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ON NATURAL FUNCTIONS AND LIPSCHITZ FUNCTIONS

Abstract

We give certain characterizations of the so-called natural functions introduced by Chistyakov. We show that the set of natural functions from $[a, b]$ into a normed space X is small in the porosity sense in the set of uniform Lipschitz 1 functions with the norm inherited from the space of BV functions. The sizes of other classes of Lipschitz functions in BV are also established.

1 Some Characterizations of Natural Functions

We use the following notation: $E \subset \mathbb{R}$ is a non-empty bounded set; for $t \in E$ we put $E_t^- = \{s \in E \mid s \le t\}$ and $E_t^+ = \{s \in E \mid t \le s\}$; for $a, b \in E$ $(a \le b)$ let $E_a^b = \{s \in E \mid a \le s \le b\}$; X is a metric space with metric d, and X^E is the set of all functions from E into X. If $f \in X^E$, we denote by $\omega(f, E) = \sup\{d(f(t), f(s)) \mid t, s \in E\}$ the diameter of the image $f(E)$, called the *oscillation* of f on E. Let $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{Q}$ stand for the set of all rationals.

Let $\mathcal{T}(E) = \{T = \{t_i\}_{i=0}^m \subset E \mid m \in \mathbb{N},\ t_{i-1} \le t_i,\ i = 1, \dots, m\}$ be the set of all partitions of E by finite ordered collections of points in E. For a function $f\colon E \to X$ and a partition $T = \{t_i\}_{i=0}^m \in \mathcal{T}(E)$, we set

$$V(f, T) = \sum_{i=1}^m d(f(t_i), f(t_{i-1}))$$

and we extend this definition to the entire set E by the formula

$$V(f, E) = \sup\{V(f, T) \mid T \in \mathcal{T}(E)\}.$$

We call the quantity $V(f, E) \in [0, \infty]$ the *total variation* (in the sense of Jordan) of the function f on E. If $V(f, E)$ is finite, then we call f a *function*

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of bounded variation on E. We denote the set of all functions of bounded variation on E by $BV(E)$. Functions of the class $BV(E)$ were considered by V. V. Chistyakov in [1], [2] and [3]. He called a function $g\colon E \to X$ *natural* if $V(g, E_a^b) = b - a$ for all $a, b \in E$, $a \le b$. Obviously a natural function is a Lipschitz function and $\mathcal{L}ip(g)$, the smallest Lipschitz constant for g, equals 1. Chistyakov proved the following.

Theorem 1. *[1, Thm 3.1] A function $f\colon E \to X$ has bounded variation if and only if there exists a non-decreasing bounded function $\varphi\colon E \to \mathbb{R}$ and a natural function $g\colon \varphi(E) \to X$ such that $f = g \circ \varphi$ on E.*

Let us recall some properties of variation.

Proposition 1. *[1, Prop 2.1] For an arbitrary function $f\colon E \to X$ we have:*

(V1) if $t, s \in E$, then $d(f(t), f(s)) \le \omega(f, E) \le V(f, E)$;

(V2) if $t \in E$, then $V(f, E) = V(f, E_t^-) + V(f, E_t^+)$;

(V3) $V(f, E) = \sup\{V(f, E_a^b) \mid a, b \in E, a \le b\}$;

(V4) if $A \subset B \subset E$, then $\mathcal{T}(A) \subset \mathcal{T}(B)$ and $V(f, A) \le V(f, B)$.

Let us start with the following simple characterization.

Proposition 2. *Let $f\colon E \to X$. The following conditions are equivalent:*

(a) f is a natural function;

(b) $V(f, E_x^-) = x + c,\ x \in E,$ where $c = -\inf(E)$;

(c) f is a Lipschitz function such that $\mathcal{L}ip(f) \le 1$ and $V(f, E) = \sup(E) - \inf(E)$.

Proof. (a) $\Rightarrow$ (b) By property (V3) in Proposition 1 and (a) we obtain

$$\begin{aligned} V(f,E) &= \sup\{V(f, E_a^b) \mid a, b \in E, a \le b\} \\ &= \sup\{(b-a) \mid a, b \in E, a \le b\} = \sup(E) - \inf(E). \end{aligned}$$

So we have $V(f, E_x^-) = x - \inf(E) = x + c$, where $c = -\inf(E)$.

(b) $\Rightarrow$ (c) By (b) we get $V(f, E) = \sup(E) - \inf(E)$. We will show that f is a Lipschitz function such that $\mathcal{L}ip(f) \le 1$. Let $x, y \in E$, $x \le y$. By (V1), (V2) in Proposition 1 and (b), we have

$$d(f(x), f(y)) \le V(f, E_x^y) = V(f, E_y^-) - V(f, E_x^-) = y + c - x - c = y - x.$$

Consequently, f is a Lipschitz function and $\mathcal{L}ip(f) \le 1$.

(c) $\Rightarrow$ (a) Suppose that f is not a natural function. Then there exist $x, y \in E$, $x < y$, such that $V(f, E_x^y) < y - x$. Thus by condition (V2) we have

$$\begin{aligned}\sup(E) - \inf(E) =& V(f, E) = V(f, E_x^-) + V(f, E_x^y) + V(f, E_y^+)\\ \le& x - \inf(E) + V(f, E_x^y) + \sup(E) - y\\ <& x - \inf(E) + y - x + \sup(E) - y\\ =& \sup(E) - \inf(E).\end{aligned}$$

This contradiction completes the proof. $\square$

We put $V_a^b f = V(f, [a, b])$. The set of all real-valued absolutely continuous functions on $[a, b]$ will be written as AC.

Corollary 1. *Let $f\colon [a, b] \to \mathbb{R}$. Then f is a natural function if and only if $f \in AC$ and $|f'(x)| = 1$ a.e. on $[a, b]$.*

Proof. We assume that f is a natural function. Then $f \in AC$ and by [4, Thm 8.14] we have $(V_a^x f)' = |f'(x)|$ a.e. on $[a, b]$. From this and Proposition 2 it follows that $|f'(x)| = (V_a^x f)' = (x - a)' = 1$ a.e. on $[a, b]$. Now, we assume that $f \in AC$ and $|f'(x)| = 1$ a.e. on $[a, b]$. By [4, Thm 8.14] we get $V_a^x f = \int_a^x |f'| = x - a$. Applying Proposition 2 completes the proof. $\square$

From now on, we assume that $X \neq \{0\}$ is a normed space over $\mathbb{R}$, with the norm $|\cdot|$.

Theorem 2. *Let $f\colon [a, b] \to X$ be a function with $0 < V_a^b f < \infty$. The following conditions are equivalent:*

(a) f/α is a natural function, where $\alpha = V_a^b f/(b - a)$;

(b) f is continuous at points a and b, and there exists a set D dense in $[a, b]$ such that

$$(\forall t, s, p, q \in D)\ (t < s,\ p < q,\ s - t \le q - p) \Rightarrow V_t^s f \le V_p^q f;$$

(c) $(\forall t, s, p, q \in [a, b])\ (t < s,\ p < q,\ s - t = q - p) \Rightarrow V_t^s f = V_p^q f$.

Proof. (a) $\Rightarrow$ (b) This results from the definition of a natural function and (V4) in Proposition 1.

(b) $\Rightarrow$ (c) First, we will show that f is continuous on $[a, b]$. Assume to the contrary that $x_0 \in (a, b)$ is a discontinuity point of f. (By assumption (b) we know that f is continuous at points a and b.) Let $x_n', x_n'' \in D$ be such that $x_n' < x_0 < x_n''$ for all $n \in \mathbb{N}$ and $x_n' \to x_0$, $x_n'' \to x_0$. Since f is discontinuous at x_0, we have $\sup_n \omega(f, [x_n', x_n'']) = \varepsilon > 0$. Thus from Proposition 1 (V1) it follows that $V_{x_n'}^{x_n''} f \geq \varepsilon$ for all n. On the other hand, since $f \in BV([a,b])$ and D is dense in $[a,b]$, there exist points $x_1, x_2 \in D$ such that $x_1 < x_2$ and $V_{x_1}^{x_2} f < \varepsilon$. Choosing n such that $x_n'' - x_n' \leq x_2 - x_1$, we obtain a contradiction to (b). From now on f is assumed continuous on $[a,b]$.

Let $t, s, p, q \in (a,b)$ and $t < s$, $p < q$, $s - t = q - p$. Pick numbers $t_n', t_n'', s_n', s_n'', p_n', p_n'', q_n', q_n''$ in D so that

$$t_n' \leq t \leq t_n'' \leq s_n' \leq s \leq s_n'', \quad p_n' \leq p \leq p_n'' \leq q_n' \leq q \leq q_n'',$$

$$t_n' \to t,\ t_n'' \to t,\ s_n' \to s,\ s_n'' \to s,\ p_n' \to p,\ p_n'' \to p,\ q_n' \to q,\ q_n'' \to q.$$

We may additionally assume that all sequences $\{t_n{}'\}, \ldots, \{q_n{}''\}$ are monotonic and that we have $s_n{}'' - t_n' \geq q_n' - p_n'' \geq 0$ and $q_n'' - p_n' \geq s_n' - t_n'' \geq 0$ for all n. By (b), it follows that $V_{p_n''}^{q_n'} f \leq V_{t_n'}^{s_n''} f$ and $V_{t_n''}^{s_n'} f \leq V_{p_n'}^{q_n''} f$. Since f is continuous, we have $V_p^q f \leq V_t^s f \leq V_p^q f$. Consequently, $V_t^s f = V_p^q f$.

Let $t, s, p, q \in [a,b]$ and $t < s$, $p < q$, $s - t = q - p$. Choose $\varepsilon \in (0, s-t)$. By the first part of the proof we have $V_{t+\frac{\varepsilon}{2}}^{s-\frac{\varepsilon}{2}} f = V_{p+\frac{\varepsilon}{2}}^{q-\frac{\varepsilon}{2}} f$. Since f is continuous, taking $\varepsilon \to 0$ we obtain $V_t^s f = V_p^q f$.

(c) $\Rightarrow$ (a) Put $\alpha = V_a^b f/(b-a)$. Let n be a positive integer and let $x_k = a + k(b-a)/n$ for $k = 0, \ldots, n$. Since $\sum_{k=1}^{n} V_{x_{k-1}}^{x_k} f = V_a^b f$ and $V_{x_{k-1}}^{x_k} f = V_{x_{j-1}}^{x_j} f$ for any $k, j \in \{1, \ldots, n\}$, we have $V_{x_{k-1}}^{x_k} f = \alpha(b-a)/n = \alpha(x_k - x_{k-1})$ for every k. Observe that the function $\Phi(x) = V_a^x f$ is continuous on $[a,b]$. Let $x \in [a,b)$, $\varepsilon > 0$. Choose a positive integer n such that $\alpha(b-a)/n \leq \varepsilon$. Pick x_{k-1}, x_k such that $x \in [x_{k-1}, x_k)$. For each $t \in [x, x_k)$ we have

$$|\Phi(t) - \Phi(x)| = V_a^t f - V_a^x f = V_x^t f \leq V_{x_{k-1}}^{x_k} f = \alpha(b-a)/n \leq \varepsilon.$$

Hence f is continuous from the right at x. Analogously, f is continuous from the left at each point $x \in (a,b]$. The set $D = \{a + k(b-a)/n \mid n \in \mathbb{N}, k \in \{0, \ldots, n\}\}$ is dense in $[a,b]$. Let $x \in [a,b]$ and pick a sequence $x_k \in D$, $k \in \mathbb{N}$, $x_k \to x$. We have $\Phi(x) = \lim_{k\to\infty} \alpha(x_k - a) = \alpha(x - a)$. Hence $V_a^x(f/\alpha) = (V_a^x f)/\alpha = x - a$. By Proposition 2 we deduce that f/α is a natural function. $\square$

Remarks. 1. The assumption $E = [a,b]$ is essential in Theorem 2. Indeed, let $E = \{0, 1, 3\}$, and let a function $f\colon E \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ 4 & \text{if } x = 3. \end{cases}$$

Then f satisfies the assumptions (b) and (c) in Theorem 2, but f/α is not a natural function for any $\alpha > 0$.

2. We may write "for every set D dense in $[a, b]$" instead of "there exists a set D dense in $[a, b]$" in Theorem 2(b).

3. The assumption that f is continuous at points a and b is essential in Theorem 2(b). Indeed, let $f\colon [1, \sqrt{2}] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \in [1, \sqrt{2}) \\ 1 & \text{if } x = \sqrt{2} \end{cases}$$

and put $D = \mathbb{Q} \cap [1, \sqrt{2}]$. Then f/α is not a natural function (f is not continuous) but the final condition in the statement (b) is satisfied.

4. We can replace the statement (b) in Theorem 2 by the following. There exists a set D dense in $[a, b]$ with $a, b \in D$ and such that

$$(\forall t, s, p, q \in D)\ (t < s,\ p < q,\ s - t \leq q - p) \Rightarrow V_t^s f \leq V_p^q f.$$

2 Porosity Sizes of Some Classes of Lipschitz Functions

Recall the definition of a strongly porous set [5]. Let P be a metric space. The open ball with center $x \in P$ and radius $r > 0$ will be denoted by $B(x, r)$. Let $M \subset P$, $x \in P$ and $R > 0$. Then we denote by $\gamma(x, R, M)$ the supremum of the set of all $r > 0$ for which there exists $z \in P$ such that $B(z, r) \subset B(x, R) \setminus M$. The number $p(M, x) = 2 \cdot \limsup_{R \to 0^+} \dfrac{\gamma(x, R, M)}{R}$ is called the *porosity* of M at x. The set M is called *porous* (*strongly porous*) *at* x if $p(M, x) > 0$ ($p(M, x) = 1$). The set M is (*strongly*) *porous* if it is (strongly) porous at each of its points. Obviously, a porous set is nowhere dense.

If one puts $\|f\| = V_a^b f + |f(a)|$ for $f\colon [a, b] \to X$, $f \in BV([a, b])$, then $\|\cdot\|\colon BV([a, b]) \to [0, \infty)$ is a norm on $BV([a, b])$. In fact $BV([a, b])$ is a Banach space provided X is a Banach space [3]. We will consider various subsets of the space $BV([a, b])$. We will study their sizes in the language of porosity or Baire category.

In the sequel, we denote the set consisting of all natural functions from $[a, b]$ to X by $\mathcal{N}$. We write $\alpha\mathcal{N} = \{f\colon [a, b] \to X \mid (f/\alpha) \in \mathcal{N}\}$ for $\alpha > 0$. Functions from $\alpha\mathcal{N}$ will be called *α-natural* on $[a,b]$. We denote the set consisting of all Lipschitz functions from $[a,b]$ to X, with $\mathcal{L}ip(f)=\alpha$ by $\mathcal{L}ip(\alpha)$, and the set consisting of all Lipschitz functions by $\mathcal{L}ip$. We say that $f\colon [a,b]\to X$ satisfies the *uniform Lipschitz condition* with constant α if $\mathcal{L}ip(f|_{[c,d]})=\alpha$ for all $c,d\in[a,b], c<d$. We denote by u-$\mathcal{L}ip(\alpha)$ the set of all functions which satisfy the uniform Lipschitz condition with constant α.

Lemma 1. *If $\alpha>0$, then $\alpha\mathcal{N}\subset u$-$\mathcal{L}ip(\alpha)$.*

Proof. Let $f\in\alpha\mathcal{N}$ and $\alpha>0$. By (V1) in Proposition 1 we have

$$|f(t)-f(s)|\le V_t^s f=\alpha(s-t) \text{ for all } t,s\in[a,b], t\le s.$$

Hence $\mathcal{L}ip(f)\le\alpha$.

Let $c,d\in[a,b]$ and $c\le d$. For a partition $T=\{t_i\}_{i=0}^m$ of $[c,d]$, $c\le t_0\le \dots\le t_m\le d$, we have

$$\begin{aligned}V(f,T)=&\sum_{i=1}^m|f(t_i)-f(t_{i-1})|\le\sum_{i=1}^m\mathcal{L}ip(f|_{[c,d]})(t_i-t_{i-1})\\ \le&\mathcal{L}ip(f|_{[c,d]})(d-c)\le\alpha(d-c).\end{aligned}$$

Hence

$$\alpha(d-c)=V_c^d f\le\mathcal{L}ip(f|_{[c,d]})(d-c)\le\alpha(d-c).$$

Thus we obtain $\mathcal{L}ip(f|_{[c,d]})=\alpha$ and $\alpha\mathcal{N}\subset u$-$\mathcal{L}ip(\alpha)$. □

By a *Cantor set* in $[a,b]$ we mean any perfect nowhere dense subset C of $[a,b]$ with $a,b\in C$.

Lemma 2. *Let $\beta>0$ and $v\colon[a,b]\to X$, $v\in\bigcup_{0\le\alpha\le\beta}\mathcal{L}ip(\alpha)$. Let C be a Cantor set in $[a,b]$. Let $g\colon[a,b]\to X$ be a function equal to $v(x)$ for $x\in C$, and defined on a component (c,d) of $C^c=[a,b]\setminus C$ as follows. Fix $z\in X$ with $|z|=1$. Put*

$$w=\begin{cases}z & \text{if } v(c)=v(d)\\ \frac{v(d)-v(c)}{|v(d)-v(c)|} & \text{if } v(c)\ne v(d)\end{cases}$$

and

$$s=\frac{c+d}{2}+\frac{|v(d)-v(c)|}{2}.$$

Define

$$g(x)=\begin{cases}\beta(x-c)w+v(c) & \text{if } x\in(c,s]\\ \beta(d-x)w+v(d) & \text{if } x\in(s,d).\end{cases}$$

Then $g\in u$-$\mathcal{L}ip(\beta)$. Moreover, if v is constant on $[a,b]$ and $\mu(C)>0$, then $V_a^b g=\beta\mu(C^c)$.

Proof. Let (c,d) be a component of C^c. The function $g|_{[c,d]}$ is β-natural on $[c,d]$ which results from the definition of g. Let $x,y \in [a,b],\ x<y$. Then there exist $c,d \in [x,y]$, $c<d$, such that $(c,d) \subset C^c$. Observe that $\mathcal{L}ip(g|_{[x,y]}) \geq \beta$. We will show that $|g(y)-g(x)| \leq \beta(y-x)$. If $x,y \in C$, then by the definition of g we obtain $|g(y)-g(x)| = |v(y)-v(x)| \leq \beta(y-x)$. Now, let $x \in C$ and $y \in C^c$. Pick the component (c_y,d_y) of C^c such that $y \in (c_y,d_y)$. We obtain $|g(y)-g(x)| \leq |g(y)-g(c_y)|+|g(c_y)-g(x)| \leq \beta(y-c_y)+\beta(c_y-x) = \beta(y-x)$. Now, let x and y belong to different components (c_x,d_x) and (c_y,d_y) of C^c. Pick $p \in C$ such that $d_x < p < c_y$. By the previous part of the proof we have $|g(y)-g(x)| \leq |g(y)-g(p)|+|g(p)-g(x)| \leq \beta(y-p)+\beta(p-x) = \beta(y-x)$. Hence $\mathcal{L}ip(g|_{[x,y]}) = \beta$ and thus $g \in u\text{-}\mathcal{L}ip(\beta)$.

Now, assume that v is constant on $[a,b]$ and $\mu(C)>0$. Let $(c_i,d_i), i \in \mathbb{N}$, be all components of C^c. Since g is constant on C and $(g/\beta)|_{[c_i,d_i]}$ is natural for all $i \in \mathbb{N}$, we have $V_a^b g = \beta \sum_{i=1}^{\infty}(d_i-c_i) = \beta\mu(C^c)$. □

Remark 5. If $\mu(C)=0$ and v is constant in the Lemma 2, then $g \in \beta\mathcal{N}$. This results from $\mu(C)=0$ and the definition of g.

Theorem 3. *Let* $\emptyset \neq I \subset (0,\infty)$. *Then* $\bigcup_{\alpha\in I} \alpha\mathcal{N}$ *is strongly porous in* $\bigcup_{\alpha\in I} u\text{-}\mathcal{L}ip(\alpha)$.

Proof. By Lemma 1 we get $\bigcup_{\alpha\in I}\alpha\mathcal{N} \subset \bigcup_{\alpha\in I} u\text{-}\mathcal{L}ip(\alpha)$. Let $f \in \bigcup_{\alpha\in I}\alpha\mathcal{N}$; i.e., there exists $\alpha_0 \in I$ such that $f \in \alpha_0\mathcal{N}$. Let $0<R<\alpha_0(b-a)$. Define a function $v\colon [a,b] \to X$ by $v(x) = f(b-\frac{R}{2\alpha_0})$ for all $x \in [a,b]$. Consider a Cantor set in $[b-\frac{R}{2\alpha_0}, b]$ with $\mu(C) = \frac{R}{2\alpha_0} - \frac{1}{n\alpha_0}$, where $n \in \mathbb{N}$ is chosen so that $\frac{R}{4} > \frac{1}{n}$. Let $g\colon [b-\frac{R}{2\alpha_0}, b] \to X$ be a function defined as in Lemma 2, with $\beta = \alpha_0$. Consider a function $h\colon [a,b] \to X$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in [a, b-\frac{R}{2\alpha_0}) \\ g(x) & \text{if } x \in [b-\frac{R}{2\alpha_0}, b]. \end{cases}$$

From the definition of h it follows that $h \in \bigcup_{\alpha\in I} u\text{-}\mathcal{L}ip(\alpha) \setminus \bigcup_{\alpha\in I}\alpha\mathcal{N}$. We will show that $h \in B(f,R)$. By Lemma 2 and the definition of $\alpha\mathcal{N}$ we have

$$\begin{aligned} \| h-f \| &= V^b_{b-\frac{R}{2\alpha_0}}(h-f) + |(h-f)(a)| \leq V^b_{b-\frac{R}{2\alpha_0}} h + V^b_{b-\frac{R}{2\alpha_0}} f \\ &= \frac{1}{n} + \alpha_0 \frac{R}{2\alpha_0} = \frac{1}{n} + \frac{R}{2} < R. \end{aligned}$$

Now, we will show that $B(h, \frac{R}{2}-\frac{1}{n}) \subset B(f,R)$. Let $z \in B(h, \frac{R}{2}-\frac{1}{n})$. We have

$$\| z-f \| \leq \| z-h \| + \| h-f \| < \frac{R}{2} - \frac{1}{n} + \frac{1}{n} + \frac{R}{2} = R.$$

The ball $B(h, \frac{R}{2} - \frac{1}{n})$ does not contain functions from $\bigcup_{\alpha\in I} \alpha\mathcal{N}$. Indeed, let $\beta > 0$ and $z \in \beta\mathcal{N}$. First, we assume that $\beta > \alpha_0$. We obtain

$$\begin{aligned}\| z-h \| =& V_a^b(z-h) + |(z-h)(a)| \geq V^b_{b-\frac{R}{2\alpha_0}}(z-h)\\ \geq& V^b_{b-\frac{R}{2\alpha_0}} z - V^b_{b-\frac{R}{2\alpha_0}} h = \beta\frac{R}{2\alpha_0} - \frac{1}{n} > \frac{R}{2} - \frac{1}{n}.\end{aligned}$$

If $0 < \beta \leq \alpha_0$, then

$$\begin{aligned}\| z-h \| =& V_a^b(z-h) + |(z-h)(a)| \geq V_a^b(z-h)\\ =& V_a^{b-\frac{R}{2\alpha_0}}(z-h) + V^b_{b-\frac{R}{2\alpha_0}}(z-h)\\ \geq& (\alpha_0-\beta)(b-\frac{R}{2\alpha_0}-a) + \beta\frac{R}{2\alpha_0} - \frac{1}{n}\\ =& \beta(\frac{R}{2\alpha_0} - b + \frac{R}{2\alpha_0} + a) + \alpha_0(b - \frac{R}{2\alpha_0} - a) - \frac{1}{n}\\ =& \beta(a-b+\frac{R}{\alpha_0}) + \alpha_0(b-\frac{R}{2\alpha_0}-a) - \frac{1}{n}.\end{aligned}$$

Since $(a-b+\frac{R}{\alpha_0}) < 0$, the sum $\beta(a-b+\frac{R}{\alpha_0}) + \alpha_0(b-\frac{R}{2\alpha_0}-a) - \frac{1}{n}$ is minimal for $\beta = \alpha_0$ and its minimal value equals $\frac{R}{2} - \frac{1}{n}$. Hence, we have

$$\gamma(f, R, \bigcup_{\alpha\in I} \alpha\mathcal{N}) \geq \sup\{\frac{R}{2} - \frac{1}{n} \mid n \in \mathbb{N},\ n > \frac{4}{R}\} = \frac{R}{2}.$$

Consequently, $p(\bigcup_{\alpha\in I} \alpha\mathcal{N}, f) \geq 2\frac{(R/2)}{R} = 1.$ □

Corollary 2. *The set $\mathcal{N}$ is strongly porous in u-$\mathcal{L}ip(1)$.*

Finally, we will compare the porous sizes of sets u-$\mathcal{L}ip(\beta)$, $\mathcal{L}ip(\alpha)$ and $\mathcal{L}ip$.

Lemma 3. *Let $\beta > 0$ and $c, d \in [a, b]$, $c < d$. Then $B = \{f\colon [a,b] \to X \mid \mathcal{L}ip(f|_{[c,d]}) \leq \beta\}$ is closed.*

PROOF. Let $f_n \in B$ for all $n \in \mathbb{N}$ and $f_n \to f$. Hence $V_a^b(f_n - f) \to 0$. Let $x, y \in [c, d]$ and $\varepsilon > 0$. Pick an $n \in \mathbb{N}$ such that $V_a^b(f_n - f) < \varepsilon$. By (V1) in Proposition 1 we have

$$|(f_n - f)(y) - (f_n - f)(y)| \leq V_a^b(f_n - f) < \varepsilon.$$

Hence

$$|f(y) - f(x)| < \varepsilon + |f_n(y) - f_n(x)|.$$

Because $f_n \in B$, we conclude that $|f(y) - f(x)| < \varepsilon + \beta|y - x|$. Taking $\varepsilon \to 0$ we obtain $|f(y) - f(x)| \leq \beta|y - x|$. □

Theorem 4. *Let $\beta > 0$. Then the set u-$\mathcal{L}ip(\beta)$ is a dense G_δ (thus nonporous and residual) set in the closed subspace $\bigcup_{0\le\alpha\le\beta}\mathcal{L}ip(\alpha)$ of $BV([a,b])$.*

Proof. From Lemma 3 it follows that $\bigcup_{0\le\alpha\le\beta}\mathcal{L}ip(\alpha)$ is a closed subspace of $BV([a,b])$. First, we will prove that u-$\mathcal{L}ip(\beta)$ is dense. Let $v \in \bigcup_{0\le\alpha\le\beta}\mathcal{L}ip(\alpha)$ and $\varepsilon > 0$. Consider a Cantor set in $[a,b]$ with $\mu(C) = b - a - \frac{1}{n}$, where $n \in \mathbb{N}$ is chosen such that $\frac{1}{n} < \frac{\varepsilon}{2\beta}$. Let $g \in u$-$\mathcal{L}ip(\beta)$ denote the function defined in Lemma 2. We have $(v-g)(x) = 0$ for all $x \in C$ and the both v, g are Lipschitz functions such that $\mathcal{L}ip(v) \le \beta$, $\mathcal{L}ip(g) = \beta$. Hence $\mathcal{L}ip(v-g) \le 2\beta$. Let (c_i, d_i), $i \in \mathbb{N}$, denote the components of $C^c = [a,b] \setminus C$. Then

$$\begin{aligned}\| v - g \| =& V_a^b(v-g) + |(v-g)(a)| = V_a^b(v-g)\\ \le& 2\beta\sum_{i=1}^{\infty}(d_i - c_i) = 2\beta\mu(C^c) = 2\beta\frac{1}{n} < \varepsilon.\end{aligned}$$

Hence u-$\mathcal{L}ip(\beta)$ is dense in $\bigcup_{0\le\alpha\le\beta}\mathcal{L}ip(\alpha)$.

Now, we will show that u-$\mathcal{L}ip(\beta)$ is G_δ. Let $\mathbb{Q}^* = ([a,b]\cap\mathbb{Q}) \cup \{a,b\}$. List all pairs $(c,d) \in \mathbb{Q}^* \times \mathbb{Q}^*$ with $c < d$ as (c_i, d_i) for $i \in \mathbb{N}$. We let $A_i = \{f \colon [a,b] \to X \mid \mathcal{L}ip(f|_{[c_i,d_i]}) = \beta\}$ for all $i \in \mathbb{N}$. It is easy to prove that u-$\mathcal{L}ip(\beta) = \bigcap_{i=1}^{\infty} A_i$. Finally, we will show that A_i are of type G_δ for all $i \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that $\beta - \frac{1}{n_0} > 0$. We let

$$\begin{aligned}B_i =& \{f \colon [a,b] \to X \mid \mathcal{L}ip(f|_{[c_i,d_i]}) \le \beta\},\\ C_i =& \{f \colon [a,b] \to X \mid \mathcal{L}ip(f|_{[c_i,d_i]}) < \beta\},\\ C_i^n =& \{f \colon [a,b] \to X \mid \mathcal{L}ip(f|_{[c_i,d_i]}) \le \beta - \frac{1}{n}\}\end{aligned}$$

for all $i \in \mathbb{N}$ and $n \ge n_0$, $n \in \mathbb{N}$. We have $C_i = \bigcup_{n=n_0}^{\infty} C_i^n$. Thus by Lemma 3, the set C_i is F_σ. Then we have

$$A_i = B_i \setminus C_i = \Big(\bigcup_{0\le\alpha\le\beta}\mathcal{L}ip(\alpha) \setminus C_i\Big) \cap B_i,$$

which shows that A_i is G_δ. Consequently u-$\mathcal{L}ip(\beta)$ is G_δ. $\square$

Corollary 3. *Let $A \subset (0,\infty)$ be an unbounded and countable set. Then $\bigcup_{\beta\in A} u$-$\mathcal{L}ip(\beta)$ is a dense $G_{\delta\sigma}$ (thus nonporous) set in the F_σ subspace $\mathcal{L}ip$ of $BV([a,b])$.*

On the other hand, we have the following assertion.

Theorem 5. *Let $\beta > 0$. The set $\bigcup_{0\le\alpha\le\beta}\mathcal{L}ip(\alpha)$ is strongly porous in $\mathcal{L}ip$.*

PROOF. Let $f \in \bigcup_{0\le\alpha\le\beta} \mathcal{L}ip(\alpha)$, $0 < R < \beta(b-a)$ and $n \in \mathbb{N}, n \ge 3$. Define $g\colon [a,b] \to X$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in [a, b - \frac{R}{\beta n}) \\ (\frac{\beta n(x-b)+R}{2|f(b-\frac{R}{\beta n})|} + 1) f(b - \frac{R}{\beta n}) = & \text{if } x \in [b - \frac{R}{\beta n}, b]. \end{cases}$$

It is easy to show that $g \in \mathcal{L}ip \setminus \bigcup_{0\le\alpha\le\beta} \mathcal{L}ip(\alpha)$. As in the proof of Theorem 3 we obtain $g \in B(f,R)$ and $B(g, \frac{R}{2} - \frac{R}{n}) \subset B(f,R)$. The ball $B(g, \frac{R}{2} - \frac{R}{n})$ does not contain functions from $\bigcup_{0\le\alpha\le\beta} \mathcal{L}ip(\alpha)$. Indeed, let $v \in \bigcup_{0\le\alpha\le\beta} \mathcal{L}ip(\alpha)$. Then

$$\begin{aligned} \| g - v \| &\ge V_{b-\frac{R}{\beta n}}^{b}(g-v) \ge V_{b-\frac{R}{\beta n}}^{b} g - V_{b-\frac{R}{\beta n}}^{b} v \\ &\ge \frac{\beta n}{2}\frac{R}{\beta n} - \beta\frac{R}{\beta n} = \frac{R}{2} - \frac{R}{n}. \end{aligned}$$

As in the proof of Theorem 3 we obtain $p(\bigcup_{0\le\alpha\le\beta} \mathcal{L}ip(\alpha), f) \ge 1$. □

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