only finitely many points of E .

Enumerate E as $\{e_k\}$. We shall define the modified trajectory \bar{y} by inserting each e_k between two carefully chosen terms in \bar{x} . Since E is scattered, it is a countable \mathcal{G}_δ , and we may write $E = \bigcap_{i=1}^\infty G_i$, where each G_i is open and $G_1 \supset G_2 \supset \dots$.

We first choose where to insert e_1 . Let $r_1 > 0$ so that $B(e_1, r_1) \subseteq G_1$. Let S_1 denote the hypersphere which is the boundary of $B(e_1, 2r_1/3)$. Since S_1 is compact, we may cover it with a finite collection of balls $\mathcal{B}_1 = \{B(z_{1,i}, r_1/3) : z_i \in S_1, i = 1, 2, \dots, p_1\}$. Choose j_1 so large that each ball in \mathcal{B}_1 contains an x_j for some $j < j_1$. Note that if $x \notin G_1$, then there is an x_j with $j < j_1$ for which $\|x - x_j\| < \|x - e_1\|$. To see this, let z denote the nearest point in S_1 to x . Then $\|x - e_1\| = \|x - z\| + 2r_1/3$. Next, choose a ball from \mathcal{B}_1 such that $z \in B(z_{1,i}, r_1/3)$. There is a $j < j_1$ such that $x_j \in B(z_{1,i}, r_1/3)$. Then $\|x - x_j\| \leq \|x - z\| + \|z - x_j\| < \|x - z\| + 2r_1/3 = \|x - e_1\|$. We begin the definition of \bar{y} by inserting e_1 between x_{j_1} and x_{j_1+1} ; that is, \bar{y} begins as $\{x_1, x_2, \dots, x_{j_1}, e_1, x_{j_1+1}\}$.

Next, we choose where to insert e_2 . Let $r_2 > 0$ so that $B(e_2, r_2) \subseteq G_2$. Let S_2 denote the hypersphere which is the boundary of $B(e_2, 2r_2/3)$. Since S_2 is compact, we may cover it with a finite collection of balls $\mathcal{B}_2 = \{B(z_{2,i}, r_2/3) : z_i \in S_2, i = 1, 2, \dots, p_2\}$. Choose $j_2 > j_1$ so large that each ball in \mathcal{B}_2 contains an x_j for some $j < j_2$. Note that if $x \notin G_2$, then there is an x_j with $j < j_2$ for which $\|x - x_j\| < \|x - e_2\|$. We extend the definition of the initial string in \bar{y} by inserting e_2 between x_{j_2} and x_{j_2+1} ; that is, \bar{y} begins as $\{x_1, x_2, \dots, x_{j_1}, e_1, x_{j_1+1}, \dots, x_{j_2}, e_2, x_{j_2+1}\}$.

We continue this process, inductively inserting e_k between an x_{j_k} and x_{j_k+1} . The key feature to note is that if $x \notin G_k$, then e_k is not in the

first return route to x based on the trajectory \bar{y} . Thus, if $\{e_{k_t}\}_{t=1}^\infty \subseteq \mathcal{R}_{x,\bar{y}}$, then $x \in \bigcap_{t=1}^\infty G_{k_t} = E$. Therefore, if $x \in \mathbb{I}^n \setminus E$, then $\mathcal{R}_{x,\bar{y}}$ can contain only finitely many points of E and, thus, from some point on $\mathcal{R}_{x,\bar{y}}$ and $\mathcal{R}_{x,\bar{x}}$ agree. Hence, \bar{y} recovers f on \mathbb{I}^n . \square

If one compares the above proof with that given in [6] for $n = 1$, it will be noted that the basic idea prevails, but the “blocking” of the e_k ’s is more involved due to the richer geometry of \mathbb{I}^n .

As an example of a situation where a virtually cosmetic change to the one-variable proof is sufficient, we consider the problem of characterizing consistently recoverable functions as addressed by Evans, Humke, and O’Malley in [7].

Definition 3.2. Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$. Let D be a support set. We shall say that D *consistently recovers* f provided that f is first-return recoverable with respect to every ordering of D . A function is said to be *consistently recoverable* ($f \in \mathcal{CR}$) if there exists a support set D which consistently recovers f .

Theorem 3.2. *A function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ is consistently first-return recoverable if and only if f is continuous except at countably many points.*

PROOF. Suppose that f is consistently recoverable with respect to D . Suppose there is a point $s \in \mathbb{I}^n \setminus D$ at which f is discontinuous. Hence, there is an $\epsilon > 0$ and a sequence $\{s_k\}$ converging to s such that $\|s_{k+1} - s\| < \frac{\|s_k - s\|}{3}$ and $|f(s_k) - f(s)| > \epsilon$ for all k . Let $\bar{x} = \{x_j\}$ be an arbitrary but fixed ordering of D . We shall inductively define a rearrangement $\{y_k\}$ of $\{x_j\}$ which fails to first-return recover f at s .

Since f restricted to D , denoted $f|_D$, is dense in f , there is a point $y_1 \in D$ such that $\|y_1 - s_1\| < \frac{\|s_1 - s\|}{3}$ and $|f(y_1) - f(s_1)| < \frac{\epsilon}{2}$. There are finitely many, say K , integers j less than $\bar{x}^{(-1)}(y_1)$ for which $\|x_j - s\| \geq \|y_1 - s\|$. List these in any order as y_2, y_3, \dots, y_{k_1} , where $k_1 = K + 1$.

Next, suppose that for a natural number n , an integer k_n has been chosen, and y_1, y_2, \dots, y_{k_n} have been defined. There is a point $y_{k_n+1} \in D$ such that $\|y_{k_n+1} - s_{n+1}\| < \frac{\|s_{n+2} - s\|}{3}$ and $|f(y_{k_n+1}) - f(s_{n+1})| < \frac{\epsilon}{2}$. There are finitely many, say P , integers j less than $\bar{x}^{(-1)}(y_{k_n+1})$ for which $\|x_j - s\| \geq \|y_{k_n+1} - s\|$ and $x_j \notin \{y_k : k \leq k_n + 1\}$. List these in any order as $y_{k_n+2}, y_{k_n+3}, \dots, y_{k_{n+1}}$, where $k_{n+1} = P + k_n + 1$.

In this inductive manner we have defined a rearrangement $\{y_k\}$ of $\{x_j\}$. Furthermore, with respect to the trajectory $\{y_k\}$, we have that the first-return route to s contains the sequence $\{y_{k_n+1}\}$. For each n , we have

$$|f(y_{k_n+1}) - f(s)| \geq |f(s_n) - f(s)| - |f(y_{k_n+1}) - f(s_n)| > \frac{\epsilon}{2},$$

indicating that f is not first-return recoverable with respect to $\{y_k\}$ at s .

Conversely, suppose that f is continuous except at countably many points. Let $D = \{x : f \text{ is not continuous at } x\}$. Extend D to a countable, dense set $S \subseteq \mathbb{I}^n$. For every ordering of \bar{x} , f is obviously first-return recoverable with respect to \bar{x} . \square

In [6], Evans and Humke examined what type of functions $f : \mathbb{I} \rightarrow \mathbb{R}$ can be obtained if the requirement that f be first-return recoverable on \mathbb{I} is weakened to that of being first-return recoverable except on a “small” set, where various interpretations of “small” were explored. For example, as noted previously, there is no difference between being first-return recoverable except at a scattered set and being first-return recoverable everywhere. Other examples of small sets considered in [6] include countable sets, first category sets, measure zero sets, and so on. For example, there it was shown that a function $f : \mathbb{I} \rightarrow \mathbb{R}$ has the Baire property if and only if there is a trajectory \bar{x} such that f is first-return recoverable via \bar{x} except at a first category set of points. Furthermore, it was shown that any reordering of \bar{x} will work as well. We shall now show that the analogous results carry over easily to \mathbb{I}^n . We first need a few relevant definitions.

Definition 3.3. A set S has the *property of Baire* if and only if it can be represented in the form $A = F \triangle Q$, where F is closed, Q is of first category, and $F \triangle Q \equiv (F \cup Q) \setminus (F \cap Q) \equiv (F \setminus Q) \cup (Q \setminus F)$.

It is easy to see that the collection of sets that have the Baire property form a σ -algebra [10].

Definition 3.4. A function f has the *property of Baire* if for each open set \mathcal{O} , $f^{-1}(\mathcal{O})$ has the property of Baire.

Definition 3.5. Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$. We say that f is

1. *typically recoverable* ($f \in \mathcal{TR}$) if there exists a trajectory \bar{x} which recovers f at each point of $\mathbb{I}^n \setminus S$, where S is of first category.
2. *typically consistently recoverable* ($f \in \mathcal{TCR}$) if there is a first category set S and a support set D , every ordering of which recovers f at each point of $\mathbb{I}^n \setminus S$.
3. *consistently typically recoverable* ($f \in \mathcal{CTR}$) if there is a support set D such that every ordering \bar{x} of D recovers f at each point of $\mathbb{I}^n \setminus S(\bar{x})$, where $S(\bar{x})$ is of first category.

We call the next several results “observations” because they are known to be true, but are not generally treated in a standard undergraduate real analysis course. The proofs, which are given in [12], closely follow those given in standard texts, such as [11], for the analogous results concerning Lebesgue measurable functions. These proofs will be omitted here.

Observation 3.1. *Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$. Then the following statements are equivalent:*

1. *For each real number α , the set $\{x : f(x) > \alpha\}$ has the Baire property.*
2. *For each real number α , the set $\{x : f(x) \geq \alpha\}$ has the Baire property.*
3. *For each real number α , the set $\{x : f(x) < \alpha\}$ has the Baire property.*
4. *For each real number α , the set $\{x : f(x) \leq \alpha\}$ has the Baire property.*

Just as is the case with Lebesgue measurability, we may recast the definition of a function having the Baire property as follows:

Corollary 3.1. *A function $f : \mathbb{I}^n \rightarrow \mathbb{R}$ has the Baire property if and only if it satisfies one of the four statements in Observation 3.1.*

Observation 3.2. *Let $\{f_k\}$ be a sequence of functions from $\mathbb{I}^n \rightarrow \mathbb{R}$. If all the functions in the sequence $\{f_k\}$ have the Baire property, and for all $x \in \mathbb{I}^n$,*

$$\lim_{k \rightarrow \infty} f_k(x) = f(x),$$

then f also has the Baire property.

Observation 3.3. *If f has the Baire property and $f = g$ on a residual subset of \mathbb{I}^n (where $f, g : \mathbb{I}^n \rightarrow \mathbb{R}$), then g has the Baire property.*

Observation 3.4. *For each $k \in \mathbb{N}$, let $f_k : \mathbb{I}^n \rightarrow \mathbb{R}$ have the Baire property, and let S be a residual set. Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$ and suppose that for all $x \in S$,*

$$\lim_{k \rightarrow \infty} f_k(x) = f(x).$$

Then f also has the Baire property.

Definition 3.6. If $\bar{x} = \{x_k\}_{k=1}^{\infty}$ is a trajectory in \mathbb{R}^n , then call the set

$$\mathcal{U}(\bar{x}, x_k) \equiv \{x : \|x - x_k\| < \|x - x_i\| \text{ for every } i \in \{1, \dots, k-1\}\}$$

the open set of influence of x_k . Note that for $x \in \mathbb{R}^n$, $x_k \in \mathcal{R}(\bar{x}, x)$ if and only if $x \in \mathcal{U}(\bar{x}, x_k)$.

Proposition 3.1. *If $f \in \mathcal{TR}$, then f has the Baire property.*

PROOF. Let \bar{x} be a trajectory and P a first category set such that \bar{x} recovers f at each point of $\mathbb{I}^n \setminus P$. Define the sequence $f_k : \mathbb{I}^n \rightarrow \mathbb{R}$ inductively by $f_1(x) = f(x_1)$, and for every $k > 1$,

$$f_k(x) = \begin{cases} f(x_k) & \text{if } x \in \mathcal{U}(\bar{x}, x_k) \\ f_{k-1}(x) & \text{otherwise.} \end{cases} \tag{2}$$

Then each f_k has the Baire property, and we claim that for all $x \in \mathbb{I}^n \setminus P$, $f_k(x) \rightarrow f(x)$. To see this, let $x \in \mathbb{I}^n \setminus P$. If the sequence \mathcal{R}_x is eventually constant; i.e., if $x = x_{k_0}$ for some k_0 , then $f_k(x) = f(x)$ for all $k \geq k_0$. On the other hand, if \mathcal{R}_x is not eventually constant, then it is a subsequence of \bar{x} , say $\mathcal{R}_x = \{x_{k_j}\}_{j=1}^\infty$. Note that $f_k(x) = f_{k_j}(x)$ for all $k_j \leq k < k_{j+1}$. Therefore,

$$\lim_{k \rightarrow \infty} f_k(x) = \lim_{j \rightarrow \infty} f_{k_j}(x) = \lim_{j \rightarrow \infty} f(x_{k_j}) = f(x).$$

Thus, the proof follows from Observation 3.4. □

Now, we are ready to state the following theorem.

Theorem 3.3. *Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$. The following are equivalent:*

1. f has the Baire property.
2. $f \in \mathcal{TR}$.
3. $f \in \mathcal{CTR}$.
4. $f \in \mathcal{TCR}$.

PROOF. That (4) \Rightarrow (3) \Rightarrow (2) follows directly from the definitions. That (2) \Rightarrow (1) follows from Proposition 3.2. It remains to show that (1) \Rightarrow (4). To this end, let f have the Baire property. Then there is a residual set S such that $f|_S$ is continuous. Let D be any support set lying entirely in S . Clearly, every ordering of D recovers f at each point of S . Hence, $f \in \mathcal{TCR}$, thus completing our proof. □

Next, we could explore what happens when we replace the first-category exceptional sets in the above with countable exceptional sets.

Definition 3.7. Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$. We say that f is

1. *nearly consistently recoverable* ($f \in \mathcal{NCR}$) if there is a countable set S and a support set D , every ordering of which recovers f at each point of $\mathbb{I}^n \setminus S$.

2. *consistently nearly recoverable* ($f \in \mathcal{CN}\mathcal{R}$) if there is a support set D such that every ordering \bar{x} of D recovers f at each point of $\mathbb{I}^n \setminus S(\bar{x})$, where $S(\bar{x})$ is countable.

The following theorem was proved in [12] by following the single-variable proof of Evans and Humke in [6], making only the obvious alterations.

Theorem 3.4. *Let $f : \mathbb{I}^n \rightarrow \mathbb{R}$. The following are equivalent:*

1. $f \in \mathcal{NCR}$.
2. $f \in \mathcal{CN}\mathcal{R}$.
3. *There is a co-countable set $T \subseteq \mathbb{I}^n$ such that $f|_T$ is continuous.*

4 In Search of a Definition for First-Return Continuity in the Several Variables Case.

The one-variable notion of first-return approachability can be strengthened to first-return continuity. The following definition is taken from the recent survey article of Evans and O'Malley [8].

Definition 4.1. Let $\{x_n\}$ be a trajectory. For $0 < x \leq 1$, the *left first return path to x based on $\{x_n\}$* , $\mathcal{P}_x^l = \{t_k\}$, is defined recursively via

$$t_1 = r(0, x), \text{ and } t_{k+1} = r(t_k, x).$$

For $0 \leq x < 1$, the *right first return path to x based on $\{x_n\}$* , $\mathcal{P}_x^r = \{s_k\}$, is defined analogously. We say that f is *first return continuous from the left [right] at x with respect to the trajectory $\{x_n\}$* provided

$$\lim_{k \rightarrow \infty} f(t_k) = f(x) \quad \left[\lim_{k \rightarrow \infty} f(s_k) = f(x) \right].$$

We say that for any $x \in (0, 1)$, f is *first return continuous at x with respect to the trajectory $\{x_n\}$* provided it is both left and right first return continuous at x with respect to the trajectory $\{x_n\}$. (For $x = 0$ or $x = 1$, we only require the appropriate one-sided first-return continuity.)

We say that f is *first return continuous with respect to $\{x_n\}$* provided it is first return continuous with respect to $\{x_n\}$ at each $x \in [0, 1]$. Likewise, f is said to be *first return continuous* provided there exists a trajectory with respect to which f is first return continuous.

The Baire one, Darboux functions turn out to be precisely the first-return continuous functions as evidenced by the following result of Darji, Evans, and O'Malley [4].

Theorem 4.1. *A function $f : I \rightarrow \mathbb{R}$ is Baire one, Darboux if and only if it is first-return continuous.*

Thus, first-return continuous functions of one variable have the intermediate value property, taking connected sets to connected sets. Note that, unlike the situation in the previous section, there is not an obvious analog in \mathbb{I}^n of the concept of being first-return approachable from both the left and right. Indeed, there are numerous possibilities for analogs of this notion for functions of several variables. Even for functions of two variables, there are many options. In the one variable case, first-return continuity at a point forces approachability relative to every connected set having that point as a limit point. Which connected sets and of what type should be utilized in the plane? Here we shall explore several of the more natural choices for a function of two variables.

Definition 4.2. A sector S at x has the form:

$$S = \{z: \theta_1 < \arg(z - x) < \theta_2\}, \text{ where } x \text{ is viewed in the complex plane.}$$

Definition 4.3. A function $f : \mathbb{I}^2 \rightarrow \mathbb{R}$ will be said to be *sectorially first-return approachable* at $x \in \mathbb{I}^2$ with respect to a trajectory \bar{x} if it is first-return approachable relative to every sector at x . If a function is sectorially first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory \bar{x} for all $x \in \mathbb{I}^n$, we say it is *sectorially first-return approachable*.

Definition 4.4. Let $x, y \in \mathbb{I}^2$. An *arc from y to x* is a continuous one-to-one function $g : [0, 1] \rightarrow \mathbb{I}^2$ with $g(0) = y$ and $g(1) = x$. An *arc at x* is a continuous one-to-one function $g : [0, 1] \rightarrow \mathbb{I}^2$ with $g(1) = x$. When the context makes the usage clear, we shall not distinguish between an arc and its range. If the function g is linear, we call it a *ray at x* . If g is an arc at x , then any open set containing $g \setminus \{x\}$ is called an *envelope for g* . Likewise, if g is piecewise linear, we call it a *polygonal arc*. Specifically, g is *piecewise linear* if there is a partition $P = (0 = a_0 < a_1 < a_2 < \dots < a_n = 1)$ of $[0, 1]$ such that g is linear on each $[a_{i-1}, a_i]$, $i = 1, \dots, n$.

Definition 4.5. A function $f : \mathbb{I}^2 \rightarrow \mathbb{R}$ will be said to be *radially-sectorially first-return approachable* at $x \in \mathbb{I}^2$ with respect to a trajectory \bar{x} if for every ray g at x , there is a sector A at x which is an envelope for g such that f is first-return approachable relative to every sector $B \subseteq A$ at x which is also an envelope for g . If a function is radially-sectorially first-return approachable at $x \in \mathbb{I}^n$ with respect to a trajectory \bar{x} for all $x \in \mathbb{I}^2$, we say it is *radially-sectorially first-return approachable*.

Definition 4.6. A function $f : \mathbb{I}^2 \rightarrow \mathbb{R}$ will be said to be *radially first-return approachable* at $x \in \mathbb{I}^2$ with respect to a trajectory \bar{x} if for every ray g at x , there is an envelope G_g such that f is first-return approachable relative to every envelope G of g for which $G \subset G_g$. If a function is radially first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory \bar{x} for all $x \in \mathbb{I}^2$, we say it is *radially first-return approachable*.

Definition 4.7. Let $f : \mathbb{I}^2 \rightarrow \mathbb{R}$, and let \bar{x} be a trajectory. We say that f is *arcwise first-return approachable with respect to \bar{x} at x* if for each arc g at x , there is an envelope G_g such that for every subenvelope $G \subseteq G_g$ of g , we have that f is first-return approachable at x with respect to \bar{x} relative to G . If a function is arcwise first-return approachable at $x \in \mathbb{I}^2$ with respect to a trajectory \bar{x} for all $x \in \mathbb{I}^2$, we say it is *arcwise first-return approachable*.

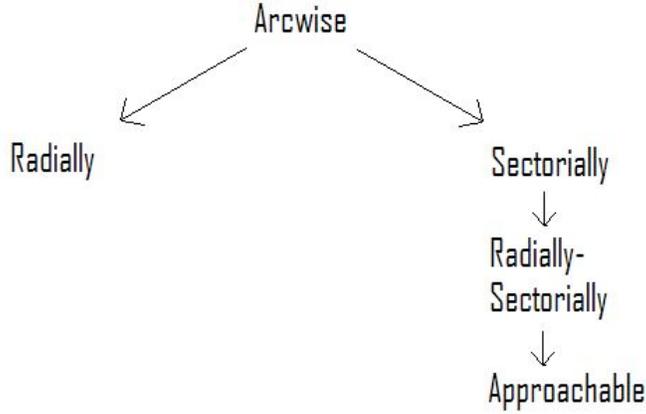
The relative strength of these concepts is clear in several instances. For example, it is immediate that if a function is sectorially first-return approachable at a point x with respect to a trajectory \bar{x} , then it is first-return approachable at x with respect to \bar{x} . Similarly, in an *a priori* fashion, we see that if a function is arcwise first-return approachable at a point x with respect to a trajectory \bar{x} , then it is radially first-return approachable at x with respect to \bar{x} . Also, it is obvious that if a function is sectorially first-return approachable at x with respect to the trajectory \bar{x} , then it is radially-sectorially first-return approachable at x with respect to the trajectory \bar{x} . We shall next examine other pointwise relationships among the above-listed concepts. When we are finished we will have shown that the following relationships exist, with each arrow representing a relationship and no non-trivial arrows possible:

It is important to keep in mind that all of these relationships are pointwise and with respect to one given trajectory.

First, we shall observe that arcwise first-return approachability with respect to \bar{x} at a point implies sectorially first-return approachability with respect to \bar{x} at that point.

Proposition 4.1. *If a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ is arcwise first-return approachable with respect to a trajectory \bar{x} at a point x , then f is sectorially first-return approachable with respect to \bar{x} at x .*

PROOF. Suppose a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ is arcwise first-return approachable with respect to a trajectory \bar{x} at a point x , but is not sectorially first-return approachable with respect to \bar{x} at x . Then, for some sector S at x , f fails to be first-return approachable at x relative to S . Thus, there is an $\epsilon > 0$ and a subsequence $\{z_k\}$ of the first-return approach to x relative to S such that $|f(z_k) - f(x)| > \epsilon$ for all k . For each k , we let g_k be an arc from z_k to z_{k+1}



such that

$$g_k \subset S \cap \overline{B(x, \|z_k - x\|)} \setminus B(x, \|z_{k+1} - x\|).$$

Then the union of the g_k and the singleton $\{x\}$ yields an arc g at x . Furthermore, if G_g is any envelope for g , then $S \cap G_g$ is a subenvelope for g and each z_k will belong to the first-return approach to x relative to $S \cap G_g$, contradicting the assumption that f is arcwise first-return approachable at x . This contradiction completes our proof. \square

We continue by observing that the similar-looking notions of radial and radially-sectorial first-return approachability are independent in the sense that neither implies the other. To see that radially-sectorially first-return approachable does not imply radially first return approachable, we shall actually show that sectorially first-return approachable does not imply radially first-return approachable.

We shall find it useful to define the following “dunce-cap” auxiliary function:

Definition 4.8. Given p a point, r a radius, and a ball $\overline{B(p, r)}$, let $d_{p,r}$ be defined as the function which has the value 1 at p , 0 on the circle centered at p of radius r , and is the linear interpolation on the rest of $\overline{B(p, r)}$.

Example 4.1. *There is a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ which is sectorially first-return approachable everywhere with respect to a trajectory \bar{x} , but f is not radially first-return approachable at least at one point with respect to \bar{x} .*

PROOF. Let $A = \{(x, y) \in \mathbb{I}^2 : x > 0, 0 < |y| < x^2\}$. Let S be a support set that is dense in each circle of rational radius at most one centered at the origin and that contains the points of the form $(\frac{1}{n}, 0)$, $n \in \mathbb{N}$, on the positive x -axis. Let $\{t_k\}$ be an enumeration of S . Let $f : \mathbb{I}^2 \rightarrow \mathbb{R}$ be the function given by:

$$f(x) = \begin{cases} d_{(1/n, 0), (1/3n^2)}(x), & \text{if } x \in B((1/n, 0), (1/3n^2)), n \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Note that f is continuous everywhere except at the origin. We shall order S into a trajectory \bar{x} in such a way that f will be radially-sectorially first-return approachable at the origin with respect \bar{x} , but not radially first-return approachable at the origin with respect to \bar{x} . We shall define the trajectory \bar{x} inductively in stages. We shall find it convenient to let $C_n = \{(x, y) \in \mathbb{I}^2 : x^2 + y^2 = 1/n^2\} \cap S \setminus \bar{A}$.

Step 1: Select four points from C_1 , one from each quadrant, and begin the sequence that will become the trajectory \bar{x} by listing these four points as x_1, x_2, x_3, x_4 . Then set $x_5 = (1, 0)$. Now look at t_1 . If it lies more than one unit from the origin and has not yet been appended to the sequence, do it now as x_6 , define $j_1 = 6$, and proceed to stage 2. Otherwise, do not yet append it, set $j_1 = 5$, and move on to stage 2.

Step n : Assume that $n > 1$ and stage $n - 1$ has been completed. In particular, assume that the ordering $\{x_j\}_{j=1}^{j_{n-1}}$ has been defined. Select a sufficiently large but finite number, say i_n , of points from C_n so that for every sector T at the origin with vertex angle at least π/n , we have that $T \cap C_n \neq \emptyset$. Starting with $x_{j_{n-1}+1}$, append these points to $\{x_j\}$ in any order, then append the point $(1/n, 0)$. If any of the points t_1, t_2, \dots, t_n lie more than $1/n$ from the origin and have not yet been appended to the sequence, append them now in any order. Let j_n denote the total number of points in the partial trajectory \bar{x} through this stage. This completes stage n , and by induction completes the definition of the sequence \bar{x} .

Clearly, \bar{x} is an ordering of S and thus is a trajectory. Note that f is not radially first-return approachable with respect to the trajectory \bar{x} at the origin. To see this, consider the ray g at the origin along the positive x -axis. If G_g is any envelope for g , then $A \cap G_g$ is a subenvelope of G_g , and the

first-return approach to the origin relative to $A \cap G_g$ is a tail of the sequence $\{(1/n, 0)\}$ at each point of which f has value 1, but $f(0) = 0$. On the other hand, for any sector T at the origin, we have that the first return approach to $(0, 0)$ relative to T has a tail, all points of which lie outside of A . Since f has value 0 at these points, f is first-return approachable at $(0, 0)$ relative to T . Furthermore, since f is continuous at each point other than the origin, it follows from the definition that it is first-return approachable everywhere. \square

Next we shall show that there is a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ which is radially first-return approachable everywhere with respect to a trajectory \bar{x} , but f is not radially-sectorially first-return approachable at least at one point x with respect to \bar{x} . In fact, we shall show more by first showing that radial-sectorial first-return approachability with respect to \bar{x} at a point implies first-return approachability with respect to \bar{x} at that point. Then, in Example 4.4, we show that radial first-return approachability everywhere with respect to \bar{x} does not imply first-return approachability at every point.

Proposition 4.2. *If a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ is radially-sectorially first-return approachable with respect to a trajectory \bar{x} at a point x , then f is first-return approachable with respect to \bar{x} at x .*

PROOF. Suppose a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ is radially-sectorially first-return approachable with respect to a trajectory \bar{x} at a point x , but is not first-return approachable with respect to \bar{x} at x . Then there exists a sequence of points z_k in the first-return approach to x and an $\epsilon > 0$ such that $|f(z_k) - f(x)| > \epsilon$ for all k . Expressing each z_k in polar form (r_k, θ_k) with the origin at x , we see there is a subsequence $\{z_{k_j}\}$ and a $\theta_0 \in [0, 2\pi)$ such that $\lim_{j \rightarrow \infty} \theta_{k_j} = \theta_0$. Now, if A is any sector at x containing the ray at x in the direction θ_0 , then a tail of the sequence $\{z_{k_j}\}$ will lie in the first-return approach to x relative to A , implying that f is not radially-sectorially first-return approachable with respect to \bar{x} at x . This contradiction completes our proof. \square

Next, we shall observe that there is a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ which is first-return approachable everywhere with respect to a trajectory \bar{x} , but is not radially-sectorially first-return approachable with respect to \bar{x} at least at one point.

Example 4.2. *There is a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ which is first-return approachable everywhere with respect to a trajectory \bar{x} , but f is not radially-sectorially first-return approachable at least at one point x with respect to \bar{x} .*

PROOF. Let $A = \{(x, y) \in \mathbb{I}^2 : x > 0, 0 < |y| < x^2\}$. Consider the function f , the support set S , and the initial ordering $\{t_k\}$ from Example 1. Now,

we will order S into a trajectory \bar{x} in such a way that f will be first-return approachable at every point with respect to \bar{x} , but not radially-sectorially first-return approachable at the origin with respect to \bar{x} . We shall define the trajectory \bar{x} inductively in stages. We shall again find it convenient to let $C_n = \{(x, y) \in \mathbb{I}^2 : x^2 + y^2 = \frac{1}{n^2}\} \cap S \setminus (A \cup \{(x, y) : y = 0, x < 0\})$.

Step 1: Select $x_1 = (-1, 0)$ and $x_2 = (1, 0)$. Then select 4 points from C_1 , one from each quadrant, and add them to the sequence by listing these four points as x_3, x_4, x_5, x_6 . Now look at t_1 . If it lies more than one unit from the origin and has not yet been appended to the sequence, do it now as x_7 , define $j_1 = 7$, and proceed to stage 2. Otherwise, do not yet append it, set $j_1 = 6$, and move on to stage 2.

Step n : Assume that $n > 1$ and stage $n - 1$ has been completed. In particular, assume that the ordering $\{x_j\}_{j=1}^{j_{n-1}}$ has been defined. Select the points $x_{j_{n-1}+1} = (-\frac{1}{n}, 0)$, and $x_{j_{n-1}+2} = (\frac{1}{n}, 0)$. Now, select a sufficiently large but finite number, say i_n , of points from C_n so that for every sector T at the origin with vertex $\frac{\pi}{n}$, we have that $T \cap C_n \neq \emptyset$. Starting with $x_{j_{n-1}+3}$, append these points to $\{x_j\}$ in any order. If any of the points t_1, t_2, \dots, t_n lie more than $\frac{1}{n}$ from the origin and have not yet been appended to the sequence, append them now in any order. Let j_n denote the total number of points in the partial trajectory \bar{x} through this stage. This completes stage n , and by induction completes the definition of the sequence \bar{x} .

Clearly, \bar{x} is an ordering of S and thus is a trajectory. Note that f is not radially-sectorially first-return approachable with respect to the trajectory \bar{x} at the origin. To see this, consider the ray \vec{g} at the origin along the positive x -axis. If T_g is any sector that is an envelope for \vec{g} , then the first-return approach to the origin in this sector along \bar{x} is by construction of the points of the form $(\frac{1}{n}, 0)$, and f assumes the value of 1 at each such point, but $f(0) = 0$. On the other hand, the first-return approachability at the origin is obvious since the points of the form $(-\frac{1}{n}, 0)$ form the first-return approach to the origin, and f assumes the value of 0 at each of these points. Also, because of the structure of f , it is obvious that it is first-return approachable at every point other than the origin since it is actually continuous at each such point as noted in Example 1. So we have that f is first-return approachable everywhere but is not radially-sectorially first-return approachable at the origin, thus completing our proof. (Note that we now see how a simple re-ordering of the trajectory can yield quite different results concerning first-return limits.) \square

Example 4.3. *There is a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ which is radially-sectorially first-return approachable everywhere with respect to a trajectory \bar{x} , but f is not sectorially first-return approachable at least at one point with respect to \bar{x} .*

PROOF. Again, let $A = \{(x, y) \in \mathbb{I}^2 : x > 0, 0 < |y| < x^2\}$. Note that for each n , there exists a unique point a_n in the first quadrant where the parabola $y = x^2$ intersects the circle $x^2 + y^2 = 1/n^2$. Likewise, for each n , we let b_n denote the unique point in the fourth quadrant where $y = -x^2$ intersects the circle $x^2 + y^2 = 1/n^2$. Let S be a support set of \mathbb{I}^2 which contains all a_n and b_n and is dense in each C_n where $C_n = \{(x, y) \in \mathbb{I}^2 : x^2 + y^2 = 1/n^2\} \cap S \setminus \bar{A}$. Let f be defined by:

$$f(x) = \begin{cases} d_{a_n, (1/3n^2)}(x), & \text{if } x \in B(a_n, (1/3n^2)); \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Note that f is continuous everywhere except at the origin, and therefore it is both sectorially and radially-sectorially first-return approachable at least at every point other than the origin with respect to any trajectory. Let $\{t_k\}$ be an arbitrary but fixed enumeration of S . We shall define the trajectory \bar{x} inductively in stages.

Step 1: Select $x_1 = (-1, 0)$, $x_2 = b_1$, and $x_3 = a_1$. Then choose four points from C_1 , one from each quadrant, and append them to \bar{x} as x_4, x_5, x_6, x_7 . Now look at t_1 . If it lies more than one unit from the origin and has not yet been appended to the sequence, do it now as x_8 , set $j_1 = 8$, and proceed to stage 2. Otherwise, do not yet append it, set $j_1 = 7$ and move on to stage 2.

Step n : Assume that $n > 1$ and stage $n - 1$ has been completed. In particular, assume that the ordering $\{x_j\}_{j=1}^{j_{n-1}}$ has been defined. Append the points $(-\frac{1}{n}, 0)$, b_n , and a_n to the sequence as $x_{j_{n-1}+1}, x_{j_{n-1}+2}$, and $x_{j_{n-1}+3}$, respectively. Select a sufficiently large but finite number, say i_n , of points from C_n so that for every sector T at the origin with vertex at least π/n , we have that $T \cap C_n \neq \emptyset$. Starting with $x_{j_{n-1}+4}$, append these points to $\{x_j\}$ in any order. If any of the points t_1, t_2, \dots, t_n lie more than $1/n$ from the origin and have not yet been appended to the sequence, append them now in any order. Let j_n denote the total number of points in the partial trajectory \bar{x} through this stage. This completes stage n , and by induction completes the definition of the sequence \bar{x} .

Consider the sector T defined to be Quadrant 1. With respect to the trajectory \bar{x} , the first-return approach to the origin is $\{a_1, a_2, a_3, \dots\}$, and

since f assumes a value of 1 at each such point, and yet $f(0) = 0$, clearly f is not sectorially first-return approachable to the origin with respect to T and \bar{x} .

Now consider any ray \vec{g} at the origin. If \vec{g} is not the positive x -axis, then, by construction of \bar{x} , clearly f is first-return approachable inside any sector E that is an envelope for \vec{g} because by the construction of f , $E \cap A \cap B(0, \delta) = \emptyset$ for δ small enough. If \vec{g} is the positive x -axis, then for any sector E that is an envelope for \vec{g} , the first-return approach to the origin relative to E has the tail sequence $\{b_j, b_{j+1}, b_{j+2}, \dots\}$ for some j , and since f assumes the value of 0 at each such points, clearly f is first-return approachable at the origin with respect to E and \bar{x} . So we have that f is not sectorially first-return approachable at the origin with respect to T and \bar{x} , but f is radially-sectorially first-return approachable everywhere with respect to \bar{x} . \square

Example 4.4. *There is a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ which is radially first-return approachable everywhere with respect to a trajectory \bar{x} , but f is not first-return approachable at least at one point with respect to \bar{x} .*

PROOF. For each natural number n , let p_n be the unique point in the first quadrant where the circle centered at the origin with radius $1/n$ intersects the graph of $y = x^2$. Let S be any support set in \mathbb{I}^2 containing all the p_n 's. Let $\{t_j\}$ be an arbitrary but fixed ordering of S . We shall order S as a trajectory $\{x_n\}$ inductively in stages.

Step 1: Set $x_1 = p_1$ and if $t_1 \neq x_1$ and $|t_1| > p_1$, set $x_2 = t_1$ and $j_1 = 2$. Otherwise, leave x_2 undefined at this stage and proceed to stage 2.

Step k : Assume $k > 1$ and that stage $k - 1$ has been completed. So, $x_1, \dots, x_{j_{k-1}}$ have been specified. Set $x_{j_{k-1}+1} = p_k$. If any of the points t_1, t_2, \dots, t_k lie more than p_n units from the origin and have not yet been appended to the sequence, append them now in any order. Let j_k denote the total number of points in the partial trajectory \bar{x} through this stage. This completes stage k , and by induction completes the definition of the sequence \bar{x} .

Let f be the function defined as:

$$f(x) = \begin{cases} d_{p_k, (1/3k^2)}(x), & \text{if } x \in B(p_k, (1/3k^2)); \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Note that the first-return approach to the origin consists of the sequence $\{p_k\}$, implying that f is not first-return approachable with respect to \bar{x} at the origin. For every ray at the origin except the positive x -axis, we have that

any sector A at the origin which contains that ray, but misses the positive x -axis will have the feature that there exists an $N_A \in \mathbb{N}$ such that $A \cap \{p_k : k > N_A\} = \emptyset$. Thus, A provides an envelope for that ray. For the ray at the origin in the direction of the positive x -axis, we select the envelope $G \equiv \{(x, y) \in \mathbb{I}^2 : |y| < x^2\}$. Note that no $p_k \in G$. Thus, f is radially first-return approachable at the origin. \square

4.1 Negative Results Concerning Connectivity.

Now we shall investigate connectivity properties of functions which are approachable in the above senses at every point in \mathbb{I}^2 . Only one has the feature that it maps *every* connected set to a connected set, and this is due to the fact that it is actually continuity in disguise. Each of the others fail to preserve the connectedness of some fairly nice connected sets. We begin by showing that sectorial and radial first-return approachability are too weak to preserve the connectedness of some sets.

Example 4.5. *There is a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ which is both radially first-return approachable and sectorially first-return approachable with respect to the same trajectory \bar{x} , and a connected set T such that $f(T)$ is not connected.*

PROOF. Consider the set $A = \{(x, y) \in \mathbb{I}^2 : x > 0, (1/2)x^2 < y < (3/2)x^2\}$. Let the function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ be defined by:

$$f(x) = \begin{cases} (2y/x^2) - 1, & \text{if } (x, y) \in A \text{ and } (1/2)x^2 < y \leq x^2; \\ -(2y/x^2) + 3, & \text{if } (x, y) \in A \text{ and } x^2 < y < (3/2)x^2; \\ 0, & \text{otherwise.} \end{cases} \quad (6)$$

Now let $S = (\mathbb{Q} \times \mathbb{Q}) \setminus \{(0, 0)\}$, and consider an initial ordering $\{t_k\}$. We shall define the trajectory \bar{x} inductively in stages. We shall find it convenient to let $C_n = \{(x, y) \in \mathbb{I}^2 : x^2 + y^2 = 1/n\} \cap S \setminus \bar{A}$.

Step 1: Select $x_1 = (1, 0)$. Then choose four points from C_1 , one from each quadrant, and append them to \bar{x} as x_2, x_3, x_4, x_5 . Now look at t_1 . If it lies more than one unit from the origin and has not yet been appended to the sequence, do it now as x_6 , set $j_1 = 6$, and proceed to stage 2. Otherwise, do not yet append it, set $j_1 = 5$, and move on to stage 2.

Step n : Assume that $n > 1$ and stage $n - 1$ has been completed. In particular, assume that the ordering $\{x_j\}_{j=1}^{j_{n-1}}$ has been defined. Append the point $(\frac{1}{n}, 0)$ to the sequence as $x_{j_{n-1}+1}$. Select a sufficiently large

but finite number, say i_n , of points from C_n so that for every sector T at the origin with vertex at least π/n we have that $T \cap C_n \neq \emptyset$. Starting with $x_{j_{n-1}+2}$, append these points to $\{x_j\}$ in any order. If any of the points t_1, t_2, \dots, t_n lie more than $1/n$ from the origin and have not yet been appended to the sequence, append them now in any order. Let j_n denote the total number of points in the partial trajectory \bar{x} through this stage. This completes stage n , and by induction completes the definition of the sequence \bar{x} .

First note that f is continuous everywhere except at the origin, which implies that it is both sectorially and radially first-return approachable at least at every point other than the origin. Clearly, \bar{x} is an ordering of S and thus is a trajectory. Let \vec{g} denote the positive x -axis. Now, consider any sector E with vertex at the origin. If E is not an envelope for \vec{g} , then, by construction of \bar{x} , f is first-return approachable inside E because by the construction of f , $E \cap A \cap B(0, \delta) = \emptyset$ for δ small enough. If E is an envelope for \vec{g} , then the first-return approach to the origin is $\{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$. Since $f = 0$ at each such point, clearly f is first-return approachable at the origin with respect to any such sector. Thus, we have shown that f is sectorially first-return approachable everywhere.

To see that f is radially first-return approachable at the origin, we employ a similar argument. If the determining ray $\vec{\ell} \neq \vec{g}$, then given any envelope G , there exists a subenvelope $G_{\vec{\ell}}$ such that $G_{\vec{\ell}} \cap A \cap B(0, \delta) = \emptyset$ for δ small enough. If $\vec{\ell} = \vec{g}$, then again the first-return approach to the origin with respect to any envelope is $\{(\frac{1}{n}, 0) : n \in \mathbb{N}\}$, and since $f = 0$ at each such point, clearly f is first-return approachable at the origin with respect to any such envelope. So we have that f is radially first-return approachable everywhere.

Now, note that f disconnects the connected set $\{(x, y) : y = x^2, x \geq 0\}$. Yet f is radially and sectorially first-return approachable everywhere with respect to \bar{x} , thus completing our proof. \square

Note that the connected set T in the previous example is not convex. One might hope that one of our notions of approachability might at least preserve the connectivity of convex connected sets. Our next example shows that sectorial first-return approachability will not even do that.

Example 4.6. *There is a function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ which is sectorially first-return approachable and a convex, connected set T such that $f(T)$ is not connected.*

PROOF. Consider the set $A = \{(x, y) \in \mathbb{I}^2 : x > 0, 0 < |y| < x^2\}$. Let the function $f : \mathbb{I}^2 \rightarrow \mathbb{I}$ be defined by:

$$f(x, y) = \begin{cases} 1 - |y|/x^2, & \text{if } (x, y) \in A; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Now let $S = (\mathbb{Q} \times \mathbb{Q}) \setminus \{(0, 0)\}$, and consider an initial ordering $\{t_k\}$. We shall define the trajectory \bar{x} inductively in stages. Note that for each n there exists a unique point a_n in the first quadrant where the parabola $y = x^2$ intersects the circle $x^2 + y^2 = 1/n^2$. Likewise, for each n , we let b_n denote the unique point in the fourth quadrant where $y = -x^2$ intersects the circle $x^2 + y^2 = 1/n^2$. Let S be a support set of \mathbb{I}^2 which contains all a_n and b_n and is dense in each $C_n = \{(x, y) \in \mathbb{I}^2 : x^2 + y^2 = 1/n\} \cap S \setminus \bar{A}$.

Step 1: Select $x_1 = a_1$ and $x_2 = b_1$. Then choose four points from C_1 , one from each quadrant, and append them to \bar{x} as x_3, x_4, x_5, x_6 . Now look at t_1 . If it lies more than one unit from the origin and has not yet been appended to the sequence, do it now as x_7 , set $j_1 = 7$, and proceed to stage 2. Otherwise, do not yet append it, set $j_1 = 6$ and move on to stage 2.

Step n : Assume that $n > 1$ and stage $n - 1$ has been completed. In particular, assume that the ordering $\{x_j\}_{j=1}^{j_{n-1}}$ has been defined. Append the points a_n and b_n to the sequence as $x_{j_{n-1}+1}$ and $x_{j_{n-1}+2}$, respectively. Select a sufficiently large but finite number, say i_n , of points from C_n so that for every sector T at the origin with vertex at least π/n , we have that $T \cap C_n \neq \emptyset$. Starting with $x_{j_{n-1}+3}$, append these points to $\{x_j\}$ in any order. If any of the points t_1, t_2, \dots, t_n lie more than $1/n$ from the origin and have not yet been appended to the sequence, append them now in any order. Let j_n denote the total number of points in the partial trajectory \bar{x} through this stage. This completes stage n , and by induction completes the definition of the sequence \bar{x} .

First note that f is continuous everywhere except at the origin, and therefore it is sectorially first-return approachable everywhere but the origin. To see that f is sectorially first-return approachable at the origin, consider any sector T . If T is not an envelope for the positive x -axis, then by construction of \bar{x} , clearly f is first-return approachable at the origin with respect to T . If T is an envelope for the positive x -axis, then the first-return approach to the origin with respect to T has the tail sequence $\{a_j, a_{j+1}, a_{j+3}, \dots\}$ for some j , and since f has the value of 0 at each such point, it is clearly first-return approachable at the origin with respect to any such sector.

Thus, we have that f is sectorially first-return approachable everywhere with respect to \bar{x} . However, f clearly disconnects the convex connected set $\{(x, y) : y = 0, x \geq 0\}$, thus completing our proof. \square

We next show that arcwise first-return connectivity is too strong for our purposes in the sense that the concept is equivalent to continuity.

Theorem 4.2. *Let \bar{x} be a trajectory and $f : \mathbb{I}^2 \rightarrow \mathbb{R}$. Then f is arcwise first-return approachable with respect to \bar{x} if and only if f is continuous.*

PROOF. Clearly, a continuous function must be arcwise first-return approachable. Next, let f be arcwise first-return approachable with respect to the trajectory $\bar{x} = \{x_n\}_{n=1}^\infty$ and suppose that f is discontinuous at some point $z \in \mathbb{I}^2$. Then there exist an $\epsilon > 0$ and a sequence $\{z_k\}_{k=1}^\infty$ converging to z such that for each k , $\|z_{k+1} - z\| < \|z_k - z\|$ and $|f(z_k) - f(z)| > 2\epsilon$.

For each $k = 0, 1, 2, \dots$, we shall define δ_k , n_k , g_k , and $G_{g_k}(z_k)$.

Step 0: To initialize the inductive process, we set $z_0 = (0, 0)$, $\delta_0 = 1$, $n_0 = 1$, $g_0(t) = (0, 1 - t)$ for $t \in [0, 1]$, and $G_{g_0}(z_0) = \mathbb{I}^2$.

Step k : Assume $k > 1$ and steps 1 through $k - 1$ have been completed. Let $g_k : \mathbb{I} \rightarrow \mathbb{I}^2$ denote the linear function from z to z_k , and let $G_{g_k}(z_k)$ denote an envelope of g_k relative to which f is first-return approachable at z_k via the trajectory \bar{x} . Let $\delta_k > 0$ be chosen so small that $B(z_k, \delta_k) \cap \{x_n : n \leq n_{k-1}\} = \emptyset$, and for each x in the first return approach to z_k relative to $G_{g_k}(z_k)$, we have $|f(x) - f(z_k)| < \epsilon$. Then choose n_k so that $x_{n_k} = r(\bar{x}, G_{g_k}(z_k) \cap B(z_k, \delta_k))$.

Now, for each $k \in \mathbb{N}$, let h_k be an arc from $x_{n_{k-1}}$ to x_{n_k} which misses the set $\{x_n : n < n_k, n \neq n_{k-1}\} \cup \overline{B(z, |z - x_{n_k}|)}$. Then cover h_k with an open set K_k such that $K_k \cap (\{x_n : n < n_k, n \neq n_{k-1}\} \cup \overline{B(z, |z - x_{n_k}|)}) = \emptyset$, $K_k \cap B(z_{k-1}, \delta_{k-1}) \subseteq G_{g_{k-1}}(z_{k-1})$, and $K_k \cap B(z_k, \delta_k) \subseteq G_{g_k}(z_k)$.

Then there is an arc h at z for which $h(1) = z$ and $\{h(t) : t \in [0, 1] = \bigcup_{k=1}^\infty \{h_k(t) : t \in [0, 1]\}$.

The open set $G_h(z) \equiv \bigcup_{k=1}^\infty K_k$ forms an envelope for h . If G is any envelope of h at z , then the first-return approach to z relative to $G \cap G_h(z)$ via the trajectory \bar{x} contains the sequence x_{n_k} . Furthermore, since for each $k \in \mathbb{N}$ we have $|f(x_{n_k}) - f(z)| > \epsilon$, f is not arcwise first-return approachable at z , and this contradiction completes our proof. \square

4.2 A Positive Connectivity Result and Recent Contributions of Evans and Humke.

In the previous section, we saw that a radially first-return approachable function need not take connected sets to connected sets. We close by noting that such a function will take a polygonally connected set to a connected set, however. Recall the following definition.

Definition 4.9. We say that a set $H \subseteq \mathbb{R}^2$ is polygonally connected if for each x and y in H , there is a polygonal arc g from y to x such that $g([0, 1]) \subseteq H$.

A proof for the following theorem is given in [12].

Theorem 4.3. *Let $f : \mathbb{I}^2 \rightarrow \mathbb{R}$ be radially first-return approachable and first-return recoverable on \mathbb{I}^2 with respect to a trajectory \bar{x} . Then, if $H \subseteq \mathbb{I}^2$ is polygonally connected, $f(H)$ is connected.*

We shall not present the proof of this result since it has recently been shown by Evans and Humke [5] that the assumption that f be first-return recoverable on \mathbb{I}^2 with respect to the trajectory \bar{x} can be dropped. Specifically, the following theorem is proved in [5].

Theorem 4.4. *If $f : \mathbb{I}^2 \rightarrow \mathbb{R}$ is radially first-return approachable, then f is of Baire class one and if $H \subseteq \mathbb{I}$ is polygonally connected, then $f(H)$ is connected.*

In the same paper [5], Evans and Humke show that although sectorially first-return approachable functions need not preserve the connectedness of a line segment (Example 4.6), they do preserve the connectedness of closed, convex sets having nonempty interior, a type of Darboux property which Malý showed to be possessed by derivatives (gradients) [9].

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