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György Gát,* Institute of Mathematics and Computer Science, College of Nyíregyháza, P.O. Box 166, Nyíregyháza, H-4400 Hungary.
email: gatgy@nyf.hu

Ushangi Goginava, Department of Mechanics and Mathematics, Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia.
email: z_goginava@hotmail.com

MAXIMAL CONVERGENCE SPACE OF A SUBSEQUENCE OF THE LOGARITHMIC MEANS OF RECTANGULAR PARTIAL SUMS OF DOUBLE WALSH-FOURIER SERIES

Abstract

The main aim of this paper is to prove that the maximal operator of the logarithmic means of rectangular partial sums of double Walsh-Fourier series is of type $(H^{\#}, L_1)$ provided that the supremum in the maximal operator is taken over some special indices. The set of Walsh polynomials is dense in $H^{\#}$, so by the well-known density argument we have that $t_{2^n,2^m} f\left(x^1, x^2\right) \to f\left(x^1, x^2\right)$ a. e. as $m, n \to \infty$ for all $f \in H^{\#}(\supset L\log^+ L)$. We also prove the sharpness of this result. Namely, For all measurable function $\delta : [0, +\infty) \to [0, +\infty)$, $\lim_{t\to\infty} \delta(t) = 0$ we have a function f such as $f \in L\log^+ L\,\delta(L)$ and the two-dimensional Nörlund logarithmic means do not converge to f a.e. (in the Pringsheim sense) on I^2.

1 Introduction.

The rectangular partial sums of the Walsh-Fourier series $S_{n,m}(f)$, of the function $f \in L_p\left(I^2\right)$, $1 < p < \infty$ converge in L^p norm to the function f as

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$n, m \to \infty$, [16, 23]. In the $L_1\left(I^2\right)$ case this result does not hold [10, 17]. But in the one-dimensional case the operators S_n are of weak type $(1,1)$ [22]; that is, the analogue of the estimate of Kolmogorov for conjugate function [12]. This estimate implies the convergence of $S_n(f)$ in measure on I to the function $f \in L_1(I)$. However, for double Walsh-Fourier series this result [6, 19] fails to hold.

Classical regular summation methods often improve the convergence of Walsh-Fourier series. For instance, the Fejér means $\sigma_{n,m}(f)$ of the Walsh-Fourier series of the function $f \in L_1\left(I^2\right)$, converge in norm $L_1\left(I^2\right)$ to the function f, as $n, m \to \infty$ [14, 23]. These means present the particular case of the Nörlund means [11].

In 1992 Móricz, Schipp and Wade [15] proved with respect to the Walsh-Paley system that

$$\sigma_{n,k} f = \frac{1}{nk} \sum_{i=1}^{n-1} \sum_{k=1}^{m-1} S_{i,k}(f) \to f$$

a.e. for each $f \in L \log^+ L(I^2)$, when $\min\{n,k\} \to \infty$. In 2000 Gát proved [2] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0, +\infty) \to [0, +\infty)$ be a measurable function with the property $\lim_{t\to\infty} \delta(t) = 0$. Gát proved [2] the existence of a function $f \in L_1(I^2)$ such that $f \in L \log^+ L\, \delta$, and $\sigma_{n,k} f$ does not converge to f a.e. as $\min\{n,k\} \to \infty$. That is, the maximal convergence space for the $(C,1)$ means of two-dimensional partial sums is $L \log^+ L(I^2)$.

We consider the method of Nörlund logarithmic means $t_{n,m}(f)$, which is weaker than the Cesàro method of any positive order [11]. In [3] it is proved, that these means of double Walsh-Fourier series in general do not converge in 2-dimensional measure on I^2 even for functions from Orlicz spaces wider than the Orlicz space $L \log^+ L(I^2)$. In the one-dimensional case [4] it is established that the means $t_n(f)$, of Walsh-Fourier series is of weak type $(1,1)$, and this fact implies their convergence in measure on I. In particular, from this theorem [4] follows the similarity of convergence of $S_n(f)$ and $t_n(f)$, in contrast to convergence of Fejér means $\sigma_n(f)$. United these results with a statement from [20] we obtain, that the partial sums of double Walsh-Fourier series converge in measure for all functions from Orlicz space if and only if their Nörlund logarithmic means converge in measure. Thus, not all classic regular summation method improve the convergence in measure of double Walsh-Fourier series. Thus, the behavior of convergence in measure of logarithmic means of quadrilateral partial sums of double Walsh-Fourier series differs from the behavior of the Fejér means and is like the usual quadrilateral partial sums of double Walsh-Fourier series.

The main aim of this paper is to prove that the maximal operator of the logarithmic means of rectangular partial sums of double Walsh-Fourier series is of type $(H^{\#}, L_1)$ provided that the supremum in the maximal operator is taken over special indices. The set of Walsh polynomials is dense in $H^{\#}$, so by the well-known density argument we have that $t_{2^n,2^m} f\left(x^1, x^2\right) \to f\left(x^1, x^2\right)$ a.e. as $m, n \to \infty$ for all $f \in H^{\#}(\supset L\log^+ L)$. We also prove the sharpness of this result. Namely, for all measurable function $\delta : [0, +\infty) \to [0, +\infty)$, $\lim_{t\to\infty} \delta(t) = 0$ we have a function f such that $f \in L\log^+ L\,\delta(L)$ and the two-dimensional Nörlund logarithmic means do not converge to f a.e. (in the Pringsheim sense) on I^2. In the paper of Getsadze [7] the Nörlund logarithmic summation method is studied in a more general context (for arbitrary bounded ONS).

2 Definitions and Notation.

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N} := \mathbb{P} \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let I be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of I are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}\,(k \in \mathbb{N})$. The group operation on I is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group I is called the Walsh group. A base for the neighborhoods of I can be given by

$$I_0(x) := I,\ \ I_n(x) := \{y \in I : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\},$$

$$(x \in I, n \in \mathbb{N}).$$

These sets are called the dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in I$ denote the null element of I, $I_n := I_n(0)\ (n \in \mathbb{N})$. Set $\overline{I}_n := I \backslash I_n$. Let $\mathcal{A}_n$ be the σ-algebra generated by the sets $I_n(x)(x \in I)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$, $n \in \mathbb{N}\,(f \in L_1)$. For $t = (t^1, t^2) \in I^2, b = (b_1, b_2) \in \mathbb{N}^2$ set the two-dimensional dyadic interval $I_b^2 := I_{b_1}(t^1) \times I_{b_2}(t^2)$. If $b \in N$, then $I_b^2(t) := I_b(t^1) \times I_b(t^2)$. For $n = (n_1, n_2) \in \mathbb{N}^2$ denote by $E_n = E_{(n_1,n_2)}$ the two-dimensional expectation operator with respect to the $\mathcal{A}_n = \mathcal{A}_{(n_1,n_2)} = \mathcal{A}_{n_1} \times \mathcal{A}_{n_2}$.

For $k \in \mathbb{N}$ and $x \in I$ let

$$r_k(x) := (-1)^{x_k}\ \ (x \in I, k \in \mathbb{N})$$

denote the k-th Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0,1\}$ $(i \in \mathbb{N})$; i.e., n is expressed in the number system of base 2. Let $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$; that is, $2^{|n|} \leq n < 2^{|n|+1}$.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions.

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in I, n \in \mathbb{P}).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \in \overline{I}_n. \end{cases} \tag{1}$$

The rectangular partial sums of the 2-dimensional Walsh-Fourier series are defined by

$$S_{M,N}(f; x^1, x^2) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j) w_i(x^1) w_j(x^2),$$

where the number

$$\widehat{f}(i,j) = \int_{I^2} f(x^1, x^2) w_i(x^1) w_j(x^2) d\mu(x^1, x^2)$$

is said to be the (i,j)th Walsh-Fourier coefficient of the function f.

The norm of the space $L_p(I^2)$ is defined by

$$\|f\|_p := \left(\int_{I^2} \left| f(x^1, x^2) \right|^p d\mu(x^1, x^2) \right)^{1/p} \quad (1 \leq p < \infty),$$

and for $p = \infty$ we have $\|f\|_\infty = \sup \text{ess} |f(x)|$. The space weak-$L_1(I^2)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_1(I^2)} := \sup_{\lambda > 0} \lambda \text{mes}(|f| > \lambda) < +\infty.$$

The hybrid Hardy space $H^{\#}\left(I^2\right)$ is based on the maximal function

$$f^{\#}\left(x^1,x^2\right)=\sup_{n\geq 1}\frac{1}{\operatorname{mes}\left(I_n(x^2)\right)}\left|\int\limits_{I_n(x^2)} f\left(x^1,u^2\right)d\mu\left(u^2\right)\right|,$$

defined for all $f\in L_1\left(I^2\right)$ and $\left(x^1,x^2\right)\in I^2$. Thus $H^{\#}\left(I^2\right)$ represents the collection of functions $f\in L_1\left(I^2\right)$ which satisfy

$$\|f\|_{\#}:=\int\limits_{I^2} f^{\#}<\infty.$$

The positive logarithm $\log^+$ is defined as

$$\log^+(x):=\begin{cases}\log(x), \text{ if } x>1\\ 0, \text{ otherwise.}\end{cases}$$

We say that the function $f\in L_1(I^2)$ belongs to the logarithmic space $L\log^+ L$ if the integral

$$\int_{I^2}|f(x^1,x^2)|\log^+|f(x^1,x^2)|d\mu(x^1,x^2)$$

is finite.

The logarithmic means of partial sums of the double Walsh-Fourier series are defined as follows

$$t_{n,m}f\left(x^1,x^2\right)=\frac{1}{l_n l_m}\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{S_{i,j}\left(f,x^1,x^2\right)}{(n-i)\,(m-j)},$$

where $l_n=\sum\limits_{k=1}^{n}\frac{1}{k}$. Let

$$F_n\left(x\right)=\frac{1}{l_n}\sum_{k=1}^{n-1}\frac{D_k\left(x\right)}{n-k} \text{ and } K_n\left(x\right)=\frac{1}{n}\sum_{k=1}^{n}D_k\left(x\right),$$

For $N=(N_1,N_2)$ and $x\in I^2$ let

$$t_N f(x):=t_{N_1,N_2}f(x^1,x^2),\ \ F_N(x):=F_{N_1}(x^1)F_{N_2}(x^2).$$

For the function f we consider the maximal operator

$$t^{\#}f:=\sup_{n,m\in\mathbb{N}}|t_{2^n,2^m}f|.$$

3 Formulation of Main Results.

Theorem 1. *Let $1 < p \leq +\infty$. Then*

$$\left\| t^{\#} f \right\|_p \leq c_p \left\| f \right\|_p \quad \left(f \in L^p \left(I^2 \right) \right).$$

Moreover, if $f \in H^{\#} \left(I^2 \right) \supset \left(L \log^+ L \left(I^2 \right) \right)$, then

$$mes \left\{ t^{\#} f > \lambda \right\} \leq \frac{c}{\lambda} \left\| f^{\#} \right\|_1.$$

Corollary 1. *Let $f \in L \log^+ L \left(I^2 \right)$. Then*

$$t_{2^n, 2^m} f \left(x^1, x^2 \right) \to f \left(x^1, x^2 \right) \text{ a.e. as } \min (n, m) \to \infty.$$

Theorem 2. *For all measurable function $\delta : [0, +\infty) \to [0, +\infty)$, $\lim_{t \to \infty} \delta(t) = 0$ we have a function f such that $f \in L \log^+ L \, \delta(L)$ and the two-dimensional Nörlund logarithmic means does not converge to f a.e. (in the Pringsheim sense) on I^2.*

4 Auxiliary Propositions.

Lemma 1 ([8])**.** *Let $A \in \mathbb{N}$. Then*

$$\int_{\overline{I}_A} \sup_{n \geq 2^A} \left| K_n (x) \right| d\mu (x) \leq c < \infty.$$

Lemma 2 ([8])**.** *The maximal operator $\sup_{n \in \mathbb{N}} \left| \int_I f(x) \left| F_{2^n} (x + y) \right| d\mu (x) \right|$ is of weak type $(1, 1)$ and of type (H, L).*

Define a subset of the set of the two-dimensional intervals $\mathcal{I} \times \mathcal{I}$:

$$\mathcal{I}_{n,a}(x) := \{ I_{n+j}(x^1) \times I_{n+a-j}(x^2) : j = 0, 1, \ldots, a \} \quad (x \in I^2, a, n \in \mathbb{N}).$$

It is easy to verify that

$$\bigcap \mathcal{I}_{n,a}(x) = I_{n+a}(x^1) \times I_{n+a}(x^2), \quad \mu(\bigcap \mathcal{I}_{n,a}(x)) = 2^{-2n-2a},$$

$F \in \mathcal{I}_{n,a}(x)$ implies $\mu(F) = 2^{-2n-a}$. In paper [2, Lemma 1] one can find that $\mu(\bigcup \mathcal{I}_{n,a}(x)) = (a/2 + 1) 2^{-2n-a}$. For $t \in I^2$, $a, b, k \in \mathbb{N}$ define the sets $J^k_{a,b}(t)$, $\Omega^k_{a,b}(t)$ recursively by

$$J^0_{a,b}(t) := \{t\}, \ \Omega^0_{a,b}(t) := \bigcup \mathcal{I}_{b,a}(t).$$

Suppose that the sets $J^j_{a,b}(t)$, $\Omega^j_{a,b}(t)$ are defined for $j < k$. Then decompose

$$I^2_b(t) \setminus \bigcup_{j=0}^{k-1} \Omega^j_{a,b}(t)$$

as the disjoint union of dyadic squares of the form $I^2_{b+ka}(x)$. Take from each dyadic rectangle an element to represent. The set of x's corresponding to these squares is $J^k_{a,b}(t)$. That is,

$$I^2_b(t) \setminus \bigcup_{j=0}^{k-1} \Omega^j_{a,b}(t) = \bigcup_{x \in J^k_{a,b}(t)} I^2_{b+ka}(x).$$

Then, set

$$\Omega^k_{a,b}(t) := \bigcup_{x \in J^k_{a,b}(t)} \bigcup \mathcal{I}_{b+ka,a}(x).$$

It is evident that

$$\texttt{card}(J^k_{a,b}(t)) = \left(2^a \left(2^a - (1 + a/2)\right)\right)^k . \tag{2}$$

This gives the a.e. equality $I^2_b(t) = \bigcup_{j=0}^{\infty} \Omega^j_{a,b}(t)$. By this we get

$$I^2 = \bigcup_{\substack{t^1_i, t^2_i \in \{0,1\} \\ i=0,1,\dots,b-1}} \bigcup_{j=0}^{\infty} \Omega^j_{a,b}(t),$$

For $a, b, d \in \mathbb{N}$ $(b \geq 2)$, $b^\circ := [b/2] - 1$ define the functions $f^d_{a,b}$ by

$$f^d_{a,b}(x) := \begin{cases} (-1)^{x^1_{b^\circ} + x^2_{b^\circ}} 2^a, & \text{if there exists } t \in I^2, k \leq d, y \in J^k_{a,b}(t) \\ & \text{for which } x \in \bigcap \mathcal{I}_{b+ka,a}(y), \\ 0, & \text{otherwise.} \end{cases}$$

Denoting by 1_B the characteristic function of any set $B \subset I^2$ we have

$$f^d_{a,b}(x) = 2^a (-1)^{x^1_{b^\circ} + x^2_{b^\circ}} \sum_{\substack{t^1_i, t^2_i \in \{0,1\} \\ i=0,1,\dots,b-1}} \sum_{k=0}^{d} \sum_{y \in J^k_{a,b}(t)} 1_{I^2_{b+(k+1)a}(y)}(x) =: \sum_{k=0}^{d} g_k(x),$$

where

$$g_k(x) = (-1)^{x^1_{b^\circ} + x^2_{b^\circ}} 2^a \sum_{u \in \Lambda_k} 1_{I^2_{b+(k+1)a}(u)}(x).$$

(This equality gives the definition of the set Λ_k, $\Lambda = \cup \Lambda_k$.) The next lemma is the base of the proof of the divergence result.

Lemma 3. *Let $\gamma, a, b, d \in \mathbb{P}$ and γ, b so large that $C\left(2^{-\gamma}+\frac{a}{b}+\frac{a^2}{b^2}+\frac{1}{2^a}+\frac{2^a}{b}\right)<\frac{1}{8}$, where C is some positive constant discussed later and*

$$y \in \bigcup_{\substack{t_i^1, t_i^2 \in \{0,1\} \\ i=0,1,\ldots,b-1}} \bigcup_{k=0}^{d} \widetilde{\Omega}_{a,b}^k(t),$$

where

$$\widetilde{\Omega}_{a,b}^k(t) := \bigcup_{x \in J_{a,b}^k(t)} \bigcup_{j=\gamma}^{a-\gamma} I_{b+ka+j}(x^1) \times I_{b+(k+1)a-j}(x^2).$$

Then there exists a unique $k \le d$ for which

$$y \in I_{b+ka+j}(z^1) \times I_{b+(k+1)a-j}(z^2)$$

for some $z \in J_{a,b}^k(t)$. Setting $N := (2^{b+ka+j}, 2^{b+(k+1)a-j}) \in \mathbb{P}^2$ we have

$$|t_N(f_{a,b}^d)(y)| \ge \frac{1}{8}.$$

Proof. The first part of the lemma is a straightforward consequence of the definition of the sets $\widetilde{\Omega}_{a,b}^k(t)$. For the second part let

$$\begin{aligned} g_k(x) &= (-1)^{x_{b^\circ}^1 + x_{b^\circ}^2}\left(2^a 1_{I_{b+(k+1)a}^2(z)}(x) + 2^a \sum_{u \in \Lambda_k, u \ne z} 1_{I_{b+(k+1)a}^2(u)}(x)\right) \\ &=: g^1(x) + g^2(x). \end{aligned}$$

Clearly $x \in I_{b+(k+1)a}^2(z) := I_{b+(k+1)}(z^1) \times I_{b+(k+1)}(z^2)$ and $y \in I_{b+ka+j}(z^1) \times I_{b+(k+1)a-j}(z^2)$ gives $F_N(y-x) = F_N(0) \ge 2^{b+ka+j-1}2^{b+(k+1)a-j-1}$ and consequently

$$|t_N(g^1)(y)| = 2^a \int_{I_{b+(k+1)a}^2(z)} F_N(y-x)d\mu(x^1, x^2) \ge 2^{-2}.$$

It is easy to see that

$$D_{2^n-k} = D_{2^n} - w_{2^n-1}D_k$$

for every $k = 0, 1, \ldots, 2^n - 1$ and $n \in \mathbf{N}$. Consequently,

$$F_{2^n} = D_{2^n} - \frac{w_{2^n-1}}{l_{2^n}} \sum_{k=1}^{2^n-1} \frac{D_k}{k},$$

and by the Abel transformation we have

$$|F_{2^n}| \leq D_{2^n} + \frac{1}{l_{2^n}} \sum_{k=1}^{2^n-1} \frac{|K_k|}{k} + \frac{1}{l_{2^n}} |K_{2^n}|.$$

Next, we give an upper bound for the absolute value of $t_N(g^2)(y)$.

Investigate

$$|t_N(g^2)(y)| \leq |t_N(g_1^2)(y)| + |t_N(g_2^2)(y)| + |t_N(g_3^2)(y)|,$$

where

$$g_1^2 = 2^a \sum_{\substack{u \in \Lambda_k \\ u^1 = z^1, u^2 \neq z^2}} 1_{I^2_{b+(k+1)a}(u)}, \quad g_2^2 = 2^a \sum_{\substack{u \in \Lambda_k \\ u^1 \neq z^1, u^2 = z^2}} 1_{I^2_{b+(k+1)a}(u)},$$

$$g_3^2 = 2^a \sum_{\substack{u \in \Lambda_k \\ u^1 \neq z^1, u^2 \neq z^2}} 1_{I^2_{b+(k+1)a}(u)}.$$

First, we discuss $t_N(g_1^2)$. Recall that $y \in I_{b+ka+j}(z^1) \times I_{b+(k+1)a-j}(z^2)$. Then, $F_{N_1}(y^1 - x^1) = F_{N_1}(0) = N_1 - \frac{N_1}{l_{N_1}}$. So,

$$\frac{N_1}{2} \leq F_{N_1}(0) \leq N_1, \; D_{2^{b+(k+1)a-j}}(y^2 - x^2) = 0 \text{ for } x^2 \in I_{b+(k+1)a}(u^2).$$

It follows that

$$\begin{aligned}
|t_N(g_1^2)(y)| \leq & \sum_{\substack{u \in \Lambda_k \\ u^1 = z^1, u^2 \neq z^2}} \int_{I^2_{b+(k+1)a}} 2^a \cdot 2^{b+ka+j} \\
& \times \left(\frac{1}{l_{N_2}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2 - x^2)|}{s} + \frac{1}{l_{N_2}} \sum_{s=2^{b+ka}}^{N_2-1} \frac{|K_s(y^2 - x^2)|}{s} \right. \\
& \left. + \frac{|K_{N_2}(y^2 - x^2)|}{l_{N_2}} \right) d\mu(x^1, x^2) \\
= & I + II + III.
\end{aligned}$$

If one takes two different $u, \tilde{u} \in \Lambda_k, u^1 = z^1, u^2 \neq z^2, \tilde{u}^1 = z^1, \tilde{u}^2 \neq z^2$, then

$I_{b+ka}(u^2) \cap I_{b+ka}(\tilde{u}^2) = \emptyset$. Hence

$$\begin{aligned}
I &\leq \sum_{\substack{u\in\Lambda_k\\ u^1=z^1, u^2\neq z^2}} 2^j \int_{I_{b+(k+1)a}(u^2)} \frac{1}{l_{N_2}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2-x^2)|}{s}\, d\mu(x^2) \\
&= \sum_{\substack{u\in\Lambda_k\\ u^1=z^1, u^2\neq z^2}} 2^j 2^{-b-(k+1)a} \frac{1}{l_{N_2}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2-u^2)|}{s} \\
&= 2^{-a} 2^j \sum_{\substack{u\in\Lambda_k\\ u^1=z^1, u^2\neq z^2}} \int_{I_{b+ka}(u^2)} \frac{1}{l_{N_2}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2-t)|}{s}\, d\mu(t) \\
&\leq 2^{j-a} \int_0^1 \frac{1}{l_{2^{b+ka}}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2-t)|}{s} d\mu(t) \leq C 2^{j-a} \leq C 2^{-\gamma}.
\end{aligned}$$

Similarly

$$\begin{aligned}
II &\leq \sum_{\substack{u\in\Lambda_k\\ u^1=z^1, u^2\neq z^2}} 2^j \int_{I_{b+(k+1)a}(u^2)} \frac{1}{l_{N_2}} \sum_{s=2^{b+ka}}^{2^{b+(k+1)a-j}-1} \frac{|K_s(y^2-u^2)|}{s}\, d\mu(x^2) \\
&= \sum_{\substack{u\in\Lambda_k\\ u^1=z^1, u^2\neq z^2}} \int_{I_{b+(k+1)a-j}(u^2)} \frac{1}{l_{N_2}} \sum_{s=2^{b+ka}}^{2^{b+(k+1)a-j}-1} \frac{|K_s(y^2-t)|}{s} d\mu(t) \\
&\leq \int_0^1 \frac{1}{l_{N_2}} \sum_{s=2^{b+ka}}^{2^{b+(k+1)a-j}-1} \frac{|K_s(y^2-t)|}{s} d\mu(t) \\
&\leq C \frac{\log 2^{a-j}}{\log N_2} \leq C \frac{a}{b+ka} \leq C \frac{a}{b}.
\end{aligned}$$

Also, we get

$$\begin{aligned}
III &\leq \sum_{\substack{u\in\Lambda_k\\ u^1=z^1, u^2\neq z^2}} 2^j 2^{-j} \int_{I_{b+(k+1)a-j}(u^2)} \frac{1}{l_{N_2}} |K_{N_2}(y^2-t)|\, d\mu(t) \\
&\leq \frac{1}{l_{N_2}} \|K_{N_2}\|_1 \leq \frac{C}{b}.
\end{aligned}$$

Consequently, $|t_N(g_1^2)(y)| \leq C(2^{-\gamma} + \frac{a}{b})$. Since $\gamma \leq j \leq a-\gamma$, we also have the same bound for $|t_N(g_2^2)(y)|$.

Next, we discuss $|t_N(g_3^2)(y)|$. If $u \in \Lambda_k, u^i \neq z^i$, then we have $D_{N_i}(y^i - x^i) = 0$ $(i = 1, 2)$. (Recall that $N = (N_1, N_2) = (2^{b+ka+j}, 2^{b+(k+1)a-j})$.) Thus

$$\begin{aligned}
|F_N(y-x)| \le & \frac{1}{l_{N_1} l_{N_2}} \Bigg(\sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^1-x^1)|}{s} + \sum_{s=2^{b+ka}}^{2^{b+ka+j}-1} \frac{|K_s(y^1-x^1)|}{s} \\
& \qquad + |K_{2^{b+ka+j}}(y^1-x^1)| \Bigg) \\
& \times \Bigg(\sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2-x^2)|}{s} + \sum_{s=2^{b+ka}}^{2^{b+(k+1)a-j}-1} \frac{|K_s(y^2-x^2)|}{s} \\
& \qquad + |K_{2^{b+(k+1)a-j}}(y^2-x^2)| \Bigg) \\
=: & \frac{1}{l_{N_1} l_{N_2}} (J_1^L + J_2^L + J_3^L)(J_1^R + J_2^R + J_3^R)
\end{aligned}$$

First, discuss the case given by $J_1^L J_1^R$. We have the upper bound $\frac{C}{2^a}$.

$$\begin{aligned}
& \sum_{\substack{u \in \Lambda_k \\ u^i \neq z_i, i=1,2}} \int_{I^2_{b+(k+1)a}(u)} \frac{2^a}{l_{N_1} l_{N_2}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^1-x^1)|}{s} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2-x^2)|}{s} d\mu(x^1, x^2) \\
&= \sum_{\substack{u \in \Lambda_k \\ u^i \neq z_i, i=1,2}} \frac{1}{2^{2a}} \int_{I^2_{b+ka}(u)} \frac{2^a}{l_{N_1} l_{N_2}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^1-t^1)|}{s} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2-t^2)|}{s} \, d\mu(t^1, t^2) \\
&\le \frac{1}{2^a} \left(\int_0^1 \frac{1}{l_{N_1}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^1-t^1)|}{s} \, d\mu(t^1) \right) \left(\int_0^1 \frac{1}{l_{N_1}} \sum_{s=1}^{2^{b+ka}-1} \frac{|K_s(y^2-t^2)|}{s} \, d\mu(t^2) \right) \\
&\le \frac{C}{2^a}.
\end{aligned}$$

For the case given by $J_1^L(J_2^R+J_3^R)$ we have

$$\begin{aligned}
&\sum_{\substack{u\in\Lambda_k\\ u^i\neq z_i,i=1,2}} \left(\int_{I_{b+ka}(u^1)} \frac{1}{2^a}2^a\frac{1}{l_{N_1}}\sum_{s=1}^{2^{b+ka}-1}\frac{|K_s(y^1-u^1)|}{s}d\mu(x^1)\right)\\
&\times\frac{1}{2^j}\left(\int_{I_{b+(k+1)a-j}(u^2)}\left(\frac{1}{l_{N_2}}\sum_{s=2^{b+ka}}^{2^{b+(k+1)a-j}-1}\frac{|K_s(y^2-u^2)|}{s}\right.\right.\\
&\qquad\left.\left.+\frac{1}{l_{N_2}}|K_{2^{b+(k+1)a-j}}(y^2-u^2)|\right)d\mu(x^2)\right)\\
&\le\left(\int_0^1\frac{1}{l_{N_1}}\sum_{s=1}^{2^{b+ka}-1}\frac{|K_s(t^1)|}{s}d\mu(t^1)\right)\\
&\times\frac{1}{2^j}\left(\int_0^1\frac{1}{l_{N_2}}\sum_{s=2^{b+ka}}^{2^{b+(k+1)a-j}-1}\frac{|K_s(t^2)|}{s}+\frac{1}{l_{N_2}}|K_{2^{b+(k+1)a-j}}(t^2)|d\mu(t^2)\right)\\
&\le C\frac{1}{2^j}\frac{a}{b}.
\end{aligned}$$

The same bound can be given for the case determining by $(J_2^l+J_3^L)J_1^R$, and for the case given by $(J_2^l+J_3^L)(J_2^R+J_3^R)$ we have $C\frac{a^2}{b^2}$. That is, we verified the inequality

$$|t_Ng_k(y)|\ge 2^{-2}-C\left(2^{-\gamma}+\frac{a}{b}+\left(\frac{a}{b}\right)^2\right).$$

In the sequel we give a bound for the logarithmic means of the other g_l's.

$$\begin{aligned}
|t_N(\sum_{l=k+1}^d g_l)(y)| &\le \sum_{l=k+1}^d\sum_{u\in\Lambda_l}\int_{I^2_{b+(l+1)a}(u)}2^a|F_N(y-x)|\,d\mu(x^1,x^2)\\
&=\sum_{l=k+1}^d\sum_{u\in\Lambda_l}\frac{1}{2^a}\int_{I^2_{b+la}(u)}|F_N(y-x)|\,d\mu(x^1,x^2)\\
&\le\frac{1}{2^a}\int_{I^2}|F_N|\le\frac{C}{2^a}.
\end{aligned}$$

It remains to investigate $|t_N(\sum_{l=0}^{k-1}g_l)(y)|$. We have $y\notin I^2_{b+la}(u)$ for all $u\in\Lambda_l$, $l=0,\dots,k-1$ (and even for all $l\neq k$). Then we may suppose that for instance $y^2 \notin I_{b+la}(u^2)$ which gives

$$D_{2^{b+(k+1)a-j}}(y^2 - x^2) = 0 \text{ for all } x^2 \in I_{b+la}(u^2).$$

As a result of this we get

$$F_{N_2}(y^2 - x^2) = \frac{1}{l_{N_2}} \omega_{2^{b+(k+1)a-j}-1}(y^2 - x^2) \sum_{s=1}^{N_2-1} \frac{D_s(y^2 - x^2)}{s}.$$

Since $x^2 \in I_{b+(l+1)a}(u^2)$ and $y^2 \notin I_{b+la}(u^2)$, we have $y^2 - x^2 \in I_\tau \setminus I_{\tau+1}$ for some $\tau < b + la$. By this fact we have

$$D_s(y^2 - x^2) = \omega_s(y^2 - x^2) \left(\sum_{i=0}^{\tau-1} s_i 2^i + (-1)^{s_\tau} 2^\tau \right).$$

Moreover,

$$\begin{aligned}
&\int_{I^2_{b+(l+1)a}(u)} 2^a F_{N_1}(y^1 - x^1) F_{N_2}(y^2 - x^2)\, d\mu(x^1, x^2) \\
=&\left(\int_{I_{b+(l+1)a}(u^1)} 2^a F_{N_1}(y^1 - x^1)\, d\mu(x^1) \right) \\
&\times \int_{I_{b+(l+1)a}(u^2)} \frac{1}{l_{N_2}} \sum_{s=1}^{N_2-1} \omega_{2^{b+(k+1)a-j}-1}(y^2 - x^2) \\
&\times \omega_s(y^2 - x^2) \left(\sum_{i=0}^{\tau-1} s_i 2^i + (-1)^{s_\tau} 2^\tau \right) \frac{1}{s}\, d\mu(x^2).
\end{aligned}$$

If we want this integral with respect to the second variable to be different from zero, then the following equalities must hold $s_{b+(k+1)a-j-1} = s_{b+(k+1)a-j-2} = \cdots = s_{b+(l+1)a} = 1$. This means

$$2^{b+(k+1)a-j-1} + 2^{b+(k+1)a-j-2} + \cdots + 2^{b+(l+1)a} \le s < 2^{b+(k+1)a};$$

that is,

$$2^{b+(k+1)a-j} - 2^{b+(l+1)a} \le s < 2^{b+(k+1)a}.$$

It is easy to have by the help of the Abel transformation

$$\begin{aligned}\left|\sum_{s=2^{b+(k+1)a-j}2^{b+(l+1)a}}^{2^{b+(k+1)a-j-1}}\frac{D_s}{s}\right| &= \left|\sum_{s=2^{b+(k+1)a-j}2^{b+(l+1)a}}^{2^{b+(k+1)a-j-1}}\frac{K_s}{s}+K_{2^{b+(k+1)a-j-1}}\right| \\ &\leq C\sup_{s\geq 2^{b+ka}}|K_s|\sum_{s=2^{b+(k+1)a-j}2^{b+(l+1)a}}^{2^{b+(k+1)a-j-1}}\frac{1}{s} \\ &\leq C\sup_{s\geq 2^{b+ka}}|K_s|\log\frac{1}{1-2^{(l-k)a+j}} \\ &\leq C\sup_{s\geq 2^{b+ka}}|K_s|\log\frac{1}{1-2^{-a+j}} \\ &\leq C\sup_{s\geq 2^{b+ka}}|K_s|\log\frac{1}{1-2^{-\gamma}}\leq C\sup_{s\geq 2^{b+ka}}|K_s|.\end{aligned}$$

This gives

$$\begin{aligned}\left|t_N(\sum_{l=0}^{k-1}g_l)(y)\right| \leq &\left|\sum_{l=0}^{k-1}\sum_{\substack{u\in\Lambda_l\\ y^2\notin I_{b+la}(u^2)}}\int_{I^2_{b+(l+1)a}(u)}2^aF_{N_1}(y^1-x^1)F_{N_2}(y^2-x^2)\,dx\right| \\ &+\left|\sum_{l=0}^{k-1}\sum_{\substack{u\in\Lambda_l\\ y^1\notin I_{b+la}(u^1)}}\int_{I^2_{b+(l+1)a}(u)}2^aF_{N_1}(y^1-x^1)F_{N_2}(y^2-x^2)\,dx\right| \\ =:\ & I+II.\end{aligned}$$

We have the same bound for I and II. Namely,

$$I\leq C2^a\int_{I\times\bar{I}_{b+ka}}|F_{N_1}(t^1)|\frac{1}{l_{N_2}}\sup_{s\geq 2^{b+ka}}|K_s(t^2)|\mu(t^1,t^2)\leq C2^a\frac{1}{l_{N_2}}\leq C2^a\frac{1}{b}.$$

So, at last we get

$$|t_N(f^d_{a,b})(y)|\geq 2^{-2}-C(2^{-\gamma}+\frac{a}{b}+\frac{a^2}{b^2}+\frac{1}{2^a}+\frac{2^a}{b})\geq 2^{-3}. \qquad \square$$

5 Proof of Main Results.

Using Lemma 2 the proof of Theorem 1 is the same as the proof of Theorem 2 in the paper of Simon [18]. Since the Walsh polynomials are dense in $H^{\#}\left(I^2\right)$, Corollary 1 implied by a usual density argument [13].

PROOF OF THEOREM 2. We give the definition of the sequences $(a_n), (b_n)$, (d_n), $(\beta_n), (\delta_n)$. First, for all the elements of these sequences must hold

$$C(2^{-\gamma} + \frac{a_n}{b_n} + \frac{a_n^2}{b_n^2} + \frac{1}{2^{a_n}} + \frac{2^{a_n}}{b_n}) \leq \frac{1}{8}. \tag{3}$$

Furthermore, these sequences are defined recursively in the following way (β_0), (δ_0), (a_0), (b_0), (d_0) are defined arbitrary, but satisfying (3). For $n \in \mathbb{N}$

$$\begin{aligned}
&\beta_n := C \max(n, \sum_{k=0}^{n-1} \beta_k 2^{a_k}) \text{ for a suitable but fixed } C \in \mathbb{N},\\
&\delta_n := [\sup\{t \in \mathbb{R} : \delta(t) > \frac{1}{2^n \beta_n}\}] + 1\\
&\quad (\text{if } \{t : \delta(t) > \frac{1}{2^n \beta_n}\} = \emptyset, \text{ then let } \delta_n = \delta_0),\\
&a_n := \max(2^{n+1}, \delta_n, \log^+(2\beta_n)),\\
&b_n := C \max(b_{n-1} + (d_{n-1}+1)a_{n-1} + 2, 2^{a_n}) \text{ for a suitable but fixed}\\
&\qquad 2 \leq C \in \mathbb{N}, \text{ and } d_n \in \mathbb{N} \text{ be such that } \left(1 - \frac{a_n}{2^{a_n}}\right)^{d_n} < \frac{1}{2^{n+1}}.
\end{aligned} \tag{4}$$

Define the function f as $f = \sum_{n=0}^{\infty} \beta_n f_n := \sum_{n=0}^{\infty} \beta_n f_{a_n,b_n}^{d_n}$. The proof of the fact that $f \in L \log^+ L\, \delta(L)$ is same as proof of Lemma 1 Gát in [2]. Set

$$G_{a_n,b_n,d_n} := \bigcup_{\substack{t_i^1, t_i^2 \in \{0,1\}\\ i=0,1,\ldots,b_n-1}} \bigcup_{j=0}^{d_n} \widetilde{\Omega}_{a_n,b_n}^j(t)$$

for all $n \in \mathbb{N}$ and $G := \liminf G_{a_n,b_n,d_n} = \cup_{k\in\mathbb{N}} \cap_{n \geq k} G_{a_n,b_n,d_n}$. Since

$$\begin{aligned}
&\mu\left(\bigcup_{j=0}^{\gamma-1} I_{b+ka+j}(x^1) \times I_{b+(k+1)a-j}(x^2)\right)\\
=&\mu\left(\bigcup_{j=a-\gamma+1}^{a} I_{b+ka+j}(x^1) \times I_{b+(k+1)a-j}(x^2)\right) = \frac{\gamma+1}{2^{2b+(2k+1)a+1}},
\end{aligned}$$

from (2) and (4) we have

$$\begin{aligned}\mu(G_{a_n,b_n,d_n}) :=&\mu\left(\bigcup_{\substack{t_i^1,t_i^2\in\{0,1\}\\ i=0,\dots,b_n-1}}\bigcup_{j=0}^{d_n}\widetilde{\Omega}^j_{a_n,b_n}(t)\right) = \mu\bigcup_{\substack{t_i^1,t_i^2\in\{0,1\}\\ i=0,\dots,b_n-1}}\bigcup_{j=0}^{d_n}\Omega^j_{a_n,b_n}(t)\\ &- 2\mu\left(\bigcup_{\substack{t_i^1,t_i^2\in\{0,1\}\\ i=0,\dots,b_n-1}}\bigcup_{k=0}^{d_n}\bigcup_{j=0}^{\gamma-1} I_{b_n+ka_n+j}(x^1)\times I_{b_n+(k+1)a_n-j}(x^2)\right)\\ =&1-\left(1-\frac{1+a_n/2}{2^{a_n}}\right)^{d_n+1}-\sum_{k=0}^{d_n}\frac{\gamma+1}{2^{(2k+1)a_n}}\\ =&1-\left(1-\frac{1+a_n/2}{2^{a_n}}\right)^{d_n+1}-\frac{\gamma+1}{2^{a_n}}\frac{2^{2a_n}-2^{-2a_nd_n}}{2^{2a_n}-1}\geq 1-\frac{C}{2^n}.\end{aligned}$$

Hence we can write that $\mu(G)=1$.

It remains to prove the "divergence". Namely, we verify that for a.e. $x\in I^2$

$$\limsup|t_{(2^{n_1},2^{n_2})}f(x)| = +\infty \quad (\min(n_2,n_2)\to\infty).$$

This is the same, as $|\sup_{n_1,n_2\in\mathbb{N}} t_{(2^{n_1},2^{n_2})}f(x)| = +\infty$. Let $y\in G$ (recall that $\mu(I^2\setminus G)=0$). Then there are infinitely many $n\in\mathbb{N}$ for which (even for all, but finitely many) $y\in G_{a_n,b_n,d_n}$. Then Lemma 3 gives that there exists a $t\in I^2$, $k\leq d_n$ such that $y\in\widetilde{\Omega}^k_{a_n,b_n}(t)$. Whereby, there is a $z\in J^k_{a_n,b_n}(t)$, a unique $j\in\{\gamma,\gamma+1,\dots,a_n-\gamma\}$ for which $y\in I_{b_n+ka_n+j}(z^1)\times I_{b_n+(k+1)a_n-j}(z^2)$. Set $N:=(N_1,N_2):=(2^{b_n+ka_n+j},2^{b_n+(k+1)a_n-j})\in\mathbb{N}^2$. By Lemma 3 we have

$$|t_{(N_1,N_2)}f^{d_n}_{a_n,b_n}(x)|\geq\frac{1}{8}.$$

In [9] one finds that $\|F_n\|_1\leq C$ for $n\in\mathbb{N}$ which in the standard way (see e.g. [17]) gives $\|t_nf\|_\infty\leq C\|f\|_\infty$ ($f\in L^1(I^2)$, $n\in\mathbb{N}^2$). This follows for $1\leq i\in\mathbb{N}$

$$\|t_Nf^{d_{n-i}}_{a_{n-i},b_{n-i}}\|_\infty\leq C\|f^{d_{n-i}}_{a_{n-i},b_{n-i}}\|_\infty\leq C\cdot 2^{a_{n-i}}.$$

The definition of the function $f^d_{a,b}$ gives that $E_{(b_n^\circ,b_n^\circ)}f^{d_n}_{a_n,b_n}=0$. Hence, for $1\leq s\in\mathbb{N}$ (Recall that $N=(2^{b_n+ka_n+j},2^{b_n+(k+1)a_n-j})$.)

$$t_N(f^{d_{n+s}}_{a_{n+s},b_{n+s}})(y)=t_N(E_{(b_n+(k+1)a_n,b_n+(k+1)a_n)}f^{d_{n+s}}_{a_{n+s},b_{n+s}})(y)=0,$$

because $E_{(b^\circ_{n+s},b^\circ_{n+s})}f^{d_{n+s}}_{a_{n+s},b_{n+s}}=0$, and (Use (4), and recall that $b^\circ_{n+1}:=[b_{n+1}/2]-1$.)

$$b_n+(k+1)a_n\leq b_n+(d_n+1)a_n\leq b^\circ_{n+1}\leq b^\circ_{n+s}.$$

For all $1 \le s \in \mathbb{N}$ this gives that $t_{(N_1,N_2)} f_{a_{n+s},b_{n+s}}^{d_{n+s}}(y) = 0$. Consequently, by (4) we have

$$\begin{aligned} |t_{(N_1,N_2)} f(y)| &\ge |t_{(N_1,N_2)} \beta_n f_{a_n,b_n}^{d_n}(y)| - \sum_{i=1}^{n-1} |t_{(N_1,N_2)} \beta_i f_{a_i,b_i}^{d_i}(y)| \\ &\ge \frac{1}{8}\beta_n - C\sum_{i=0}^{n-1} \beta_i \|f_{a_i,b_i}^{d_i}\|_\infty \ge \frac{1}{8}\beta_n - C\sum_{i=0}^{n-1}\beta_i 2^{a_i} \ge \frac{1}{16}\beta_n \ge Cn. \end{aligned}$$

This completes the proof of the Theorem 2. □

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