Real Analysis Exchange
Vol. (), , pp. 175–190

D. N. Sarkhel,* Department of Mathematics, University of Kalyani, Kalyani 741235, West Bengal, India.

ON APPROXIMATE DERIVATIVES AND KRZYZEWSKI–FORAN LEMMA

Abstract

Let $E \subseteq \mathbb{R}$ and $g : E \to \mathbb{R}$. We show that if $|g(E)| = 0$, then $\underline{g}_{ap}(x) \leq 0 \leq \overline{g}_{ap}(x)$ almost everywhere on E, which immediately implies a lemma of Krzyzewski [10] and Foran [7]. The function g is said to satisfy the inverse Lusin condition (N^{-1}) on E if $|g^{-1}(H)| = 0$ for every $H \subseteq g(E)$ with $|H| = 0$. We prove that if $g'_{ap}(x)$ exists almost everywhere on E, then g is an N^{-1}-function if and only if $g'_{ap}(x) \neq 0$ almost everywhere on E. We also improve upon Foran's [7] chain rule for approximate derivatives, and obtain necessary and sufficient conditions for its validity almost everywhere.

1 Introduction

Some general chain rules for derivatives and approximate derivatives over a linear interval $[a, b]$, with applications to integration, were obtained by Krzyzewski [10], Serrin and Varberg [14], Goodman [8] and Foran [7]. Their proofs are based on the following assertion.

Krzyzewski–Foran Lemma. *Let $F : [a, b] \to \mathbb{R}$ and let E be a subset of $[a, b]$ whose image under F has outer Lebesgue measure zero, $|F(E)| = 0$. If $F'(x)$ $[F'_{ap}(x)]$ exists for each x in E, then $F'(x) = 0$ $[F'_{ap}(x) = 0]$ for almost every x in E.*

The case of F' was proved by Krzyzewski [10, Lemma 1, page 99] (and, also by Serrin and Varberg [14, Theorem 1, page 515]), from which the case of F'_{ap} was deduced by Foran [7, Lemma K, p. 446]. We remark that, since the set where F'_{ap} is infinite has measure zero, Chow [4, Theorem 1], it is immaterial

Key Words: approximate derivative, chain rule, inverse Lusin condition (N^{-1}), measurable boundaries, monotone (functions), absolutely continuous.
Mathematical Reviews subject classification: 26A24, 26A48, 26A46, 28A20.
Received by the editors May 1, 2001
*The author warmly thanks the referee for his comments.

here whether $F'(x)$ (or $F'_{ap}(x)$) is assumed to be finite or not. The case of F'_{ap} was proved again by Ene [6, Lemma 2.17.2, page 70] for measurable F, with the subsequent remark that Foran's proof for arbitrary F was not clear to him. Perhaps Ene overlooked that $F'_{ap}(x)$ is assumed to exist on E, which justifies Foran's proof. Cater [3, Theorem 2.1, page 641] obtained Foran's result for measurable functions defined on arbitrary bounded measurable sets, and deduced from it an equivalent result [3, Corollary 2.2, page 642].

In this paper, for an arbitrary function g defined on an arbitrary set, in Section 3 we first obtain some very useful results about the approximate Dini derivatives of g. Without assuming the existence of $g'_{ap}(x)$, and by a new method of proof, we then obtain a general result, Theorem 1, which immediately implies the above results of Krzyzewski, Foran and Cater. Also, a related result concerning the inverse Lusin condition (N^{-1}), Theorem 2, immediately implies a result of Villani [17] and one of Cater [3].

Foran [7], extending Goodman [8], gave two sets of sufficient conditions for the validity almost everywhere of a variant of the usual chain rule for the approximate derivatives of functions defined on intervals. In Theorem 3 and Corollary 4, with a new and simpler method of proof we improve upon this result and obtain natural necessary and sufficient conditions for the validity almost everywhere of the chain rule for arbitrary functions defined on arbitrary sets, with no added hypothesis. Finally we show by example that Foran's theorems [7, page 445] are in fact false.

2 Notation and Preliminaries

We shall deal exclusively with an arbitrary set $E \subseteq \mathbb{R}$ and an arbitrary function $g : E \to \mathbb{R}$. For $x \in \mathbb{R}$, we define

$$d^+(E,x) = \limsup_{y \to x+} \frac{\big|E \cap [x,y]\big|}{y-x}, \quad d_+(E,x) = \liminf_{y \to x+} \frac{\big|E \cap [x,y]\big|}{y-x}.$$

If $d^+(E,x) = d_+(E,x)$, then this common value is called the right density of E at x. Similarly we define $d^-(E,x)$, $d_-(E,x)$ as the left density of E at x. We also define

$$\overline{d}(E,x) = \max\big\{d^+(E,x), d^-(E,x)\big\}, \quad \underline{d}(E,x) = \min\big\{d_+(E,x), d_-(E,x)\big\}.$$

If $\overline{d}(E,x) = \underline{d}(E,x)$, then this common value is called the density of E at x. If B is any measurable cover of E, then E has density 1 almost everywhere on B and 0 almost everywhere on $\mathbb{R} \setminus B$ [12, Lemma 4.1, page 245].

The infimum [supremum] of the extended real numbers K for which the set

$$\{y \in E : g(y) \geq K\} \quad \Big[\{y \in E : g(y) \leq K\}\Big]$$

has right density 0 at x, is called the upper [lower] right approximate limit of g at x and is denoted by $A^+g(x)$ [$A_+g(x)$]. Similarly we define $A^-g(x)$ and $A_-g(x)$, and put

$$\overline{A}g(x) = \max\{A^+g(x), A^-g(x)\}, \quad \underline{A}\,g(x) = \min\{A_+g(x), A_-g(x)\}.$$

If $d^+(E,x) = 0$, then $A^+g(x) = -\infty$ and $A_+g(x) = \infty$, etc. The extended real valued functions $\overline{A}g$ and $\underline{A}\,g$ defined on $\mathbb{R}$ are the measurable boundaries of g on E [1, 18, 13]. These are measurable on $\mathbb{R}$ and satisfy

$$\underline{A}\,g(x) \leq g(x) \leq \overline{A}\,g(x) \tag{1}$$

almost everywhere on E [13, page 443]. Also from the proof of Chow [4, Lemma 1, page 795] we deduce that, for almost every x in E,

$$A^+g(x) = A^-g(x) = \overline{A}\,g(x) \text{ and } A_+\,g(x) = A_-\,g(x) = \underline{A}\,g(x). \tag{2}$$

Given $x \in E$, define

$$f(y) = \frac{g(y) - g(x)}{y - x} \text{ for } y \in E \setminus \{x\}.$$

Following Ward [18] and Saks [11], we define

$$\begin{aligned} \overline{g}^+_{ap}(x) &= A^+ f(x), & \underline{g}^+_{ap}(x) &= A_+ f(x), \\ \overline{g}^-_{ap}(x) &= A^- f(x), & \underline{g}^-_{ap}(x) &= A_- f(x), \\ \overline{g}_{ap}(x) &= \overline{A}\,f(x), & \underline{g}_{ap}(x) &= \underline{A}\,f(x). \end{aligned}$$

If $\overline{g}_{ap}(x) = \underline{g}_{ap}(x)$, then this common value is called the approximate derivative of g at x and is denoted by $g'_{ap}(x)$, and we say that $g'_{ap}(x)$ exists (finite or not). If $\overline{d}(E,x) = 0$, then $\overline{g}_{ap}(x) = -\infty$ and $\underline{g}_{ap}(x) = \infty$, etc. We will often tacitly use the fact that if $x \in A \subseteq E$, then $\big(\overline{g_{\restriction A}}\big)_{ap}(x) \leq \overline{g}_{ap}(x)$, etc. Our notations should cause no confusion since all definitions are relative to the domain of the function.

Recall that [11, page 224] the function g is said to satisfy Lusin's condition (N) on E if $|g(B)| = 0$ for every $B \subseteq E$ with $|B| = 0$. We say that g satisfies the *inverse Lusin condition* (N^{-1}) on E if $\big|g^{-1}(H)\big| = 0$ for every $H \subseteq g(E)$ with $|H| = 0$.

3 Results

We begin with some crucial lemmas leading to the main theorems.

Lemma 1. (cf. [9, Theorem 7.17, page 200]). *The functions* $\overline{g}_{ap}$ *and* $\underline{g}_{ap}$ *are measurable on* E*, relatively to* E *(that is, with respect to the* σ*-algebra of sets* $\{E\cap M : M \subseteq \mathbb{R} \text{ measurable}\}$*). Also the set* $E' = \{x \in E : g'_{ap}(x) \text{ exists}\}$ *is measurable relatively to* E *and the function* g'_{ap} *is measurable on* E' *relatively to* E*.*

Proof. Let $B = \{x \in \mathbb{R} : -\infty < \underline{A}\, g(x) = \overline{A}\, g(x) < \infty\}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} g(x) & \text{if } x \in E \cap B \\ \underline{A}\, g(x) = \overline{A}\, g(x) & \text{if } x \in B \setminus E \\ 0 & \text{if } x \in \mathbb{R} \setminus B \end{cases}$$

By (1), $f(x) = \underline{A}\, g(x) = \overline{A}\, g(x)$ almost everywhere on B. Since the functions $\overline{A}\, g$ and $\underline{A}\, g$ are measurable on $\mathbb{R}$, it follows that the set B is measurable, and the function f is measurable on $\mathbb{R}$. Then the functions $\overline{f}_{ap}$ and $\underline{f}_{ap}$ are measurable on $\mathbb{R}$ [2, Lemma 3, page 349; 9, Theorem 7.18, page 201].

Now, given $x \in E \cap B$ and $K \leq \infty$, write

$$R_x = \Big\{y \in \mathbb{R} : y \neq x \text{ and } \frac{f(y) - f(x)}{y - x} \geq K\Big\},$$

$$E_x = \Big\{y \in E : y \neq x \text{ and } \frac{g(y) - g(x)}{y - x} \geq K\Big\}.$$

Since f is measurable, so the set R_x is measurable. Since $f = g$ on $E \cap B$, so $E_x \cap B = R_x \cap B \cap E$. Thus $R_x \cap B$ is a measurable set containing $E_x \cap B$. Consider any measurable set

$$M \subseteq (R_x \cap B) \setminus (E_x \cap B) = (R_x \cap B) \setminus (R_x \cap B \cap E) = (R_x \cap B) \setminus E.$$

Then $E \subseteq \mathbb{R} \setminus M$ and so E has density 0 at almost every point of M. On the other hand, since $\overline{A}g(y) = \underline{A}\, g(y)$ for each $y \in M$, so E has positive upper density at each point of M. Hence $|M| = 0$. Therefore, $R_x \cap B$ is a measurable cover of $E_x \cap B$, and $R_x \cap B$ has density 0 at x if and only if $E_x \cap B$ has density 0 at x [12, Lemma 4.1, page 245]. Thus if $\mathbb{R} \setminus B$ has density 0 at x, then R_x has density 0 at x if and only if E_x has density 0 at x. However $\mathbb{R} \setminus B$ has density 0 at almost every point x of $E \cap B$. Consequently $\overline{f}_{ap}(x) = \overline{g}_{ap}(x)$ for almost every $x \in E \cap B$. Likewise, $\underline{f}_{ap}(x) = \underline{g}_{ap}(x)$ for almost every $x \in E \cap B$.

Again, by (1) and (2), for almost every $x \in E \setminus B$ we have either

$$g(x) < A^+g(x) = A^-g(x) or g(x) > A_+g(x) = A_-g(x).$$

So clearly, $\overline{g}_{ap}(x) = \infty$ and $\underline{g}_{ap}(x) = -\infty$, almost everywhere on $E \setminus B$.

Since $\overline{f}_{ap}$ and $\underline{f}_{ap}$ are measurable on $\mathbb{R}$, and since $E \cap B$ and $E \setminus B$ are measurable relatively to E, it readily follows from above that both $\overline{g}_{ap}$ and $\underline{g}_{ap}$ are measurable on E, relatively to E. The second part is now obvious. □

Lemma 2. *Let g be nondecreasing [nonincreasing] on E with $g' \geq r > 0$ [$g' \leq -r < 0$] almost everywhere on E. Then $|g(E)| \geq r|E|$. Also, if $c \in E$ is such that $g(x) \neq g(c)$ for all $x \in E \setminus \{c\}$ and $\overline{g}_{ap}(c) < \infty$ [$\underline{g}_{ap}(c) > -\infty$], then for any set Y_c having density 0 at $g(c)$ the set $g^{-1}(Y_c)$ has density 0 at c.*

Proof. We prove the first case, leaving the analogous second case. Now, if $A = \{x \in E : r \leq g'(x) < \infty\}$ then $|E \setminus A| = 0$. Let $0 < t < r$ and $g(E) \subseteq G$, where G is open. The family $\mathcal{V}$ of the intervals $[a, b]$ with $a, b \in E$ and satisfying $g(b) - g(a) \geq t(b - a)$ and $[g(a), g(b)] \subset G$, is clearly a Vitali cover of the set A. Then, by Vitali's covering theorem [11, page 109], $\mathcal{V}$ has a disjoint subfamily $\{[a_n, b_n]\}$ that covers almost entirely the set A. Since g is nondecreasing on E, clearly $\{[g(a_n), g(b_n)]\}$ is a nonoverlapping family of intervals contained in G, and

$$|A| \leq \sum_n (b_n - a_n) \leq \frac{1}{t} \sum_n \big(g(b_n) - g(a_n)\big) \leq \frac{1}{t}|G|.$$

Thus, $|G| \geq t|A| = t|E|$, for all open sets $G \supseteq g(E)$ and all $0 < t < r$. Hence $|g(E)| \geq r|E|$, proving the first part.

Next, put $E_0 = g^{-1}(Y_c)$. If $\overline{g}_{ap}(c) < K < \infty$, then the set

$$E_1 = \left\{x \in E : x \neq c \text{ and } \frac{g(x) - g(c)}{x - c} \geq K\right\}$$

has density 0 at c. Let $y > c$. If $\big|(E_0 \setminus E_1) \cap [c, y]\big| = 0$, then E_0 has right density 0 at c. In the contrary case, we can find $x \in (E_0 \setminus E_1) \cap (c, y)$ so that

$$\big|(E_0 \setminus E_1) \cap [c, x]\big| > \frac{1}{2}\big|(E_0 \setminus E_1) \cap [c, y]\big|.$$

Then $c < x < y$ and $0 < g(x) - g(c) < K(x - c)$. Hence, using the first part,

$$\begin{aligned}\frac{\big|E_0 \cap [c,y]\big|}{y-c} &\le \frac{\big|(E_0 \cap E_1) \cap [c,y]\big|}{y-c} + \frac{\big|(E_0 \setminus E_1) \cap [c,y]\big|}{y-c} \\ &< \frac{\big|E_1 \cap [c,y]\big|}{y-c} + \frac{2\big|(E_0 \setminus E_1) \cap [c,x]\big|}{y-c} \\ &\le \frac{\big|E_1 \cap [c,y]\big|}{y-c} + \frac{2}{r} \cdot \frac{\big|g\big((E_0 \setminus E_1) \cap [c,x]\big)\big|}{x-c} \\ &\le \frac{\big|E_1 \cap [c,y]\big|}{y-c} + \frac{2K}{r} \cdot \frac{\big|Y_c \cap \big[g(c), g(x)\big]\big|}{g(x)-g(c)}.\end{aligned}$$

Since E_1 has density 0 at c and Y_c has density 0 at $g(c)$, it follows at once from above that E_0 has right density 0 at c. Similarly E_0 has left density 0 at c. Hence E_0 has density 0 at c. □

Lemma 3. (cf. [15, Lemma 51, page 120]). *Let* $\underline{g}_{ap}(x) > r > -\infty$ $\big[\overline{g}_{ap}(x) < r < \infty\big]$ *and* $\underline{d}(E,x) > 0$ *for each* x *in a set* $A \subseteq E$. *Then* A *is the union of a sequence of sets* A_n *such that no point of* $A \setminus A_n$ *is a two-sided limit point of* A_n, *and the function* $g(t) - rt$ *is strictly increasing [decreasing] on* A_n *for each* n.

Proof. Since $\underline{(-g)}_{ap}(x) = -\overline{g}_{ap}(x)$, we need only prove the first case. Now, for each $x \in A$, we have $\overline{d}(E_x, x) = 0$ where

$$E_x = \Big\{t \in E : t \neq x \text{ and } \frac{g(t)-g(x)}{t-x} \le r\Big\}.$$

Then, $A = \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} A_{k,n}$, where $A_{k,n}$ denotes the set of points $x \in A$ such that, if $u \le x \le v$ and $0 < v - u \le \frac{1}{n}$, then

$$\big|E_x \cap [u,v]\big| < \frac{1}{2k}(v-u) \text{ and } \big|E \cap [u,v]\big| > \frac{1}{k}(v-u).$$

If $x \in A_{p,m}$ and $y \in A_{q,n}$ with $0 < y - x \le \min\{\frac{1}{m}, \frac{1}{n}\}$, then

$$\big|E_x \cap [x,y]\big| < \frac{1}{2p}(y-x) \text{ and } \big|E \cap [x,y]\big| > \frac{1}{p}(y-x),$$

$$\big|E_y \cap [x,y]\big| < \frac{1}{2q}(y-x) \text{ and } \big|E \cap [x,y]\big| > \frac{1}{q}(y-x).$$

Setting $k = \min\{p,q\}$, it then follows from these that

$$\begin{aligned}\big|(E_x \cup E_y) \cap [x,y]\big| &\le \big|E_x \cap [x,y]\big| + \big|E_y \cap [x,y]\big| \\ &< \frac{1}{2k}(y-x) + \frac{1}{2k}(y-x) = \frac{1}{k}(y-x) < \big|E \cap [x,y]\big|.\end{aligned}$$

There exists points $t \in E \cap (x,y) \setminus (E_x \cup E_y)$, so that then

$$g(t) - g(x) > r(t-x) \text{ and } g(y) - g(t) > r(y-t).$$

So $g(y) - ry > g(x) - rx$, for all pairs x, y as above.

Now, for $i = 0, \pm 1, \pm 2, \ldots$, let $A^i_{k,n} = A_{k,n} \cap \left[\frac{i}{n}, \frac{i+1}{n}\right]$ and let $B^i_{k,n} = A^i_{k,n} \cup \{x \in A : c \text{ is a two-sided limit point of } A^i_{k,n}\}$. Clearly $B^i_{k,n} \subseteq \left[\frac{i}{n}, \frac{i+1}{n}\right]$ and no point of $A \setminus B^i_{k,n}$ is a two-sided limit point of $B^i_{k,n}$. Let $u, v \in B^i_{k,n}$, $u < v$. Then $0 < v - u \leq \frac{1}{n}$, and choosing $x, y \in A^i_{k,n}$ sufficiently close to u,with v respectively, with $u \leq x < y \leq v$, it readily follows from above that

$$g(v) - rv \geq g(y) - ry > g(x) - rx \geq g(u) - ru.$$

Hence the function $g(t) - rt$ is strictly increasing on each $B^i_{k,n}$. Then the proof ends by taking $A_1, A_2, \ldots$ as an enumeration of the countable family of sets $\{B^i_{k,n}\}$. □

From Lemma 3 and Lemma 2 we easily obtain:

Corollary 1. $\left|\{x \in E : \underline{g}_{ap}(x) = \infty \text{ or } \overline{g}_{ap}(x) = -\infty\}\right| = 0.$

This also follows from the next corollary.

Corollary 2. (cf. [18, Theorem II, page 344]). *At almost every point of the set* $\left\{x \in E : \underline{g}_{ap}(x) > -\infty \text{ or } \overline{g}_{ap}(x) < \infty\right\}$, $g'_{ap}(x)$ *exists and is finite.*

Proof. Let $B = \left\{x \in E : \underline{g}_{ap}(x) > -k\right\}$, where k is a fixed positive integer. By Lemma 1, B is measurable relatively to E. If $A = \left\{x \in B : \underline{d}(E,x) > 0\right\}$ then $|B \setminus A| = 0$. By Lemma 3, A is the union of a sequence of sets A_n such that no point of $A \setminus A_n$ is a two-sided limit point of A_n, and the function $g(x) + kx$ is strictly increasing on A_n for each n. Clearly A_n is measurable relatively to E and $(g_{|A_n})'(x)$ exists finitely almost everywhere on A_n for each n. Since A_n has density 1 (and $E \setminus A_n$ has density 0) at almost every point of A_n, it follows that $g'_{ap}(x)$ exists and is finite almost everywhere on A_n, B and on $\left\{x \in E : \underline{g}_{ap}(x) > -\infty\right\}$. The proof ends by noting that $(\underline{-g})_{ap}(x) = -\overline{g}_{ap}(x)$. The result also follows from [9, Theorem 7.13, page 198]. □

Lemma 4. (cf. [11, Theorem 10.14, page 239]). *Let* $-r < \underline{g}^+_{ap}(x) \leq \overline{g}^+_{ap}(x) < r < \infty$ *and* $d_+(E,x) > 0$ *for each* x *in a set* $A \subseteq E$. *Then* A *is the union of an increasing sequence of sets* (A_k) *where* A_k *is the union of an increasing* *sequence of sets* $(A_{k,n})$*, such that for each* n *and each pair of points* $x, y \in A_{k,n}$ *with* $0 < y - x \le \frac{1}{n}$ *we have* $\big|g(y) - g(x)\big| < r(y - x)$*. Also* $|g(A)| \le r|A|$*, so* g *satisfies Lusin's condition* (N) *on* A*. If, further,* $\overline{g}^{\,+}_{ap}(x) = \underline{g}^{\,+}_{ap}(x) = 0$ *for all* $x \in A$*, then* $|g(A)| = 0$*.*

Note. The left-hand analogue of Lemma 4 is also true. The last assertion extends a similar result of Ellis [5, 3.1, page 480].

Proof. The set A is the union of the increasing sequence of sets (A_k) where

$$A_k = \Big\{x \in A : d_+(E, x) > \frac{1}{k} \text{ and } -\frac{kr}{k+1} \le \underline{g}^{+}_{ap}(x) \le \overline{g}^{+}_{ap}(x) < \frac{kr}{k+1}\Big\}.$$

Fix any k. For each $x \in A_k$ we have $d^+(E_x, x) = 0$ where

$$E_x = \Big\{t \in E : t \ne x \text{ and } \Big|\frac{g(t) - g(x)}{t - x}\Big| \ge \frac{kr}{k+1}\Big\}.$$

Then A_k is the union of the increasing sequence of sets $(A_{k,n})$ where $A_{k,n}$ denotes the set of points x of A_k such that, if $0 < u - x \le \frac{2}{n}$ then

$$\big|E_x \cap [x, u]\big| < \frac{1}{4k^2}(u - x) \text{ and } \big|E \cap [x, u]\big| > \frac{1}{k}(u - x).$$

Let $x, y \in A_{k,n}$ and $0 < y - x \le \frac{1}{n}$. If $u = y + \frac{y-x}{2k}$, then

$$0 < u - y < u - x < 2(y - x) \le \frac{2}{n},$$

and so

$$\frac{\big|E_x \cap [x, u]\big|}{u - x} < \frac{1}{4k^2}, \quad \frac{\big|E_y \cap [y, u]\big|}{u - y} < \frac{1}{4k^2}, \quad \frac{\big|E \cap [y, u]\big|}{u - y} > \frac{1}{k}.$$

Noting that $u - x = (2k + 1)(u - y)$, we obtain

$$\begin{aligned}\big|(E_x \cup E_y) \cap [y, u]\big| &\le \big|E_x \cap [x, u]\big| + \big|E_y \cap [y, u]\big| \\ &< \frac{1}{4k^2}(u - x) + \frac{1}{4k^2}(u - y) \\ &= \frac{k+1}{2k^2}(u - y)) \le \frac{1}{k}(u - y) < \big|E \cap [y, u]\big|.\end{aligned}$$

Hence, there exists points $t \in E \cap (y, u) \setminus (E_x \cup E_y)$, so that

$$\begin{aligned}\big|g(y) - g(x)\big| &\le \big|g(y) - g(t)\big| + \big|g(t) - g(x)\big| \\ &< \frac{kr}{k+1}(t - y) + \frac{kr}{k+1}(t - x) \\ &< \frac{kr}{k+1}\big[(u - y) + (u - x)\big] = r(y - x).\end{aligned}$$

Next, consider any open cover $\{(a_j, b_j)\}_{j=1}^{\infty}$ of the set $A_{k,n}$, where $b_j - a_j < \frac{1}{n}$ for each j. For all x, y in $A_{k,n} \cap (a_j, b_j)$ we have then

$$\big|g(y) - g(x)\big| \leq r|y - x| < r(b_j - a_j).$$

Since $\big|g\big(A_{k,n} \cap (a_j, b_j)\big)\big|$ cannot exceed the oscillation of g on $A_{k,n} \cap (a_j, b_j)$, it follows that

$$\big|g(A_{k,n})\big| \leq \sum_{j=1}^{\infty}\big|g\big(A_{k,n} \cap (a_j, b_j)\big)\big| \leq \sum_{j=1}^{\infty} r(b_j - a_j).$$

Therefore, $\big|g(A_{k,n})\big| \leq r|A_{k,n}|$. Letting $n \to \infty$, we get $\big|g(A_k)\big| \leq r|A_k|$. Letting $k \to \infty$, we get $\big|g(A)\big| \leq r|A|$.

Finally, when $\overline{g}^{+}_{ap}(x) = \underline{g}^{+}_{ap}(x) = 0$ for all $x \in A$, for any positive integer n taking $r = \frac{1}{n} \cdot \frac{1}{2n}$, we have

$$\big|g\big(A \cap (-n, n)\big)\big| \leq r\big|A \cap (-n, n)\big| \leq \frac{1}{n},$$

which plainly implies that $|g(A)| = 0$. □

We now generalize the Krzyzewski–Foran lemma as follows.

Theorem 1. *If $|g(E)| = 0$, then $\underline{g}_{ap} \leq 0 \leq \overline{g}_{ap}$ almost everywhere on E.*

Proof. Since $\big|(-g)(E)\big| = \big|g(E)\big| = 0$ and $\underline{(-g)}_{ap} = -\overline{g}_{ap}$, clearly it is enough to show that $|A| = 0$ where $A = \big\{x \in E : \underline{d}(E, x) = 1 \text{ and } \underline{g}_{ap}(x) > 0\big\}$.

Now, $A = \cup_{k=1}^{\infty} A_k$ where

$$A_k = \big\{x \in A : \underline{g}_{ap}(x) > \frac{1}{k}\big\}.$$

By Lemma 3, each A_k is the union of a sequence of sets $(A_{k,n})$ such that the function $g(t) - \frac{t}{k}$ is strictly increasing on $A_{k,n}$ for each n. Then

$$g(t) = \Big(g(t) - \frac{t}{k}\Big) + \frac{t}{k}$$

is strictly increasing on $A_{k,n}$ and $\big(g_{|A_{k,n}}\big)' \geq \frac{1}{k}$ almost everywhere on $A_{k,n}$. So by Lemma 2, $|A_{k,n}| \leq k\big|g(A_{k,n})\big| \leq k\big|g(E)\big| = 0$. Thus, it follows that $|A| = 0$. □

Theorem 2. (compare [3, Corollary 2.3, page 642]). *Suppose $g'_{ap}(x)$ exists for almost every x in E. Then g satisfies the inverse Lusin condition (N^{-1}) on E if and only if $|E_0| = 0$, where $E_0 = \{x \in E : g'_{ap}(x) = 0\}$.*

Proof. First assume that g satisfies the condition (N^{-1}) on E. Let $A = \{x \in E_0 : \underline{d}(E, x) = 1\}$. Then $|E_0 \setminus A| = 0$, and by Lemma 4 $|g(A)| = 0$. So $|E_0| = |A| \leq |g^{-1}(g(A))| = 0$, by (N^{-1}). Hence $|E_0| = 0$.

Conversely, assume that $|E_0| = 0$. Consider any $H \subseteq g(E)$ with $|H| = 0$. Let $B = g^{-1}(H)$. Then $|g(B)| = 0$. Since $(g_{|B})'_{ap}(x) = g'_{ap}(x)$ exists for almost every x in B, Theorem 1 implies that $g'_{ap} = 0$ almost everywhere on B. Due to $|E_0| = 0$, it follows that $|B| = 0$. Hence g is an (N^{-1})-function on the set E. $\square$

Corollary 3. (Villani [17, Theorem 2, page 331]). *If $f : [a, b] \to \mathbb{R}$ is strictly monotone and continuous, then the inverse function f^{-1} is absolutely continuous if and only if $f' \neq 0$ almost everywhere on $[a, b]$.*

Below, we improve upon Foran's chain rule [7, Theorem 1, page 445].

Define [7] $g^*_{ap}(x) = g'_{ap}(x)$ if it exists and is finite, and $g^*_{ap}(x) = 0$ otherwise.

Theorem 3. *Let $f : Y \to \mathbb{R}$ where $g(E) \subseteq Y \subseteq \mathbb{R}$, and let*

$$A = \{x \in E : \text{both } g'_{ap}(x) \text{ and } f'_{ap}(g(x)) \text{ exist and } g'_{ap}(x) \neq 0\}.$$

(i) For almost every x in A, $(f \circ g)'_{ap}(x)$ exists and

$$(f \circ g)'_{ap}(x) = f'_{ap}(g(x)) \cdot g'_{ap}(x) \neq \pm\infty. \tag{3}$$

(ii) Let $Z = \{x \in E \setminus A : (f \circ g)^*_{ap}(x) \neq 0\}$. Foran's chain rule

$$(f \circ g)^*_{ap}(x) = f^*_{ap}(g(x)) \cdot g^*_{ap}(x) \tag{4}$$

holds true almost everywhere on E if and only if $|Z| = 0$.

Proof. Let $A_0 = \{x \in E : g'_{ap}(x)$ exists and is not $0\}$, and note the following. We have $\underline{d}(E, x) = 1$ for almost every x in A_0. By Corollary 1, $g'_{ap}(x)$ is finite for almost every x in A_0. Again by Corollary 1 on f, if B is the set of points x of A for which $f'_{ap}(g(x))$ is infinite, then $|g(B)| = 0$. Since $B \subseteq A \subseteq A_0$, Theorem 1 implies that $|B| = 0$. Thus both $g'_{ap}(x)$ and $f'_{ap}(g(x))$ are finite for almost every x in A.

Now, by Lemma 1, for each positive integer k the set

$$A_k = \Big\{x \in A_0 : \underline{d}(E,x) = 1 \text{ and } g'_{ap}(x) > \frac{1}{k}\Big\}$$

is measurable relatively to E. By Lemma 3, A_k is the union of a sequence of sets $A_{k,n}$ such that no point of $A_k \setminus A_{k,n}$ is a two-sided limit point of $A_{k,n}$ and the function $g(x) - \frac{x}{k}$ is strictly increasing on $A_{k,n}$ for each n. Then, clearly, the set $A_{k,n}$ is measurable relatively to E, g is strictly increasing on $A_{k,n}$ and $\big(g_{|A_{k,n}}\big)' \geq \frac{1}{k}$ almost everywhere on $A_{k,n}$.

Take any $c \in A \cap A_{k,n}$ where $\underline{d}(A_{k,n}, c) = 1$ and both $g'_{ap}(c)$ and $f'_{ap}\big(g(c)\big)$ are finite. Let $0 < \epsilon < g'_{ap}(c)$. Then the set

$$E_c = \Big\{x \in E : x = c \text{ or } \Big|\frac{g(x)-g(c)}{x-c} - g'_{ap}(c)\Big| \geq \epsilon\Big\}$$

has density 0 at c. Also the set

$$Y_c = \Big\{y \in Y : y \neq g(c) \text{ and } \Big|\frac{f(y) - f\big(g(c)\big)}{y - g(c)} - f'_{ap}\big(g(c)\big)\Big| \geq \epsilon\Big\}$$

has density 0 at $g(c)$, and so by Lemma 2 on $g_{|A_{k,n}}$ the set $A_{k,n} \cap g^{-1}(Y_c)$ has density 0 at c. Recalling that $\underline{d}(E,c) = \underline{d}(A_{k,n}, c) = 1$, the set

$$D = (E \setminus A_{k,n}) \cup E_c \cup \big(A_{k,n} \cap g^{-1}(Y_c)\big)$$

has density 0 at c. If $x \in E \setminus D$, then $x \in A_{k,n}$, $x \neq c$,

$$\Big|\frac{g(x)-g(c)}{x-c} - g'_{ap}(c)\Big| < \epsilon, \tag{5}$$

so $g(x) \neq g(c)$, and $x \notin g^{-1}(Y_c)$, that is, $g(x) \notin Y_c$, and so

$$\Big|\frac{f\big(g(x)\big) - f\big(g(c)\big)}{g(x)-g(c)} - f'_{ap}\big(g(c)\big)\Big| < \epsilon. \tag{6}$$

Evidently, (5) and (6) together imply by virtue of the relation

$$\frac{(f\circ g)(x) - (f \circ g)(c)}{x-c} = \frac{f\big(g(x)\big) - f\big(g(c)\big)}{g(x)-g(c)} \cdot \frac{g(x)-g(c)}{x-c},$$

that $(f \circ g)'_{ap}(c)$ exists and equals $f'_{ap}\big(g(c)\big) \cdot g'_{ap}(c)$. So $(f\circ g)'_{ap}(x)$ exists and satisfies (3) almost everywhere on $\{x \in A : g'_{ap}(x) > 0\}$ and similarly, on $\{x \in A : g'_{ap}(x) < 0\}$, and on A, proving (i).

Now by (i), (4) is true for almost every x in A. Also from the definitions, $f^*_{ap}\big(g(x)\big) \cdot g^*_{ap}(x) = 0$ at each point x of $E \setminus A$. Thus, (ii) follows at once. □

Note. Let $Z' = \{x \in Z : \underline{d}(E, x) = 1\}$. For each x in Z, since $(f \circ g)^*_{ap}(x) \neq 0$ so $(f \circ g)'_{ap}(x)$ exists and is finite and non-zero. So by Lemma 4 and Theorem 2, $f \circ g$ satisfies both the conditions (N) and (N^{-1}) on Z'. Hence, the condition $|Z| = 0$ is equivalent to each of the conditions $|Z'| = 0$ and $|(f \circ g)(Z')| = 0$.

We continue to use consistently the notations of Theorem 3.

Corollary 4. *Let*

$$P = \{x \in E : (f \circ g)^*_{ap}(x) \neq 0\},\ Q = \{y \in Y : f^*_{ap}(y) \neq 0\}$$

and

$$K = \{x \in P \cap g^{-1}(Q) : g^*_{ap}(x) = 0 \text{ and } \underline{d}(Y, g(x)) = 0\}.$$

The chain rule (4) *holds true almost everywhere on E if and only if g is measurable on P relatively to E and $|K| = |P \setminus g^{-1}(Q)| = 0$.*

Proof. Note that, by Lemma 1, the set P $[Q]$ and the function $(f \circ g)'_{ap}$ on P $[f'_{ap}$ on $Q]$ are measurable relatively to E $[Y]$.

Now, suppose the chain rule (4) holds true almost everywhere on E. Then obviously $|K| = 0$ and $|P \setminus g^{-1}(Q)| = 0$. Also, for almost every point x in P, we must have $g^*_{ap}(x) \neq 0$ and so g is plainly approximately continuous at x relatively to E. Hence g is measurable on P relatively to E (cf. [12, Theorem 4.2, page 245]).

Conversely, assume that the stated conditions hold. Put

$$P' = \{x \in P : \underline{d}(E, x) > 0\},\ Q' = \{y \in Q : \underline{d}(Y, y) > 0\}.$$

Note that, if a subset $H \subseteq Q'$ is measurable relatively to Y, then $H = (Y \cap M) \cup W$ where M is an F_σ-set and $|W| = 0$. Since by Lemma 4 f satisfies (N) on Q', then $|f(W)| = 0$. Therefore, since

$$(f \circ g)(P \cap g^{-1}(W)) \subseteq f(W)$$

and by Theorem 2 $f \circ g$ satisfies $(N^{-1}$ on P, we have $|P \cap g^{-1}(W)| = 0$. Also $P \cap g^{-1}(Y \cap M) = P \cap g^{-1}(M)$ is measurable relatively to E, due to hypothesis g is measurable on P relatively to E. Therefore $P \cap g^{-1}(H)$ is measurable relatively to E.

Now, using Lemma 3, we can evidently express P' $[Q']$ as the union of a sequence of sets P_n $[Q_n]$, each measurable relatively to E $[Y]$ and on each of which $f \circ g$ $[f]$ is strictly monotone (not necessarily of the same type). Then for all i, $j = 1, 2, \ldots$, by above $P_i \cap g^{-1}(Q_j)$ is measurable relatively to E and

g is strictly monotone on it (use the type of monotonicity of $f \circ g$ on P_i and then that of f on Q_j). Hence, obviously, $g'_{ap}(x)$ exists and is finite almost everywhere on $P_i \cap g^{-1}(Q_j)$, and on $P' \cap g^{-1}(Q')$.

Now let $B = \{x \in P' \cap g^{-1}(Q') : g'_{ap}(x)$ exists and is $0\}$. By Lemma 4, $|g(B)| = 0$. Since f satisfies (N) on Q', then

$$\big|(f \circ g)(B)\big| = \big|f\big(g(B)\big)\big| = 0.$$

Since $f \circ g$ satisfies (N^{-1}) on P, so $|B| = 0$.

Recalling Corollary 1, $g^*_{ap} \neq 0$ almost everywhere on $P' \cap g^{-1}(Q')$. Since $|P \setminus P'| = 0$ and $|K| = 0$, it follows that $g^*_{ap} \neq 0$ almost everywhere on $P \cap g^{-1}(Q)$. Additionally $g^*_{ap} = 0$ on $Z \cap g^{-1}(Q)$, and by hypothesis $\big|P \setminus g^{-1}(Q)\big| = 0$, so $|Z| = 0$ and the result follows from Theorem 3. □

Corollary 5. *If the set E is measurable, then the chain rule* (4) *holds true almost everywhere on E if (and only if) $|C| = 0$, where*

$$C = \big\{x \in E : (f \circ g)^*_{ap}(x) \neq 0 \text{ and } g^*_{ap}(x) = 0\big\}.$$

Proof. By Lemma 1, the set

$$D = \big\{x \in E : \underline{d}(E, x) > 0 \text{ and } g^*_{ap}(x) \neq 0 \text{ and } (f \circ g)^*_{ap}(x) \neq 0\big\}$$

and the functions g'_{ap} and $(f \circ g)'_{ap}$ on D are measurable. So Lemma 3 plainly implies that D is the union of a sequence of measurable sets D_n on each of which both g and $f \circ g$ are strictly monotone (not necessarily of the same type). Then for each n, f is strictly monotone on $g(D_n)$ (use the type of monotonicity of g^{-1} on $g(D_n)$ and that of $f \circ g$ on D_n). Also, since g is measurable on D_n and by Lemma 4 it satisfies (N) on D_n, so $g(D_n)$ is measurable [5, 2.1, page 476]. Obviously, $f'_{ap}(y)$ exists and is finite almost everywhere on $g(D_n)$, and on $g(D)$. Thus,

$$\big|g(D_0)\big| = 0 \text{ where } D_0 = \big\{x \in D \ : \ f'_{ap}\big(g(x)\big) \text{ does not exist}\big\}.$$

Since by Theorem 2 g satisfies (N^{-1}) on D, then $|D_0| = 0$. Since also by hypothesis $|C| = 0$, and $\underline{d}(E, x) = 0$ for all $x \in Z \setminus (C \cup D_0)$, so $|Z| = 0$ and the result follows from Theorem 3. □

Remark. Foran considered the following two theorems [7, page 445]:

Suppose $E = [a, b]$ and Y is any interval.

(F0) If f satisfies (N) on Y and $f'_{ap}(y)$ exists almost everywhere on

$g(E)$, then the chain rule (4) holds true almost everywhere on E.

(F1) Let $V = \{x \in E : g'_{ap}(x)$ exists and is $0\}$ and $B = \{x \in E : g'_{ap}(x)$ does not exist$\}$. If f satisfies (N) on $g(V \cup B)$ and $f'_{ap}(y)$ exists almost everywhere on $g(B)$, then the chain rule (4) holds true almost everywhere on E.

While (F1) implies (F0) the following simple example shows that both are false. Take $E = [0, 1]$ and $Y = [-1, 1]$. Express E as the union of two disjoint non-measurable sets S and T such that $|S| = |T| = 1$ [16, §1]. Let $g(x) = x$ if $x \in S$ and $g(x) = -x$ if $x \in T$, and let $f(y) = |y|$ for all $y \in Y$. Then $(f \circ g)(x) = x$ and $g^*_{ap}(x) = 0$ for all $x \in E$, and the hypotheses of (F0) are satisfied but the conclusion is not.

However, both (F0) and (F1) can be repaired in various ways in the light of Theorem 3 and its corollaries (see, also, [6, page 202–204].

References

[1] H. Blumberg, *The measurable boundaries of an arbitrary function*, Acta Math., **65** (1935), 263–282.

[2] J. C. Burkill and U. S. Haslam-Jones, *The derivates and approximate derivates of merasurable functions*, Proc. London Math. Soc., **34 No. 2** (1931), 346–355.

[3] F. S. Cater, *On derivatives vanishing almost everywhere on certain sets*, Real Analysis Exchange, **23 No. 2** (1997/98), 641–652.

[4] S. E. Chow, *On approximate derivatives*, Bull. Amer. Math. Soc., **54** (1948), 793–802.

[5] H. W. Ellis, *Darboux properties and applications to non-absolutely convergent integrals*, Canad. J. Math., **3** (1951), 471–485.

[6] V. Ene, *Real functions – current topics*, Lect. Notes in Math., **1603**, Springer-Verlag, 1995.

[7] J. Foran, *A chain rule for the approximate derivative and change of variables for the D-integral*, Real Analysis Exchange, **8 No. 2** (1982/83), 443–454.

[8] G. S. Goodman, (N)*-functions and integration by substitution*, Rend. Sem. Mat. Fis. Milano, **47** (1978), 123–134.

[9] R. L. Jeffery, *The theory of functions of a real variable*, 2nd ed., Toronto, 1953.

[10] K. Krzyzewski, *On change of variable in the Denjoy-Perron integral (I), (II)*, Coll. Math., **9** (1962), 99–104, 317–323.

[11] S. Saks, *Theory of the integral*, 2nd. rev. ed., Dover, New York, 1964.

[12] D. N. Sarkhel and T. Chakraborti, *Topological Vitali measure spaces*, Bull. Austral. Math. Soc., **32** (1985), 225–249.

[13] D. N. Sarkhel and T. Chakraborti, *Measurable boundaries of functions in topological Vitali measure spaces*, Bull. Inst. Math. Acad. Sinica, **15 No. 4** (1987), 439–448.

[14] J. Serrin and D. E. Varberg, *A general chain rule for derivatives and the change of variables formula for the Lebesgue integral*, Amer. Math. Monthly, **76** (1969), 514–520.

[15] B. S. Thomson, *Real functions*, Lect. Notes in Math., **1170**, Springer-Verlag, 1985.

[16] E. B. Van Vleck, *On non-measurable sets of points, with an example*, Trans. Amer. Math. Soc., **9** (1908), 237–244.

[17] A. Villani, *On Lusin's condition for the inverse function*, Rendi. Circ. Mat. Palermo, **33 No. 2** (1984), 331–335.

[18] A. J. Ward, *On the differential structure of real functions*, Proc. London Math. Soc., **39 No. 2** (1953), 339–362.