Real Analysis Exchange
ISSN:0147-1937
Vol. 31(2), 2005/2006, pp. 409–424

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CLOSED RELATIONS AND EQUIVALENCE CLASSES OF QUASICONTINUOUS FUNCTIONS

Abstract

This paper introduces a notion of equivalence that links closed relations and quasicontinuous functions; we examine classes of quasicontinuous functions that have the same set of continuity points. In doing so, we show that every minimal closed relation is the closure of a quasicontinuous function and vice-versa.

We also show that this notion is of use in dynamical systems. Every quasicontinuous function is equivalent to one that is measurable, and under certain circumstances—in fact, under just those circumstances that appear most often in the dynamics literature—it is equivalent to a quasicontinuous function that has an invariant measure.

1 Introduction.

This paper introduces a notion of equivalence that links closed relations and quasicontinuous functions; we also show that this notion is of use in dynamical systems. Quasicontinuous functions have been of interest to analysts for some seven decades, and much is known about the structure of these functions; see for example [5]–[12], [14], [16], [24], [17], [21], [25], [27], and [29]. Similarly, there is a rich literature on closed relations; see [1], which greatly influenced this article, [3], [19], and [28]. We connect these notions by examining classes of quasicontinuous functions that have the same set of continuity points. In

Key Words: closed relations, quasicontinuity, quasicontinuous functions, measure
Mathematical Reviews subject classification: 54C08, 54H20
Received by the editors July 1, 2005
Communicated by: Zbigniew Nitecki

doing so, we show that there is a natural connection between quasicontinuous functions and closed relations; namely, that every minimal closed relation is the closure of a quasicontinuous function and vice-versa.

Topological dynamicists have often studied special cases of functions with "mild" forms of discontinuities. The papers [26] and [13] demonstrate, among other things, the existence of a finite number of absolutely continuous invariant measures for expanding interval maps with a finite number of discontinuities. Many people have investigated the dynamics of interval exchange maps and their higher-dimension relatives (see [30], [31], [32], and the survey article [18] for example). These special cases fall into the larger class of quasicontinuous functions, and we show that equivalence classes of quasicontinuous functions have some useful applications for the study of dynamical systems.

One such application comes from iterating quasicontinuous and quopen functions; we extend the well-known result that the continuity set of a quasicontinuous function is residual. A second such application is in measure-theoretic dynamics. We show that, although a quasicontinuous function might not be measurable, it is always equivalent to a quasicontinuous function that is measurable. Also, we show that not every quasicontinuous function has an invariant measure (We give an example of such a function with only one discontinuity.), but under certain circumstances—in fact, under just those circumstances that appear most often in the dynamics literature—it is equivalent to a quasicontinuous function that does have an invariant measure.

2 Quasicontinouous Functions, Relations, and Equivalence Classes.

For any topological space X, we say $A \subset X$ is *quasi-open* if $A \subset \mathrm{cl}(\mathrm{int}(A))$. Equivalently, A is quasi-open if, for every non-empty, open set $U \subset X$, we have either $U \cap A = \emptyset$ or $\mathrm{int}(U \cap A) \neq \emptyset$. If X and Y are topological spaces, we say $f : X \to Y$ is *quasicontinuous* if, for every open $V \subset Y$, the set $f^{-1}(V)$ is quasi-open. (In some places in the literature, these are called "semi-open" and "semi-continuous" respectively; quasicontinuity should not however be confused with upper or lower semi-continuity.) We let $C_f = \{x \in X \mid f \text{ is continuous at } x\}$, and likewise $D_f = \{x \in X \mid f \text{ is discontinuous at } x\}$. It is well known (see for example [16]) that if Y is second countable and f is quasicontinuous, the set C_f is *residual*; that is, D_f is first category. When X is a compact metric space, as we assume hereafter, residual sets are precisely those that contain dense G_δ sets.

In everything that follows, X, Y, and Z are compact metric spaces. A *relation* $F : X \to Y$ (or $F \subset X \times Y$) is a subset of $X \times Y$ such that for every

$x \in X$, the set

$$F(x) = \{y \in Y \mid (x, y) \in F\}$$

is non-empty. We say that F is *closed* if it is closed as a subset of $X \times Y$ in the product topology. As with functions, we can perform composition. If $G : Y \to Z$, then

$$G \circ F = \{(x, z) \in X \times Z \mid z \in G(y) \text{ for some } y \in F(x)\}.$$

A composition of closed relations is closed. We can consider inverses of points;

$$F^{-1}(y) = \{x \in X \mid y \in F(x)\}$$

and of sets;

$$F^{-1}(A) = \{x \in X \mid A \cap F(x) \neq \emptyset\}.$$

Notice that for functions we have $f(f^{-1}(A)) = A$ whenever $A \subset f(X)$, but for relations we have $F(F^{-1}(A)) \supset A$.

The graph of any function $f : X \to Y$ is an example of a relation, and in this case we will write f to represent both the function and its graph. For this reason, we can write $\bar{f} = \mathrm{cl}(f)$ for the closure of the graph of f in the product topology of $X \times Y$. Accordingly, we have $\bar{f}(x) = \{y \mid (x, y) \in \bar{f}\}$. Thus we can say that $\bar{f}$ is *single-valued* (resp. *multivalued*) at x if card $\bar{f}(x) = 1$ (resp. card $\bar{f}(x) > 1$). If F is a relation, then a *selection function of* F is any function $f \subset F$; that is, for every $x \in X$, $f(x) \in F(x)$. Clearly, f is always a selection function of $\bar{f}$. If f is a continuous function, then $\bar{f} = f$, and f is the only selection function of $\bar{f}$.

Given a closed relation $F : X \to Y$, let $S_F \subset X$ denote the *single-valued set of* F; that is, $S_F = \{x \in X \mid F(x) \text{ is a singleton}\}$.

Proposition 1.

(i) *The single-valued set of the closure $\bar{f}$ of any function $f : X \to Y$ is exactly the set of continuity points of f; that is, $S_{\bar{f}} = C_f$.*

(ii) *If f is a selection function of F, then $S_F \subset C_f$.*

(iii) *If g is a selection function of $\bar{f}$, then $C_f \subset C_g$.*

PROOF. Note that (ii) follows immediately from (i), since $\bar{f} \subset F$ implies $S_F \subset S_{\bar{f}}$. Similarly, (iii) follows immediately from (i) and (ii). It remains to prove (i). To show that $S_{\bar{f}} \subset C_f$, let $x \in S_{\bar{f}}$ and suppose by way of contradiction that $x \notin C_f$. By the compactness of $\bar{f}$, there exists a sequence $x_n \to x$ such that $(x_n, f(x_n)) \to (x, y)$ in $\bar{f}$, with $y \neq f(x)$. But this contradicts the assumption that $\bar{f}$ is single-valued at x. We leave the proof that $C_f \subset S_{\bar{f}}$ as an easy exercise. $\square$

Definition. We say the functions $f : X \to Y$ and $g : X \to Y$ are *equivalent* (or $f \sim g$) if $\bar{f} = \bar{g}$ in $X \times Y$. The *equivalence class* of f is denoted $[f]$.

Theorem 2. *Let $f, g : X \to Y$ be quasicontinuous functions. Then we have*

(i) $f \sim g$ *if and only if* $C_f = C_g$ *and* $f|_{C_f} = g|_{C_g}$.

(ii) *Let* $h : X \to Y$. *If* $h \subset \bar{f}$, *then* h *is quasicontinuous and* $h \sim f$.

(iii) $\bar{f} = \displaystyle\bigcup_{h \sim f} h$.

PROOF. (i) Suppose first that $C_f = C_g$ and $f|_{C_f} = g|_{C_g}$, and let $(x, y) \in \bar{f}$. By the quasicontinuity of f, any neighborhood $U \times V$ of (x, y), with U, V open in X and Y, respectively, satisfies $W = \operatorname{int}(U \cap f^{-1}(V)) \neq \emptyset$. Since C_f is dense, the open set W contains a point $x_0 \in C_f$, at which f and g agree. Thus $(x_0, g(x_0)) \in U \times V$, which proves that $(x, y) \in \bar{g}$ and hence $\bar{f} \subset \bar{g}$. By the symmetry of the argument, $\bar{g} \subset \bar{f}$, so $\bar{f} = \bar{g}$.

Conversely, suppose $\bar{f} = \bar{g}$. Then $\bar{f}$ and $\bar{g}$ are single-valued on the same subset C of X, and by Proposition 1 (i), $C = C_f = C_g$ and $f|_{C_f} = g|_{C_g}$.

(ii) Clearly $\bar{h} \subset \bar{f}$. Moreover, since $\bar{f}$ is single-valued on C_f, f and h agree on C_f, and the same argument as in (i) shows that $\bar{f} \subset \bar{h}$; thus $\bar{h} = \bar{f}$. It remains to show that h is quasicontinuous. Let $(x_0, y_0) \in h$, let U and V be arbitrary neighborhoods of x_0 and y_0, respectively. Since $(x_0, y_0) \in \bar{f}$ and $U \times V$ is a neighborhood of (x_0, y_0), there exists a point $(x_1, y_1) \in f \cap (U \times V)$, and thus the set $W = \operatorname{int}(U \cap f^{-1}(V))$ is nonempty. But as in (i), the open set W must contain a point $x_2 \in C_f$, at which f and h agree. Thus $(x_2, h(x_2)) \in U \times V$, and since by Proposition 1 (iii) x_2 is also a continuity point of h, there exists a neighborhood $U_2 \subset U$ of x_2 such that $h(U_2) \subset V$. Hence h is quasicontinuous.

(iii) Let $(x_0, y_0) \in \bar{f}$, and define $h_0 : X \to Y$ by $h_0(x) = f(x)$ for $x \neq x_0$, and $h_0(x_0) = y_0$. Then $h_0 \subset \bar{f}$, and by (ii), we have $h_0 \sim f$. Thus $\bar{f} \subset \bigcup_{h \sim f} h$. To show the reverse inclusion, observe that by (ii), each $h \sim f$ satisfies $\bar{h} = \bar{f}$, and thus $h \subset \bar{f}$. $\square$

Remarks. Property (iii) says that $\bar{f}$ is a picture of the entire equivalence class $[f]$; that is, the graph of $\bar{f}$ is the union of the graphs of all the functions equivalent to f.

Property (ii) says that once the relation $\bar{f}$ is determined, any function we select from it will be quasicontinuous.

Though $\bar{f}$ and $[f]$ are closely related (that is, $f \in [g] \iff f \subset \bar{g}$), the set $\bar{f}$ is a set of points (elements of $X \times Y$), while the set $[f]$ is a set of functions (subsets of $X \times Y$). Using Theorem 2, we can combine notations to write

$$\bar{f} = \bigcup_{g \in [f]} g \text{ and } [f] = \bigcup_{g \subset \bar{f}} \{g\}.$$

We conclude this section with a lemma, theorem, and corollaries that tell us that quasicontinuous functions are, in a sense, the most fundamental class for which this notion of equivalence applies.

Lemma 3. *Let D be a dense subset of X and suppose $g : D \to Y$ is continuous. Then any selection function of the closed relation $\bar{g} : X \to Y$ is quasicontinuous.*

PROOF. Because D is dense and Y is compact, $\bar{g}$ is a relation from X to Y. Now let f be a selection function of $\bar{g}$, so that $\bar{f} \subset \bar{g}$. We take it as obvious that $\bar{g}$ is single-valued on D, and hence so is $\bar{f}$. By Proposition 1 (i) we then have $D \subset C_f$. To show that f is quasicontinuous, let $U \subset X$ and $V \subset Y$ be arbitrary open sets, and suppose there exists a point $(x_0, f(x_0)) \in U \times V$. Since $(x_0, f(x_0)) \in \bar{g}$, there exists a point $(x_1, g(x_1)) = (x_1, f(x_1)) \in (U \times V)$. But $x_1 \in D$ implies $x_1 \in C_f$, and hence the set $U \cap f^{-1}(V)$ has nonempty interior. □

In 2003, Mikucka [27] defined a function $f \colon X \to Y$ to be *graph quasi-continuous* if the closed relation $\bar{f}$ contains the graph of a quasi-continuous function $g \colon X \to Y$. Our next theorem says that in fact *every* function defined on compact metric spaces is graph quasi-continuous.

Theorem 4. *Let $h : X \to Y$ (not necessarily a quasicontinuous function). Then there exists a quasicontinuous function $f : X \to Y$ such that $f \subset \bar{h}$.*

PROOF. Using an argument from Lemma 1.1 of [28], we define a function $s : X \to [0,1]$ as follows. Let $\mathcal{C} \subset [0,1]$ denote the Cantor set. By [22], Theorem 3.28, there exists a continuous surjective function $\pi : \mathcal{C} \to Y$. Then $\pi^{-1} \circ \bar{h} : X \to [0,1]$ is a closed relation, and the selection function $s \subset \pi^{-1} \circ \bar{h}$ defined by $s(x) = \max(\pi^{-1} \circ \bar{h}(x))$ is upper semi-continuous. Accordingly, C_s is residual and thus dense [20, p. 110]. By Lemma 3, there is a quasicontinuous function $r : X \to [0,1]$ such that $r \subset \bar{s}$. Now define $f = \pi \circ r : X \to Y$. Then f is quasicontinuous because it is the composition of a continuous function with a quasicontinuous one. To see that $f \subset \bar{h}$, note that $r \subset \pi^{-1} \circ \bar{h}$, which implies $f = \pi \circ r \subset \bar{h}$. □

Corollary 5. *Every closed relation contains a quasicontinuous selection function. If F is a minimal closed relation (meaning that $G \subset F$ is a closed relation $\Rightarrow G = F$), then all selection functions of F are equivalent and quasicontinuous, and therefore F is single-valued on a residual set.*

PROOF. Let F be a closed relation, and choose a selection function $f \subset F$. By Theorem 4, there exists a quasicontinuous $g \subset \bar{f} \subset F$. If F is minimal, then $\bar{g} = F = \bar{f}$, and so by Theorem 2 (ii), f is quasicontinuous and equivalent to g. □

Corollary 6. *If a closed relation F is single-valued on a dense subset S_F, then there exists a unique equivalence class of quasicontinuous functions that are selection functions of F. That is, there is a class $[f]$ such that if g is quasicontinuous and $g \subset F$, then $g \in [f]$.*

PROOF. We repeat the argument used in the proof of Theorem 4. Define $B \subset X \times Y$ by $B = \{(x, F(x)) \mid x \in S_F\}$. The closure $\mathrm{cl}(B) \subset X \times Y$ is a minimal relation, and by Corollary 5 contains a unique equivalence class $[f]$ of quasicontinuous selection functions. Now suppose $g \subset F$ is quasicontinuous. By Proposition 1 (ii) $S_F \subset C_g$, and accordingly $f \subset \mathrm{cl}(B) \subset \bar{g}$. By Theorem 2 (ii) we see that $g \sim f$. □

Note that in this case, it does not necessarily hold that $\bar{g} = F$. Consider for example the closed relation defined by $F(0) = \{2, 3\}$ and $F(x) = 3$ for $x \in \mathbb{R} \setminus \{0\}$.

The next corollary uses relations and equivalence classes to provide an equivalent definition for quasicontinuous functions. To the best of our knowledge, this definition has not appeared before in the literature.

Corollary 7. *A function $f : X \to Y$ is quasicontinuous iff f has the property that $g \subset \bar{f} \Rightarrow f \subset \bar{g}$.*

PROOF. The forward direction of the statement follows from Theorem 2. Conversely, suppose $g \subset \bar{f} \Rightarrow f \subset \bar{g}$. By Theorem 4, we may choose a quasicontinuous $g \subset \bar{f}$; that $f \subset \bar{g}$ implies (by Theorem 2) that f is likewise quasicontinuous. □

3 Iterates of Quasicontinuous and Quopen Functions.

In [15], it was shown that some standard topological theorems for continuous dynamical systems can be extended to quasicontinuous systems. Here we expand upon those results. Quasicontinuous functions have been studied extensively in the real analysis literature (see [24] and [25] and more recently [16] and [29]). In the dynamics literature, however, the functions studied usually have the additional property that they are *almost open*; that is, the forward image of a non-empty open set contains a non-empty open set.

Definition. We will say a function $f : X \to Y$ is *quopen* if f is quasicontinuous and if for every non-empty open $U \subset X$, there is a non-empty open subset $V \subset f(U)$.

A constant real-valued function is not quopen. And even continuous functions that are 1-1 and onto might not be quopen. Consider for example the function $f(x) = x$ with this twist. The domain is $\mathbb{R}$ with the discrete topology (every set is open), and the range is $\mathbb{R}$ with the usual topology.

The next theorem extends the well-known theorem that says that if f is quasicontinuous, then C_f is residual.

Theorem 8. *Let $f : X \to X$ be quopen. Let*

$$C_f^\infty = \{x \in X \mid f^k(x) \in C_f \quad \forall k \geq 0\}.$$

That is, if $x \in C_f^\infty$, then f is continuous at every point along the orbit of x, and accordingly f^k is continuous at x for every $k > 0$. Then C_f^∞ is residual.

Before we prove this theorem, we remark upon some of its consequences. The following corollary follows immediately from Proposition 1 and Theorems 4 and 8.

Corollary 9. *If $f : X \to X$ is quopen, then for every $k > 0$, there is a quasicontinuous function $g_k \subset \overline{f^k}$ that agrees with f^k on the residual set C_{f^k}; furthermore, $C_{f^k} \subset C_{g_k}$. By Corollary 6, the equivalence class $[g_k]$ is unique.*

Remark. The theorem and corollary hold even though f^k might not be quasicontinuous; in this case, $\overline{g_k}$ is strictly smaller than $\overline{f^k}$. As an example, consider $f(x) = x + 1/2$ for $0 \leq x \leq 1/2$ and $f(x) = x - 1/2$ for $1/2 < x \leq 1$. We get $f^2(x) = x$, except that $f^2(0) = 1$. In this case, $g_2(x) = x$ is a continuous function.

Another important point to note is that it is much stronger to say "f is continuous at $f^k(x)$ $\forall k \geq 0$" than to say "f^{k+1} is continuous at x $\forall k \geq 0$". Consider the function $f : [0, 4] \to [0, 4]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1) \cup [2, 3] \\ 2 & \text{if } x \in [1, 2) \cup (3, 4] \end{cases}$$

so that

$$f^2(x) = \begin{cases} 2 & \text{if } x \in [0, 1) \cup [2, 3] \\ 1 & \text{if } x \in [1, 2) \cup (3, 4]. \end{cases}$$

For $k \geq 1$, we have $f^{2k-1} = f$, $f^{2k} = f^2$, and $C_{f^k} = C_f = [0,4] \setminus \{1,2,3\}$. On the other hand, for every $k \geq 1$ and every $x \in [0,4]$, f is discontinuous at $f^k(x)$, since $f^k([0,4]) = \{1,2\} \subset D_f$.

In general it is not true that the composition of quasicontinuous functions has a residual continuity set. Consider for example $Y = [0,1] \times [-1,1]$ and functions $f : [0,1] \to Y$ and $g : Y \to \{-1,1\}$ given by

$$f(x) = (x,0)$$

and

$$g(x,y) = \begin{cases} 1 & \text{if } y > 0 \text{ or } (x,y) \subset \mathbb{Q} \times \{0\} \\ -1 & \text{if } y < 0 \text{ or } (x,y) \subset \mathbb{Q}^c \times \{0\}. \end{cases}$$

Both f and g are quasicontinuous (indeed, f is continuous), but $g \circ f$ is discontinuous everywhere.

To prove Theorem 8, we will use the following lemma.

Lemma 10. *Let X and Y be compact metric spaces and $f : X \to Y$ be quopen.*

(i) *If $D \subset Y$ is dense, then $f^{-1}(D)$ is dense in X.*

(ii) *If $A \subset Y$ is open and dense in Y, then $int(f^{-1}(A))$ is open and dense in X.*

(iii) *If $R \subset Y$ is residual in Y, then $f^{-1}(R)$ is residual in X.*

PROOF. (i) Choose a non-empty open $U \subset X$; it follows that $f(U)$ contains a non-empty open set $V \subset Y$. Because D is dense, $V \cap D \neq \emptyset$. Therefore $U \cap f^{-1}(D) \supset U \cap f^{-1}(V \cap D) \neq \emptyset$.

(ii) By the quasicontinuity of f, we have that $B = f^{-1}(A)$ is quasi-open, so that $\mathrm{cl}(\mathrm{int}(B)) \supset B$. Taking the closure of both sides we get

$$\mathrm{cl}(\mathrm{cl}(\mathrm{int}(B))) \supset \mathrm{cl}(B).$$

By (i), $B = f^{-1}(A)$ is dense, so that $\mathrm{cl}(B) = X$. Therefore, $\mathrm{cl}(\mathrm{int}(f^{-1}(A))) = X$; that is, $\mathrm{int}(f^{-1}(A))$ is dense.

(iii) R is residual means that $R \supset \cap A_n$, a countable intersection of open dense subsets. Then

$$f^{-1}(R) \supset f^{-1}(\cap A_n) = \bigcap f^{-1}(A_n) \supset \bigcap \mathrm{int}(f^{-1}(A_n)),$$

so by (ii) $f^{-1}(R)$ contains a countable intersection of open dense subsets of X; that is, $f^{-1}(R)$ is residual. $\square$

Proof of Theorem 8. For $n \geq 0$, let $C_f^n = \bigcap_{k=0}^n f^{-k}(C_f)$. Because C_f is residual, it follows that $f^{-1}(C_f)$ and accordingly C_f^1 is residual. By induction, we see that C_f^n is residual for every n, and indeed that $\bigcap_{k=0}^\infty f^{-k}(C_f)$ is residual also.

It remains to show that $C_f^\infty = \bigcap_{k=0}^\infty f^{-k}(C_f)$; that is, that this set describes points whose orbits are all continuity points of f, as defined in the statement of the theorem. If $x \in \bigcap_{k=0}^\infty f^{-k}(C_f)$, then $x \in f^0(C_f) = C_f$, so f is continuous at x. Moreover, $x \in f^{-1}(C_f) \Rightarrow f(x) \in C_f$ so f is continuous at $f(x)$. Similarly, f is continuous at $f^k(x)$ for every $k = 0, \ldots, n-1$. It follows from the definition of continuity that f^k is continuous at x for every $k = 1, \ldots, n$ and hence $C_f^\infty \supset \bigcap_{k=0}^\infty f^{-k}(C_f)$. The reverse inclusion is obvious. $\square$

4 Quasicontinuity and Measurability.

To further motivate the construction of equivalence classes of quasicontinuous functions, we give here three examples of quasicontinuous functions that have interesting measure-theoretic properties. The main purpose of the next section of this paper is to examine questions about measurability of quasicontinuous functions. Is every quasicontinuous function defined on a compact metric space Lebesgue measurable? If X is a compact metric space and $f : X \to X$ is quasicontinuous, does f have an invariant measure? The answers are well known to be "yes" if we replace "quasicontinuous" by "continuous" (see for example [1], [22]). However, the list below provides quasicontinuous counterexamples to these questions.

1. Define $p : [0,4] \to [0,4]$ by $p(x) = x/2 + 1$ for $x \in [0,2)$ and $p(x) = x/2$ for $x \in [2,4]$. This quopen function with a single discontinuity has no invariant measure (as we will show below).

2. Let $C \subset [0,1]$ be a Cantor set with Lebesgue measure $1/2$, as constructed, for example, in [2]. The construction imitates that of the Cantor middle thirds set, except at stage $n \in \mathbb{N}$, we remove 2^{n-1} middle open intervals of length 4^{-n}. Observe that $[0,1] \setminus C$ can be written as a countable union
$$[0,1] \setminus C = \bigcup_{k=1}^{\infty} I_k$$
of deleted open intervals I_k with respective lengths $4^{-n(k)}$, where $n(k) \in$

$\mathbb{N}$. We now define $t : (0,1) \to \{0,1\}$ by

$$t(x) = \begin{cases} 1 & \text{if } x \in I_k \text{ and } n(k) \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Here $D_t = C$ has Lebesgue measure $1/2$. It is clear that t is quasicontinuous, because every neighborhood $U \times V$ of a point $(x, t(x))$ contains a line segment $I_k \times \{t(x)\} \subset t$, and hence $I_k \subset t^{-1}(V)$.

3. Refer to the previous example. Choose any set $E \subset C$ which is not Lebesgue measurable. Define $r : (0,1) \to \{0,1\}$ by

$$r(x) = \begin{cases} 1 & \text{if } x \in E, \text{ or if } x \in I_k \text{ and } n(k) \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

We see that again $D_r = C$; in this case, however, r is not a measurable function because $r^{-1}(\{1\})$ is not a measurable set. It is again easy to show that r is quasicontinuous.

Here we show that the function p in Example 1 has no invariant measure. Suppose on the contrary that m is a p-invariant measure. Then $p^{-1}([0,1)) = p^{-1}((2,4]) = \emptyset$, so $m([0,1) \cup (2,4]) = m(\emptyset) = 0$. By noting that $p^{-1}(\{2\}) = \{4\}$ and $p^{-1}(\{1\}) = \{0,2\}$, we get

$$m([0,1] \cup [2,4]) = 0.$$

Similarly, $p^{-1}([1,3/2]) = [0,1] \cup [2,3]$, so $m([1,3/2]) = 0$; $p^{-1}([1,7/4]) = [0,3/2] \cup [2,5/2]$, so $m([1,7/4]) = 0$; ... Continuing in this fashion, we may determine that $m([0,4]) = 0$, so p has no invariant measure.

5 Equivalence Classes and Measurability.

Every continuous function on a compact space is Borel measurable. Although Example 3 above shows that the same is not true for quasicontinuous functions, we nonetheless have the following generalization.

Proposition 11. *If X and Y are compact metric spaces and $f : X \to Y$ is quasicontinuous, then there exists a Borel measurable $g : X \to Y$ with $g \sim f$.*

Proof. The proof follows as a direct consequence of Theorem 2 and the fact that every closed relation on compact metric spaces has a measurable selection function (see [3] and [28]). □

In this section, we will make heavy use of the following definition and theorem that guarantee the existence of an invariant measure for closed relations.

Definition. For a compact metric space X and a relation $F : X \to X$, we say the measure m on X is *invariant under* F (or F*-invariant*) if $m(A) \leq m(F^{-1}(A))$ for every Borel set $A \subset X$.

If F is single-valued everywhere (implying F is a function), then by applying this definition to both A and A^c we see that this notion agrees with the usual definition of invariant measure. The following theorem is from [3, Theorem 2.1].

Theorem 12. (Aubin, Frankowska, and Lasota) *If $F : X \to X$ is a closed relation on a compact metric space, then there exists an F-invariant measure on X.*

In the case that F is single-valued everywhere (implying F is a continuous function), then this is known as the Krylov-Bogolyubov theorem.

In the previous section, we gave examples of a quasicontinuous function that has no invariant measure. A natural question to ask is, "Is there a member of the equivalence class with an invariant measure?" We give two different instances in which the answer is "yes". Dynamics papers on piecewise expanding maps, interval exchange maps, and their generalizations often implicitly use the following proposition in those cases when the discontinuity set D_f is invisible to the measure.

Proposition 13. *If $f : X \to X$ is a quasicontinuous function on a compact metric space, and if there exists an $\bar{f}$-invariant measure m satisfying $m(D_f) = 0$, then m is also an f-invariant measure.*

The example of the function p above shows that the hypotheses of this theorem do not always hold. On the other hand, for that particular example it is clear that there is a quasicontinuous function $\tilde{p} \sim p$ (with $\tilde{p}(2) = 2$) that has an invariant measure; namely, the point measure supported on $\{2\}$. The next proposition generalizes this example.

Proposition 14. *Let X be a compact metric space, and suppose $F : X \to X$ is a closed relation that is single-valued except on a countable set $D \subset X$. Then there exists a selection function in F that has an invariant measure. From this it follows that if $f : X \to X$ is a quasicontinuous function with a countable discontinuity set, then there is some $g \sim f$ with an invariant measure.*

Proof. Let m be the F-invariant measure guaranteed by Theorem 12. Suppose first that $m(D) = 0$. By Proposition 1 (ii) we have $D_f \subset D$ for any selection function $f \subset F$, and hence by Proposition 13, m is f-invariant.

Now suppose that $m(D) > 0$. Since $m(D)$ is the sum of the measures of its elements, there must be a point in D with positive measure. Call this point a, and let $A = \bigcup_{j=1}^{\infty} F^{-j}(a)$. Then

$$\begin{aligned} m(\{a\} \cup A) =& m\Big(\{a\} \cup \bigcup_{j=1}^{\infty} F^{-j}(a)\Big) = m\Big(\bigcup_{j=0}^{\infty} F^{-j}(a)\Big) \\ \leq& m\Big(F^{-1}\big(\bigcup_{j=0}^{\infty} F^{-j}(a)\big)\Big) = m\Big(\bigcup_{j=1}^{\infty} F^{-j}(a)\Big) = m(A). \end{aligned}$$

If a and A are disjoint, this implies $m(\{a\}) = 0$, which contradicts our hypothesis. Therefore, $a \in \bigcup_{j=1}^{\infty} F^{-j}(a)$, meaning a is a periodic point. Suppose $P = \{a_1, \dots, a_p\}$ is a shortest periodic orbit of $a = a_1 = a_{p+1}$; that is, for $k \in \{1, \dots, p\}$ we have $a_{k+1} \in F(a_k)$ and $a_{k+1} \notin F(a_j)$ for $j \in \{1, \dots, k-1\}$. To define a selection function $g \subset F$ that preserves the periodic orbit P, we let $g(a_k) = a_{k+1}$ for $k \in \{1, \dots, p\}$, and choose $g(x) \in F(x)$ arbitrarily for $x \in X \setminus P$. Because we used a shortest path, g is well-defined. Now define a measure μ on X by $\mu(S) = 1/p \cdot \mathrm{card}(S \cap P)$ for $S \subset X$. Clearly, μ is invariant under g. $\square$

It is not clear whether every quasicontinuous function is equivalent to one with an invariant measure. We note that if m is an invariant measure for a function f, then m is an invariant measure for the closed relation $\bar{f}$. This follows because $f^{-1}(A) \subset \bar{f}^{-1}(A)$ for all $A \subset X$, and accordingly $m(A) = m(f^{-1}(A)) \leq m(\bar{f}^{-1}(A))$. But the converse does not hold. Consider for example

$$F(x) = \begin{cases} \frac{1}{2}\left(1 + \sin\left(\frac{1}{x}\right)\right) & \text{for } x \in [-1, 0) \\ [0,1] & \text{for } x = 0 \\ 0 & \text{for } x \in (0, 1]. \end{cases}$$

Every selection function of this closed relation is quasicontinuous. Although the point-measure δ_0 supported at $\{0\}$ is invariant for both F and for the selection function that assigns $f(0) = 0$, we will define a slightly more elaborate measure which combines δ_0 and the Lebesgue measure λ. For $A \subset [-1, 1]$, let $m(A) = (\delta_0(A) + \lambda(A \cap [0, 1]))/2$.

Notice that m is indeed an F-invariant measure, because if $0 \in A$, then

$$m(A) \leq 1 = m([0, 1]) = m(F^{-1}(A));$$

and if $A \subset (0, 1]$, then

$$m(A) \leq \frac{1}{2} = m(\{0\}) = m(F^{-1}(A)).$$

On the other hand, there is no selection function $f \subset F$ for which m is f-invariant.

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