Real Analysis Exchange
ISSN:0147-1937
Vol. 31(2), 2005/2006, pp. 385–396

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ON $\Im$-A.E. CONTINUOUS DARBOUX FUNCTIONS MAPPING $\mathbb{R}^k$ INTO $\mathbb{R}^k$

Abstract

This paper completes the results of the paper [8]. We will investigate mutual relations between classes of functions which are continuous $\Im$-a.e. with respect to various σ-ideals of subsets of $\mathbb{R}^k$.

1 Introduction.

Throughout the paper, we will consider functions whose sets of discontinuity points belong to certain σ-ideals consisting of boundary sets (i.e., sets having the empty interior). Such functions will be called $\Im$-almost everywhere ($\Im$-a.e.) continuous with respect to a specified σ-ideal $\Im$. In particular we will consider

$\mathcal{K}_k$ – σ-ideal of first category subsets of $\mathbb{R}^k$,
$\mathcal{L}_k$ – σ-ideal of Lebesgue null subsets of $\mathbb{R}^k$,
$\mathcal{N}_k$ – σ-ideal of countable subsets of $\mathbb{R}^k$.

If $k = 1$, then we will write simply $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{N}$ rather than $\mathcal{K}_1$, $\mathcal{L}_1$ and $\mathcal{N}_1$.

From Theorem 1.4 of [4] it follows that in some spaces, every $\Im$-a.e. continuous function is $\mathcal{K}_k$-a.e. continuous one. The converse isn't true. There exists a family of σ-ideals $\Im$, such that the set of $\Im$-a.e. continuous functions is topologically small in the space of $\mathcal{K}_k$-a.e. continuous ones. In Theorem 2.3 [8] it was proved that this set is uniformly porous in the space of Darboux functions mapping $\mathbb{R}^2$ into $\mathbb{R}^2$. This paper is an extension of the paper [8].

This extension consists in increasing of the dimension of considered spaces and the number of investigated σ-ideals. Additionally we replace uniformly

Key Words: σ-ideal, product of σ-ideals, Cantor-like sets, $\Im$-a.e. continuous functions, Darboux functions
Mathematical Reviews subject classification: 54B10, 54H05, 28A05
Received by the editors June 18, 2005
Communicated by: Krzysztof Chris Ciesielski

porous set by strongly porous set. For that purpose, we make use of certain σ-ideals in $\mathbb{R}^k$ (they are investigated in Section 2) and certain sets called in this paper the box-Cantor sets in $\mathbb{R}^k$ (they are introduced in Section 3). Our main results are stated and proved in Section 4.

Basic notation used in this paper is standard. In particular, $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N} = \{1, 2, 3, \dots\}$. Let k and $j < k$ be natural numbers and let $X^{(i)}(i = 1, \dots, k)$ be a topological space. For convenience write $X'_j = X^{(1)} \times \cdots \times X^{(j)}$ and $X''_j = X^{(j+1)} \times \cdots \times X^{(k)}$. For $x' \in X'_j$, $x'' \in X''_j$ and $A \subset X'_j \times X''_j$, let $A_{x'} = \{x'' \in X''_j : (x', x'') \in A\}$.

If $A \subset \mathbb{R}^k$, $z \in \mathbb{R}^k$ and $s \in \mathbb{R}$, let $z + A = \{z + a : a \in A\}$, $s \cdot A = \{s \cdot a : a \in A\}$ and in particular $-A = (-1)A$.

Let $k \in \mathbb{N}$ and let $c^1, \dots, c^k$ be positive real numbers. The k-dimensional cube $K_z(c^1, \dots, c^k)$ centered at $z \in \mathbb{R}^k$ is defined by

$$K_z(c^1, \dots, c^k) = z + \times_{i=1}^{k} [-c^i, c^i].$$

If $z = \{0, \dots, 0\}$ and $c^1 = \cdots = c^k = 1$, the cube $K_z(c^1, \dots, c^k)$ will be denoted by $\mathbf{K}$, for simplicity.

In the space $\mathbb{R}^k$ we will use the Euclidean metric d_k and in the space of functions mapping $\mathbb{R}^k$ into $\mathbb{R}^k$ we will use the metric ρ ($\rho(f, h) = \min\{1, \sup\{x \in \mathbb{R}^k : d_k(f(x), h(x))\}\}$). In these spaces we will consider an open ball $B(z, r)$ of radius r centered at z. The symbols dia$A$, cl$A$, int$A$ and FrA stand for the diameter, closure, interior and boundary of the set A, respectively.

By $C_f(D_f)$ we will denote the set of continuity (discontinuity) points of the function f and its oscillatory will be denoted by osc_f.

In the last section we will need the definition of strong porosity of a set in a metric space (see [11]). Let P be a metric space, $S \subset P$, $x \in P$, $R > 0$ and $\gamma(x, R, S) = \sup\{r > 0; \exists_{z \in P} B(z, r) \subset B(x, R) \setminus S\}$. The number $p(S, x) = 2 \cdot \limsup_{R \to 0^+} \frac{\gamma(x, R, S)}{R}$ is called the porosity of S at x. We say that the set S is strongly porous if $p(S, x) \geq 1$ at each point $x \in S$.

We say that $f : \mathbb{R}^k \to \mathbb{R}^k$ is a Darboux function if the image of any arc belonging to $\mathbb{R}^k$ is a connected set (see [5],[6]).

2 Products of σ-Ideals.

Definition 2.1. *Let $k > 2$ be a natural number and let $\Im^{(i)}$ be a σ-ideal of subsets of a topological space $X^{(i)}(i = 1, 2, \dots, k)$. The product of σ-ideals is defined by*

$$\Im^{(1)} \times \Im^{(2)} = \{A \subset X^{(1)} \times X^{(2)} : \{x^{(1)} \in X^{(1)} : A_{x^{(1)}} \notin \Im^{(2)}\} \in \Im^{(1)}\}.$$

Let us suppose that $\Im^{(1)} \times \cdots \times \Im^{(n)}$ *has been defined for* $2 \le n < k$*. For* $n+1$ *we put*

$$\Im^{(1)} \times \cdots \times \Im^{(n)} \times \Im^{(n+1)} = (\Im^{(1)} \times \cdots \times \Im^{(n)}) \times \Im^{(n+1)}.$$

Of course the family $\Im^{(1)} \times \cdots \times \Im^{(k)}$ forms a σ-ideal. If $X^{(1)} = \cdots = X^{(k)}$ and $\Im^{(1)} = \cdots = \Im^{(k)} = \Im$, then the σ-ideal $\Im^{(1)} \times \cdots \times \Im^{(k)}$ will be denoted simply by $\Im^k$. If $k = 1$, then $\Im^1 = \Im$.

Theorem 2.2 (see [9], Th.2.3 and Th.2.5)**.** *For any natural number* $k > 1$ *and for a subset* $A \subset X^{(1)} \times \cdots \times X^{(k)}$ *the following conditions are equivalent:*

(a) $A \in \Im^{(1)} \times \cdots \times \Im^{(k)}$,

(b) $\{x^{(1)} : \{\ldots\{x^{(k)} : (x^{(1)}, \ldots, x^{(k)}) \in A\} \notin \Im^{(k)} \ldots\} \notin \Im^{(2)}\} \in \Im^{(1)}$

(c) $\forall_{m<k}$ $\{(x^{(1)}, \ldots, x^{(m)}) \in X^{(1)} \times \cdots \times X^{(m)} : (A)_{(x^{(1)},\ldots,x^{(m)})} \notin \Im^{(m+1)} \times \cdots \times \Im^{(k)}\} \in \Im^{(1)} \times \cdots \times \Im^{(m)}$

(d) $\exists_{m<k}$ $\{(x^{(1)}, \ldots, x^{(m)}) \in X^{(1)} \times \cdots \times X^{(m)} : (A)_{(x^{(1)},\ldots,x^{(m)})} \notin \Im^{(m+1)} \times \cdots \times \Im^{(k)}\} \in \Im^{(1)} \times \cdots \times \Im^{(m)}$

The following definitions and properties will be needed in the next section.

A σ-ideal $\Im$ of subsets of a topological space X is called **admissible** if it is contained in the family of boundary subsets of X and contains all singleton subsets of X.

Theorem 2.3. *Let* $k \in \mathbb{N}$*. If* $\Im^{(i)}$ $(i = 1, 2, \ldots, k)$ *are admissible* σ*-ideals, then the product* $\Im^{(1)} \times \cdots \times \Im^{(k)}$ *is an admissible* σ*-ideal as well.*

PROOF. It is easy to prove the above theorem by induction with use of the method presented in [9] (Proposition 2.1). □

We say that a family $\Im$ of subsets of the space $\mathbb{R}^k$ is ***a*-invariant** if for any $A \subset \mathbb{R}^k$, $z \in \mathbb{R}^k$ and $s \in \mathbb{R} \setminus \{0\}$, the sets $z + A$ and $s \cdot A$ belong to $\Im$.

Theorem 2.4. *Let* $k \in \mathbb{N}$*. If* σ*-ideals* $\Im^{(j)}$ *are a-invariant for any* $j = 1, \ldots, k$*, then the* σ*-ideal* $\Im^{(1)} \times \cdots \times \Im^{(k)}$ *is a-invariant as well.*

PROOF. The proof is by induction with respect to k. We use Theorem 2.2 and the following properties of sections. For any set $A \subset \mathbb{R}^n \times \mathbb{R}^m$ $(n, m \in \mathbb{N})$, points $a = (a_1, a_2)$, $x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m$ and a number $s \in \mathbb{R} \setminus \{0\}$ we have (see [1], Lemma 2.1):

$$(a + A)_{x_1} = a_2 + (A)_{x_1 - a_1} \tag{1}$$

and

$$(s \cdot A)_{x_1} = s \cdot (A)_{x_1/s}. \tag{2}$$

For $k = 1$ the a-invariance is obvious. Let $k \in \mathbb{N}$, $A \subset \mathbb{R}^k \times \mathbb{R}$, and $a = (a_1, a_2)$, $x = (x_1, x_2) \in \mathbb{R}^k \times \mathbb{R}$. Assume the σ-ideals $\Im^{(1)}, \ldots, \Im^{(k+1)}$ and $\Im^{(1)} \times \cdots \times \Im^{(k)}$ are a-invariant. From (1) and the above assumptions we have

$$\begin{aligned}&\{(x_1 \in \mathbb{R}^k : (a + A)_{x_1} \notin \Im^{(k+1)}\} = \{(x_1 \in \mathbb{R}^k : (A)_{x_1 - a_1} \notin \Im^{(k+1)}\}\\ &= a_1 + \{(x_1 - a_1 \in \mathbb{R}^k : (A)_{x_1 - a_1} \notin \Im^{(k+1)}\} \in \Im^{(1)} \times \cdots \times \Im^{(k)}.\end{aligned}$$

By Theorem 2.2(d) we have shown that $a + A \in \Im^{(1)} \times \cdots \times \Im^{(k+1)}$. In the same manner, owing to (2), we can see that $s \cdot A \in \Im^{(1)} \times \cdots \times \Im^{(k+1)}$, for any $s \in \mathbb{R} \setminus \{0\}$. □

In the end of this section we compare σ-ideals constructed on the basis of σ-ideals $\mathcal{L}$ and $\mathcal{K}$. We will denote by $\mathcal{L}^k$ ($\mathcal{K}^k$) the product of k σ-ideals $\mathcal{L}$ ($\mathcal{K}$). Moreover use:

L_k– the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^k$

K_k– the σ-algebra of subsets of $\mathbb{R}^k$ having the property of Baire.

If $k = 1$, then we will write simply $\mathcal{L}$, $\mathcal{K}$, L and K rather than $\mathcal{L}^1$, $\mathcal{K}^1$, L_1 and K_1 . Observe that above-mentioned families are a-invariant.

In order to prove the next theorem we need the following lemmas.

Lemma 2.5. *If a subset $A \subset \mathbb{R}^m \times \mathbb{R}^n$ belongs to the σ-algebra L_{m+n}, then the set $\{x \in \mathbb{R}^m : A_x \notin \mathcal{L}_n\}$ belongs to the σ-algebra L_m.*

Lemma 2.6. *If a subset $A \subset \mathbb{R}^m \times \mathbb{R}^n$ belongs to the σ-algebra K_{m+n}, then the set $\{x \in \mathbb{R}^m : A_x \notin \mathcal{K}_n\}$ belongs to the σ-algebra K_m.*

Theorem 2.7. *For any natural number k*

$$\text{(a)} \quad \mathcal{L}_k = \mathcal{L}^k \cap L_k \text{ and (b)} \quad \mathcal{K}_k = \mathcal{K}^k \cap K_k$$

Proof. (a) Let us first show that for any natural number k

$$\mathcal{L}_k \subset \mathcal{L}^k. \tag{3}$$

The proof is by induction on k. For $k = 1$ the inclusion (3) is obvious, because $\mathcal{L}_1 = \mathcal{L} = \mathcal{L}^1$. Assume the inclusion (3) holds for a natural number $k \geq 1$. Let A be a Lebesgue null subset of $\mathbb{R}^{k+1}$. From Fubini's Theorem (Th. 21.12 [2]) and induction assumption we have

$$\{(x^{(1)}, \ldots, x^{(k)}) \in \mathbb{R}^k : (A)_{(x^{(1)}, \ldots, x^{(k)})} \notin \mathcal{L}\} \in \mathcal{L}_k \subset \mathcal{L}^k.$$

By Theorem 2.2(d) we have shown that $A \in \mathcal{L}^{k+1}$ and finally that inclusion (3) is true.

Now it is enough to prove that for any natural number k

$$\mathcal{L}^k \cap L_k \subset \mathcal{L}_k. \tag{4}$$

The proof is by induction on k. For $k = 1$ the inclusion (4) is obvious. Assume the inclusion (4) holds for a natural number $k \geq 1$. Let A be a Lebesgue measurable subset of $\mathbb{R}^{k+1}$. From Theorem 2.2(c), Lemma 2.5 and induction assumption we have

$$\{(x^{(1)}, \ldots, x^{(k)}) \in \mathbb{R}^k : (A)_{(x^{(1)},\ldots,x^{(k)})} \notin \mathcal{L}\} \in \mathcal{L}^k \cap L_k \subset \mathcal{L}_k.$$

By the above and Fubini's Theorem (Th. 21.12 [2]) we have proved that the set A belongs to the σ-ideal $\mathcal{L}_{k+1}$. This gives the inclusion (4) and the proof is complete.

(b) The proof is similar. We use the Kuratowski - Ulam Theorem (Th.15.1 [3]), Theorem 15.5 [3], Theorem 15.4 [3] and Lemma 2.6 . □

In [9] and [10] it was also proved that $\mathcal{L}_k \subsetneq \mathcal{L}^k$ and $\mathcal{K}_k \subsetneq \mathcal{K}^k$ for any $k > 1$.

3 Box-Cantor Sets in $\mathbb{R}^k$.

A sketch of the construction of a symmetric Cantor set in $\mathbb{R}$ can be found, for example, in [11]. It is similar to the construction of the Cantor ternary set.

Let $\xi = (\xi_n)_{n\in\mathbb{N}}$ be a sequence of real numbers $\xi_n \in (0, 1)$ and let $\mathbf{I} = [0, 1]$ be a closed interval whose length will be denoted by δ_1. In the first step of the construction, we remove from $\mathbf{I}$ the concentric open interval (a_{11}, b_{11}) of the length $\delta_1 \cdot \xi_1$. In the m-th ($m > 1$) step of the construction, from the remaining 2^{m-1} closed intervals of length equal to δ_m we remove the concentric open intervals $(a_{mi}, b_{mi})(i = 1, 2, \ldots, 2^{m-1})$ of the length $\delta_m \cdot \xi_m$. Additionally, we assume $a_{m1} < a_{m2} < \ldots . < a_{m2^{m-1}}$.

The set

$$C(\xi) = \mathbf{I} \setminus \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{2^{m-1}} (a_{mi}, b_{mi}) \tag{5}$$

is called the symmetric Cantor set with respect to the sequence $\xi = (\xi_n)_{n\in\mathbb{N}}$. Throughout the paper we assume that numbers 0 and 1 are one-side accumulation points of every symmetric Cantor set $C(\xi)$. Of course, if $\xi_n = \frac{1}{3}$ for $n = 1, 2, \ldots .$, then the set $C(\xi)$ is the classical Cantor ternary set.

From the construction it appears that the set $C(\xi)$ is closed, nowhere dense and uncountable. Of course, some properties are connected with the sequence $\xi = (\xi_n)_{n\in\mathbf{N}}$. The following fact will be needed to investigate box-Cantor sets in $\mathbb{R}^k$.

Lemma 3.1 ([7],[11])**.** *The set $C(\xi)$ has Lebesgue measure zero iff*

$$\sum_{n=1}^{\infty} \xi_n = \infty.$$

Now, we are going to define the box-Cantor set in $\mathbb{R}^k$ ($k \in \mathbb{N}$). For simplicity we construct this set in the k-dimensional closed cube $\mathbf{K} = K(1, \ldots .1) \subset \mathbb{R}^k$. Let $C(\xi^{(j)}) = \mathbf{I} \setminus \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{2^{m-1}} (a_{mi}^{(j)}, b_{mi}^{(j)})$ $(j = 1, \ldots, k)$ be a symmetric Cantor set with respect to a sequence $\xi^{(j)} = (\xi_n^{(j)})_{n\in\mathbb{N}}$ of real numbers $\xi_n^{(j)} \in (0, 1)$. Throughout the paper we assume that for any $m \in \mathbb{N}$ and $j = 1, \ldots, k$ we have $a_{m1}^{(j)} < a_{m2}^{(j)} < \ldots . < a_{m2^{m-1}}^{(j)}$.

Definition 3.2. *The set*

$$R(\xi^{(1)}, \ldots, \xi^{(k)}) = \mathbf{K} \setminus \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{2^{m-1}} \operatorname{int}(K(b_{mi}^{(1)}, \ldots, b_{mi}^{(k)}) \setminus K(a_{mi}^{(1)}, \ldots, a_{mi}^{(k)}))$$

is called the box-Cantor set with respect to sequences $\xi^{(j)} = (\xi_n^{(j)})_{n\in\mathbb{N}}$ of real numbers $\xi_n^{(j)} \in (0, 1)$ $(j = 1, \ldots, k)$.

The box-Cantor set $R(\xi^{(1)}, \ldots, \xi^{(k)})$ can be also defined as follows. Let $\phi_j : \mathbf{I} \to \mathbf{I}, (j = 1, \ldots, k)$ be an increasing homeomorphism such that $\phi_j(C(\xi^{(1)})) = C(\xi^{(j)})$, $\phi_j(a_{mi}^{(1)}) = a_{mi}^{(j)}$, $\phi_j(b_{mi}^{(1)}) = b_{mi}^{(j)}$ for any $j = 1, \ldots, k$ and

$$\operatorname{Fr}(K(\phi_1(x_1), \ldots, \phi_k(x_1))) \cap \operatorname{Fr}(K(\phi_1(x_2), \ldots, \phi_k(x_2))) = \emptyset$$

for any $x_1 \neq x_2$. Observe that

$$R(\xi^{(1)}, \ldots, \xi^{(k)}) = \bigcup_{c\in C(\xi^{(1)})} \operatorname{Fr}(K(\phi_1(c), \ldots, \phi_k(c))). \tag{6}$$

Theorem 3.3. *Let $k \in \mathbb{N}$. If σ-ideals $\Im^{(j)}$ are a-invariant and admissible for any $j = 1, \ldots, k$, then the box-Cantor set $R(\xi^{(1)}, \ldots, \xi^{(k)})$ belongs to the σ-ideal $\Im^{(1)} \times \cdots \times \Im^{(k)}$ iff $C(\ \xi^{(j)}) \in \Im^{(j)}$ for any $j = 1, \ldots, k$.*

PROOF. Let $C^* = C(\xi^{(1)}) \setminus \{0\}$ and

$$F_j = \bigcup_{c \in C^*} \{(x^{(1)}, \dots, x^{(k)}) \in K(\phi_1(c), \dots, \phi_k(c)) : x^{(j)} = \phi_j(c)\}$$

for $j \leq k$. We first show the following equivalency for any $m \leq k$.

$$F_m \in \Im^{(1)} \times \dots \times \Im^{(k)} \Longleftrightarrow C(\xi^{(m)}) \in \Im^{(m)} \tag{7}$$

Let $1 < m < k$. Then

$$F_m = \bigcup_{c \in C^*} \left((\times_{i=1}^{m-1} [-\phi_i(c), \phi_i(c)]) \times \{\phi_m(c)\} \times (\times_{i=m+1}^{k} [-\phi_i(c), \phi_i(c)]) \right).$$

For any $x = (x^{(1)}, \dots, x^{(k)}) \in F_m$ there exists $c \in C^*$ such that

$$x \in \left(\times_{i=1}^{m-1} [-\phi_i(c), \phi_i(c)] \right) \times \{\phi_m(c)\} \times (\times_{i=m+1}^{k} [-\phi_i(c), \phi_i(c)]\right).$$

Observe that

$$(F_m)_{(x^{(1)}, \dots, x^{(m-1)})} = \{\phi_m(c)\} \times (\times_{i=m+1}^{k} [-\phi_i(c), \phi_i(c)]) \tag{8}$$

and

$$\left(\{\phi_m(c)\} \times (\times_{i=m+1}^{k} [-\phi_i(c), \phi_i(c)]) \right)_{x^{(m)}} = \times_{i=m+1}^{k} [-\phi_i(c), \phi_i(c)]. \tag{9}$$

Since $\times_{i=m+1}^{k} [-\phi_i(c), \phi_i(c)]$ isn't a boundary set in the $(k-m)$-dimension cube $\mathbf{K}$ we have

$$\times_{i=m+1}^{k} [-\phi_i(c), \phi_i(c)] \notin \Im^{(m+1)} \times \dots \times \Im^{(k)}$$

and

$$\begin{aligned} &\left\{ x^{(m)} : \left(\{\phi_m(c)\} \times (\times_{i=m+1}^{k} [-\phi_i(c), \phi_i(c)]) \right)_{x^{(m)}} \notin \Im^{(m+1)} \times \dots \times \Im^{(k)} \right\} \\ &= C(\xi^{(m)}) \setminus \{0\}. \end{aligned} \tag{10}$$

We will consider the following cases:

A) $C(\xi^{(m)}) \in \Im^{(m)}$; then $C(\xi^{(m)}) \setminus \{0\} \in \Im^{(m)}$. From above and Theorem 2.2(d) we have

$$(F_m)_{(x^{(1)}, \dots, x^{(m-1)})} \in \Im^{(m)} \times \dots \times \Im^{(k)}$$

and

$$\begin{aligned} &\left\{ (x^{(1)}, \dots, x^{(m-1)}) : (F_m)_{(x^{(1)}, \dots, x^{(m-1)})} \notin \Im^{(m)} \times \dots \times \Im^{(k)} \right\} \\ &= \emptyset \in \Im^{(1)} \times \dots \times \Im^{(m-1)}. \end{aligned}$$

Therefore $F_m \in \Im^{(1)} \times \cdots \times \Im^{(k)}$ by Theorem 2.2(d).

B) $C(\xi^{(m)}) \notin \Im^{(m)}$. Since $\Im^{(m)}$ is an admissible σ-ideal $C(\xi^{(m)}) \setminus \{0\} \notin \Im^{(m)}$.

By the above, (8) - (10) and Theorem 2.2(c) we have

$$(F_m)_{(x^{(1)},\dots,x^{(m-1)})} \notin \Im^{(m)} \times \cdots \times \Im^{(k)}$$

and

$$\begin{aligned}&\left\{(x^{(1)},\dots,x^{(m-1)}) : (F_m)_{(x^{(1)},\dots,x^{(m-1)})} \notin \Im^{(m)} \times \cdots \times \Im^{(k)}\right\}\\ &= \times_{i=1}^{m-1}[-\phi_i(c),\phi_i(c)] \notin \Im^{(1)} \times \cdots \times \Im^{(m-1)}.\end{aligned}$$

Consequently, $F_m \notin \Im^{(1)} \times \cdots \times \Im^{(k)}$.

In this way we have shown equivalency (7) (for $m = 1$ and $m = k$ the proof is similar). By equivalency (7) and Theorem 2.4 we have

$$-F_m \in \Im^{(1)} \times \cdots \times \Im^{(k)} \Longleftrightarrow C(\xi^{(m)}) \in \Im^{(m)}.$$

From Theorem 2.3 it follows that $\{(0,\dots,0)\} \in \Im^{(1)} \times \cdots \times \Im^{(k)}$. It is easily seen that

$$R(\xi^{(1)},\dots,\xi^{(k)}) = \{(0,\dots,0)\} \cup \bigcup_{i=1}^{k} F_i \cup \bigcup_{i=1}^{k} (-F)_i.$$

This equality and the above remarks complete the proof. □

It appears that the set $R(\xi^{(1)},\dots,\xi^{(k)})$ has similar properties as the set $C(\xi^{(j)})$.

Lemma 3.4. *The box-Cantor set $R(\xi^{(1)},\dots,\xi^{(k)})$ with respect to sequences $\xi^{(j)} = (\xi_n^{(j)})_{n\in\mathbb{N}} \subset (0,1)$ $(j = 1,\dots,k)$ is closed and nowhere dense in $\mathbb{R}^k$.*

Proof. Observe that $C(\xi^{(i)}) \in \mathcal{K}$ for any $i = 1,\dots,k$. By Theorem 3.3, the set $R(\xi^{(1)},\dots,\xi^{(k)})$ belongs to the σ-ideal $\mathcal{K}^k$. Hence $R(\xi^{(1)},\dots,\xi^{(k)})$ is a boundary set. From (6) it follows that $R(\xi^{(1)},\dots,\xi^{(k)})$ is a closed set and, in consequence, it is a nowhere dense subset of the space $\mathbb{R}^k$. □

Corollary 3.5. *The box-Cantor set $R(\xi^{(1)},\dots,\xi^{(k)})$ with respect to sequences $\xi^{(j)} = (\xi_n^{(j)})_{n\in\mathbb{N}} \subset (0,1)$ $(j = 1,\dots,k)$ belongs to the σ-ideal $\mathcal{K}_k$ of first category subsets of $\mathbb{R}^k$.*

Lemma 3.6. *The box-Cantor set* $R(\xi^{(1)},\ldots,\xi^{(k)})$ *with respect to sequences* $\xi^{(j)}=(\xi_n^{(j)})_{n\in\mathbb{N}}\subset(0,1)$ $(j=1,\ldots,k)$ *has the Lebesgue measure zero iff*

$$\sum_{n=1}^{\infty}\xi_n^{(j)}=\infty,\ \text{for } j=1,\ldots,k.$$

PROOF. Let $\sum_{n=1}^{\infty}\xi_n^{(j)}=\infty$, for any $j=1,\ldots,k$. By Lemma 3.1, it appears that $C(\xi^{(i)})$ belongs to the σ-ideal of Lebesgue null subsets of $\mathbb{R}$ (for any $i=1,\ldots,k$). Since the σ-ideal $\mathcal{L}$ is a-invariant and admissible, Theorem 3.3 shows that $R(\xi^{(1)},\ldots,\xi^{(k)})\in\mathcal{L}^k$. Additionally $R(\xi^{(1)},\ldots,\xi^{(k)})$ is closed (see(6)), and hence Lebesgue measurable. From Theorem 2.7(a) we conclude that $R(\xi^{(1)},\ldots,\xi^{(k)})$ has Lebesgue measure zero. □

4 On $\Im$-A.E. Continuous Darboux Functions.

From Theorem 1.4 [4] we obtain the following.

Theorem 4.1. *If a function* $f:\mathbb{R}^k\longrightarrow\mathbb{R}^k$ *is continuous* $\Im$*-a.e. with respect to a* σ*-ideal* $\Im$ *of subsets of the space* $\mathbb{R}^k$*, then this function is continuous* $\mathcal{K}_k$*-a.e.*

If $\Im$ is the σ-ideal of countable, σ-porous or of Lebesgue measure zero subsets of the space $\mathbb{R}^k$, then the converse is not true, because there exist functions continuous $\mathcal{K}_k$-a.e. which are not continuous $\Im$-a.e. with respect to any of the above-mentioned σ-ideals. Moreover, these functions form a set which is not topologically small in the space of functions continuous $\mathcal{K}_k$-a.e. In the next theorem we obtain a more general case. Before we formulate this theorem we must first introduce the following definition.

We say that a σ-ideal $\Im$ of subsets of the space $\mathbb{R}^k$ has the **property** $(\mathcal{T})$ if it is a-invariant and admissible and if there exists a box-Cantor set $R(\xi^{(1)},\ldots,\xi^{(k)})$ which doesn't belong to the σ-ideal $\Im$. The existence of such ideals is guaranteed by Theorem 3.3.

Theorem 4.2. [1] *The set* $D_{(\Im)}$ *of functions* $\Im$*-a.e. continuous with respect to a* σ*-ideal* $\Im$ *having the property* $(\mathcal{T})$ *is strongly porous in the space* $D_{(\mathcal{K}_k)}$ *of Darboux* $\mathcal{K}_k$*-a.e. continuous functions.*

PROOF. Let $h\in D_{(\Im)}\cap D_{(\mathcal{K}_k)}$, $x_0\in C_h$, $R\in(0,1)$ and $r\in(0,\frac{R}{2})$. Put $s=\frac{R}{2}-r$. Let δ be a positive real number such that

$$h(K_{x_0}(\delta,\ldots,\delta))\subset B(h(x_0),s). \tag{11}$$

[1] This Theorem is an extension of Theorem 3.3 [8]

Let $\Im$ be a σ-ideal having the property $(\mathcal{T})$. We will construct a Darboux $\mathcal{K}_k$-a.e. continuous function $f : \mathbb{R}^k \to \mathbb{R}^k$ satisfying the condition $B(f, r) \subset B(h, R)$ and show that the set of discontinuity points of an arbitrary function $f_1 \in B(f, r)$ doesn't belong to the σ-ideal $\Im$.

From the property $(\mathcal{T})$ of the σ-ideal $\Im$ it follows that there exists a box-Cantor set $R(\xi^{(1)}, \ldots, \xi^{(k)})$ which doesn't belong to the σ-ideal $\Im$. For the set $R(\xi^{(1)}, \ldots, \xi^{(k)})$ considered in this proof we take notions as for box-Cantor sets defined in Section 3.

In particular (see (6))

$$R(\xi^{(1)}, \ldots, \xi^{(k)}) = \bigcup_{c \in C(\xi^{(1)})} \mathrm{Fr}(K(\phi_1(c), \ldots, \phi_k(c))).$$

Because the σ-ideal $\Im$ has the property $(\mathcal{T})$, it is a-invariant and as a result the set $x_0 + \delta \cdot R(\xi^{(1)}, \ldots, \xi^{(k)})$ doesn't belong to the σ-ideal $\Im$ either. We have

$$x_0 + \delta \cdot R(\xi^{(1)}, \ldots, \xi^{(k)}) \subset x_o + \delta \cdot \mathbf{K} = K_{x_0}(\delta, \ldots, \delta).$$

For simplicity we let

$$\begin{aligned} &C = C(\xi^{(1)}) \setminus \{1\}, \\ &K = \mathrm{int} K_{x_0}(\delta, \ldots, \delta), \\ &F = x_0 + \delta \cdot R(\xi^{(1)}, \ldots, \xi^{(k)}) \setminus \mathrm{Fr} K. \end{aligned}$$

Observe that

$$F = \bigcup_{c \in C} \mathrm{Fr}(K_{x_0}(\delta \cdot \phi_1(c), \ldots, \delta \cdot \phi_k(c))) \tag{12}$$

and

$$K \setminus F = \bigcup_{x \in [0,1) \setminus C} \mathrm{Fr}(K_{x_0}(\delta \cdot \phi_1(x), \ldots, \delta \cdot \phi_k(x))).$$

Let $\mathcal{U}$ be a family (of power continuum) of pairwise disjoint and dense subsets of C such that $\bigcup_{U \in \mathcal{U}} U = C$. Without loss of generality we may assume that all one-sided accumulation points of the set C belong to certain set $U_0 \in \mathcal{U}$. Let $g : \mathcal{U} \to B(h(x_0), \frac{R}{2})$ be a one-to-one function such that $g(U_0) = \{h(x_0)\}$. From (12) it follows that for any $x \in F$ there exists exactly one point $c_x \in C$ such that $x \in Fr K_{x_0}(\delta \cdot \phi_1(c_x), \ldots, \delta \cdot \phi_k(c_x))$ and there exists exactly one subset $U_{[c_x]} \in \mathcal{U}$ such that $c_x \in U_{[c_x]}$.

Now, we can define the function $f:\mathbb{R}^k\to\mathbb{R}^k$ by

$$f(x)=\begin{cases} h(x) & \text{if } x\in\mathbb{R}^k\setminus K,\\ g(U_{[c_x]}) & \text{if } x\in F,\\ h(x_0) & \text{if } x\in K\setminus F.\end{cases}$$

Observe that the function f is $\mathcal{K}_k$-a.e. continuous Darboux function (this follows by the same method as in the proof of Theorem 2.3 [8]). We will now show that

$$B(f,r)\subset B(h,R) \tag{13}$$

By the definition of the functions f and g and by (11) we have

$$\sup_{x\in K}(d_k(f(x),h(x))\le\frac{R}{2}+s\le R-r<1$$

and consequently

$$\rho(f,h)=\min\{1,\sup_{x\in K}(d_k(f(x),h(x))\}\le R-r$$

Thus for any function $f_1\in B(f,r)$

$$\rho(f_1,h)\le\rho(f_1,f)+\rho(f,h)<R,$$

and so (13) is proved.

It remains to prove that

$$B(f,r)\cap D_{(\Im)}=\emptyset. \tag{14}$$

Let us take $f_1\in B(f,r)$, $z\in x_0+\delta\cdot R(\xi^{(1)},\dots,\xi^{(k)})$ and $\epsilon>0$. By the definition of f, it follows that $f(K\cap B(z,\epsilon))=B(h(x_0),\frac{R}{2})$ and consequently

$$\mathrm{osc}_{f_1}(z)=\inf_{\epsilon>0}\mathrm{dia}(f_1(B(z,\epsilon)))\ge R-2r>0$$

Hence

$$x_0+\delta\cdot R(\xi^{(1)},\dots,\xi^{(k)})\subset D_{f_1}.$$

By our assumption, $x_0+\delta\cdot R(\xi^{(1)},\dots,\xi^{(k)})$ doesn't belong to $\Im$. We conclude from the above that the function f_1 isn't $\Im$-a.e. continuous, which implies (14).

We have proved that for a function $h\in D_{(\mathcal{K}_k)}\cap D_{(\Im)}$ and a number $R>0$ there exists the function $f\in D_{(\mathcal{K}_k)}$ such that $B(f,r)\subset B(h,R)\setminus D_{(\Im)}$ for any $r\in(0,\frac{R}{2})$. Hence $\gamma(h,R,D_{(\Im)})=\frac{R}{2}$ and finally we conclude that the set $D_{(\Im)}$ is strongly porous in the space $D_{(\mathcal{K}_k)}$. □

Corollary 4.3. *In the space $D_{(\mathcal{K}_k)}$ of Darboux $\mathcal{K}_k$-a.e. continuous functions (with the metric of uniform convergence) mapping $\mathbb{R}^k$ into $\mathbb{R}^k$, the set $D_{(\Im)}$ of functions continuous $\Im$-a.e. with respect to σ-ideal $\mathcal{L}_k$ or $\mathcal{N}_k$ is strongly porous.*

Proof. Let $\xi^{(j)} = \xi_n^{(j)}$ $(j = 1, \ldots, k)$ be sequences with a general term $\xi_n^{(j)} = \frac{1}{2^n}$ for $j = 1, \ldots, k$. Let us consider the box-Cantor set $R(\xi^{(1)}, \ldots, \xi^{(k)})$ with respect to these sequences. From Lemma 3.6 the set $R(\xi^{(1)}, \ldots, \xi^{(k)})$ doesn't belong to σ-ideal $\mathcal{L}_k$ (and $\mathcal{N}_k$) so the σ-ideal $\mathcal{L}_k$ (and $\mathcal{N}_k$) has the property $(\mathcal{T})$. The corollary follows from Theorem 4.2. □

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