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A CHARACTERIZATION OF H_1-INTEGRABLE FUNCTIONS

Abstract

We characterize the family of all H_1-integrable functions and solve several problems related to the H_1-integral.

1 Preliminaries

Let $E \subset \mathbb{R}$. The symbols χ_E and $\mu(E)$ denote the characteristic function and the outer Lebesgue measure of E. If $f\colon E \to \mathbb{R}$ and $A \subset E$ is nonvoid, then $\operatorname{osc}(f, A) = \sup f(A) - \inf f(A)$; i.e., $\operatorname{osc}(f, A)$ is the *oscillation of f on A*.

Let $\langle a, b\rangle$ be a nondegenerate compact interval. By a *partial tagged partition* of $\langle a, b\rangle$ we understand any finite collection $\mathcal{P}$ of pairs (I, x), where I is a compact subinterval of $\langle a, b\rangle$ and $x \in I$, such that for all $(I, x), (J, y) \in \mathcal{P}$, if $(I, x) \neq (J, y)$, then $\operatorname{int} I \cap J = \emptyset$. If $\mathcal{P}$ is a partial tagged partition of $\langle a, b\rangle$, then we put $\sigma(\mathcal{P}, f) = \sum_{(I,x)\in\mathcal{P}} f(x)\mu(I)$.

If δ is a *gauge* on $\langle a, b\rangle$; i.e., $\delta\colon \langle a, b\rangle \to (0, \infty)$, then we say that a partial tagged partition $\mathcal{P}$ is *δ-fine*, if $I \subset \big(x - \delta(x), x + \delta(x)\big)$ for every $(I, x) \in \mathcal{P}$. For partial tagged partitions $\mathcal{P}$ and $\mathcal{R}$ of $\langle a, b\rangle$, we write $\mathcal{P} \sqsupseteq \mathcal{R}$, if for every $(I, x) \in \mathcal{P}$ there is $(J, y) \in \mathcal{R}$ with $I \subset J$. If $\mathcal{P}$ is a partial tagged partition of $\langle a, b\rangle$ and $\bigcup_{(I,x)\in\mathcal{P}} I = \langle a, b\rangle$, then $\mathcal{P}$ is called a *tagged partition* of $\langle a, b\rangle$.

We say that a function $f\colon \langle a, b\rangle \to \mathbb{R}$ is *H_1-integrable on $\langle a, b\rangle$* [3] to a number $\mathbf{I} \in \mathbb{R}$, if there exists a gauge δ on $\langle a, b\rangle$ such that for every $\varepsilon > 0$, one can find a tagged partition π_0 of $\langle a, b\rangle$ with the property that $|\sigma(\pi, f) - \mathbf{I}| < \varepsilon$ for every δ-fine tagged partition $\pi \sqsupseteq \pi_0$. In this case we say that $\mathbf{I}$ is the *H_1-integral of f on $\langle a, b\rangle$* and write $\mathbf{I} = \int_a^b f$. If $E \subset \langle a, b\rangle$, then we say that f is *H_1-integrable on E*, if $f\chi_E$ is H_1-integrable on $\langle a, b\rangle$.

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We say that a function $f\colon E \to \mathbb{R}$ is a *Baire*1 function* [7], if

> for every nonvoid set $P \subset E$, closed in E, there is an open interval J with $P \cap J \neq \emptyset$ such that the restriction $f\restriction(P \cap J)$ is continuous.

The character $\mathcal{I}$ denotes the σ-ideal of all subsets of $\mathbb{R}$ which are contained in some $\mathcal{F}_\sigma$ set of measure zero.

2 Introduction

The notion of H_1-integrability was introduced by I. J. L. Garces, P. Y. Lee, and D. Zhao in 1998. The authors claimed to prove that this integral is in some sense close to the Henstock integral [3]. It was shown later by P. Sworowski [8] that a few of the results contained in [3] are not valid. The main goal of the present paper is to give a complete characterization of H_1-integrable functions (see Theorem 3.3), and to answer several questions asked in [8].

As a consequence, we obtain that each Henstock integrable Baire*1 function is H_1-integrable. Next, we characterize the sets whose characteristic function is H_1-integrable (see Theorem 3.8), which is the answer to [8, Problem 4.8]. Though condition (1) characterizing H_1-integrable functions resembles the definition of a Baire*1 function, there is an H_1-integrable Baire one function f such that $\operatorname{int}\{x\colon f(x) \neq g(x)\} \neq \emptyset$ for each Baire*1 function g (Example 4.1). Moreover we show that there is a bounded approximately continuous function f such that $\mu\bigl(\{x\colon f(x) \neq g(x)\}\bigr) > 0$ for each H_1-integrable function g (Example 4.2). So, the answer both to the question [8, Problem 6.2], and to the question [8, Problem 6.3] is negative. Finally, we construct a bounded Baire two function f such that $\mu\bigl(\{x\colon f(x) \neq g(x)\}\bigr) > 0$ for each function g, which is the limit of some uniformly convergent sequence of H_1-integrable functions (Example 4.7). These examples show that the H_1-integral is not only far from the Henstock integral, but also from the Lebesgue integral.

We will need the following theorems, whose proofs can be found in [8]. (See also [1].)

Theorem 2.1. [8, Theorem 4.2] *Let $f\colon \langle a,b\rangle \to \mathbb{R}$. Assume that $\mu(A) = 0$, where $A = \bigl\{x \in \langle a,b\rangle\colon f(x) \neq 0\bigr\}$. Then f is H_1-integrable if and only if $A \in \mathcal{I}$.*

Theorem 2.2. [1, Theorem 1], [8, Lemma 5.3] *Let $f\colon \langle a,b\rangle \to \mathbb{R}$. Assume that f is H_1-integrable on every interval $\langle c,b\rangle$, where $a < c < b$, and that the limit $\mathbf{I} = \lim_{c\to a^+} \int_c^b f$ exists and is finite. Then f is H_1-integrable on $\langle a,b\rangle$ and $\int_a^b f = \mathbf{I}$.*

Theorem 2.3. [1, Theorem 2], [8, Lemma 5.5] *Let P be a nonvoid perfect subset of $\langle a,b\rangle$ and let $(I_n)_{n=1}^{\infty}$ be a sequence of the closures of all intervals contiguous to P in $\langle a,b\rangle$. If a function $f\colon \langle a,b\rangle \to \mathbb{R}$ is H_1-integrable on each I_n and $\sum_{n=1}^{\infty} \operatorname{osc}(F_n, I_n) < \infty$, where F_n is any H_1-primitive of $f\restriction I_n$, then f is H_1-integrable on $\langle a,b\rangle \setminus P$ and $\int_a^b f\chi_{\langle a,b\rangle\setminus P} = \sum_{n=1}^{\infty}\int_{I_n} f$.*

Theorem 2.4. [8, Theorem 5.6] *Suppose that a function $f\colon \langle a,b\rangle \to \mathbb{R}$ is H_1-integrable. If $P \subset \langle a,b\rangle$ is closed and f is Henstock integrable on P, then f is H_1-integrable on P.*

3 Main Results

Lemma 3.1. *Let $E = \bigcup_{n=1}^{\infty} E_n$ be a $\mathcal{G}_\delta$ set and $f\colon E \to \mathbb{R}$. If the sequence $(E_n)_{n=1}^{\infty}$ is ascending and the restriction $f\restriction E_n$ is continuous for each n, then there exists an open interval J such that $E \cap J \neq \emptyset$ and the restriction $f\restriction(E\cap J)$ is continuous.*

PROOF. By the Baire Category Theorem, there are an n and an open interval J with $E\cap J \neq \emptyset$ such that E_n is dense in $E \cap J$. Let $x_0, x_1, x_2, \dots \in E\cap J$ satisfy $\lim_{k\to\infty} x_k = x_0$. For each $k \geq 0$ choose an $m(k) \geq n$ such that $x_k \in E_{m(k)}$. We may assume that $m(k) \geq m(0)$ for $k > 0$. For each k, the restriction $f\restriction E_{m(k)}$ is continuous and $E_{m(0)}$ is dense in $E_{m(k)} \cap J$, so there is a $t_k \in E_{m(0)}$ such that

$$|t_k - x_k| < k^{-1} \text{ and } |f(t_k) - f(x_k)| < k^{-1}.$$

Then $\lim_{k\to\infty} t_k = x_0$. So, since the restriction $f\restriction E_{m(0)}$ is continuous,

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} f(t_k) = f(x_0).$$

It follows that the restriction $f\restriction(E\cap J)$ is continuous. □

The next lemma is obvious.

Lemma 3.2. *Let $E \subset \mathbb{R}$, and assume that $f\colon E \to \mathbb{R}$ is bounded and continuous. Define*

$$g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ \liminf_{t\to x, t\in E} f(t) & \text{if } x \in \operatorname{cl} E \setminus E. \end{cases}$$

Extend g linearly on the closure of each bounded interval contiguous to $\operatorname{cl} E$, and let g be constant on the closure of each unbounded interval contiguous to $\operatorname{cl} E$. Then g is bounded and the set of discontinuity points of g is contained in $\operatorname{cl} E \setminus E$.

Now we are ready to prove the main result.

Theorem 3.3. *Let* $f\colon \langle a,b\rangle \to \mathbb{R}$. *The following are equivalent*:

a) *the function* f *is* H_1*-integrable on* $\langle a,b\rangle$,

b) *the function* f *is Henstock integrable on* $\langle a,b\rangle$ *and*

$$\text{for every nonvoid closed set } P \subset \langle a,b\rangle, \text{ there are an open interval } J \text{ and an } A \in \mathfrak{I} \text{ such that } P \cap J \setminus A \neq \emptyset \text{ and the restriction } f\restriction(P\cap J\setminus A) \text{ is continuous.} \tag{1}$$

Proof. a) ⇒ b). Assume that the function f is H_1-integrable on $\langle a,b\rangle$ using a gauge δ. Clearly f is Henstock integrable on $\langle a,b\rangle$. Let $P \subset \langle a,b\rangle$ be a nonvoid closed set. Without loss of generality we may assume that $P\cap J \notin \mathfrak{I}\setminus\{\emptyset\}$ for every open interval J. Fix an $n\in\mathbb{N}$. Put

$$E_n = \bigl\{x\in P\colon \delta(x)\geq n^{-1}\bigr\}$$

and

$$D_n = \bigl\{x\in E_n\colon f\restriction E_n \text{ is not continuous at } x\bigr\},$$

and suppose that $D_n\notin\mathfrak{I}$. For every $x\in D_n$ denote by $\omega(x)$ the oscillation of $f\restriction E_n$ at x; i.e.

$$\omega(x) = \lim_{h\to 0^+} \operatorname{osc}\bigl(f,(x-h,x+h)\cap E_n\bigr),$$

and notice that $\omega(x)>0$. Since $D_n\notin\mathfrak{I}$, there is an $m\in\mathbb{N}$ such that $M=\mu(\operatorname{cl} C)>0$, where

$$C=\bigl\{x\in D_n\colon \omega(x)>m^{-1}\bigr\}.$$

By assumption, there is a tagged partition π_0 of $\langle a,b\rangle$ such that

$$\Bigl|\sigma(f,\pi)-\int_a^b f\Bigr| < \frac{M}{4m} \quad \text{for every } \delta\text{-fine tagged partition } \pi\sqsupseteq\pi_0. \tag{2}$$

Every interval from π_0 can be written as the union of a finite family of nonoverlapping intervals of length less than n^{-1}. Let $\mathcal{A}$ be the family of all these intervals. Put $\mathcal{B}=\bigl\{I\in\mathcal{A}\colon C\cap\operatorname{int} I\neq\emptyset\bigr\}$. For each $I\in\mathcal{B}$, we can pick an $x_I\in C\cap\operatorname{int} I$, and, since $\omega(x_I)>m^{-1}$, a $y_I\in I\cap E_n$ such that $|f(y_I)-f(x_I)|>m^{-1}$. Let

$$\mathcal{B}_1=\bigl\{I\in\mathcal{B}\colon f(x_I)-f(y_I)>m^{-1}\bigr\}.$$

Notice that $\mu\big(\bigcup_{I\in\mathcal{B}} I\big) \geq \mu(\operatorname{cl} C) = M$. Consequently, $\mu\big(\bigcup_{I\in\mathcal{B}_1} I\big) \geq M/2$ or $\mu\big(\bigcup_{I\in\mathcal{B}\setminus\mathcal{B}_1} I\big) \geq M/2$. Assume that, e.g., the first case holds. Let $\mathcal{P}_1 \sqsupseteq \pi_0$ be a δ-fine partial tagged partition of $\langle a,b\rangle$ such that $\bigcup_{(I,x)\in\mathcal{P}_1} I = \langle a,b\rangle \setminus \bigcup_{I\in\mathcal{B}_1} I$. Both $\mathcal{P}_2 = \big\{(I,x_I)\colon I \in \mathcal{B}_1\big\}$ and $\mathcal{P}_3 = \big\{(I,y_I)\colon I \in \mathcal{B}_1\big\}$ are δ-fine partial tagged partitions of $\langle a,b\rangle$ and $\mathcal{P}_2, \mathcal{P}_3 \sqsupseteq \pi_0$. So, $\pi_1 = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\pi_2 = \mathcal{P}_1 \cup \mathcal{P}_3$ are δ-fine tagged partitions of $\langle a,b\rangle$ such that $\pi_1, \pi_2 \sqsupseteq \pi_0$. Then by (2),

$$\frac{M}{2m} > |\sigma(\pi_1,f) - \sigma(\pi_2,f)| = \sigma(\mathcal{P}_2,f) - \sigma(\mathcal{P}_3,f) > \frac{M}{2m},$$

an impossibility. Consequently, $D_n \in \mathfrak{I}$.

Let $A \in \mathfrak{I}$ be an $\mathcal{F}_\sigma$ set such that $A \supset \bigcup_{n=1}^\infty D_n$. Then by assumption, $P \setminus A$ is a nonvoid $\mathcal{G}_\delta$ set dense in P. As $P \setminus A = \bigcup_{n=1}^\infty (E_n \setminus A)$ and the restrictions $f{\restriction}(E_n \setminus A)$ are continuous, by Lemma 3.1, there exists an open interval J such that $P \cap J \setminus A \neq \emptyset$ and the restriction $f{\restriction}(P\cap J\setminus A)$ is continuous. We have shown that condition (1) is fulfilled.

b) $\Rightarrow$ a). Assume that f is a Henstock integrable function which fulfills (1). Suppose that f is not H_1-integrable on $\langle a,b\rangle$. Denote by P the set of all points $x \in \langle a,b\rangle$ such that f is H_1-integrable on no open interval containing x. Then $P \neq \emptyset$, and by Theorem 2.2, f is H_1-integrable on the closure of every interval contiguous to P in $\langle a,b\rangle$, whence P is perfect.

If F is the Henstock primitive of f, then F is an ACG_*-function. So, there is a nondegenerate open interval I such that F is an AC_*-function on $P \cap I$. By (1), there are an open interval $J \subset I$ and an $A \in \mathfrak{I}$ such that $P' \setminus A \neq \emptyset$ and the restriction $f{\restriction}(P' \setminus A)$ is continuous, where $P' = P \cap J$; clearly we may assume that $f{\restriction}(P' \setminus A)$ is bounded. Let g be an extension of $f{\restriction}(P' \setminus A)$ defined in Lemma 3.2. Then the set of discontinuity points of g is contained in A, so g is a bounded almost everywhere continuous function. By Riemann–Lebesgue theorem, g is Riemann integrable on J, whence H_1-integrable on J. Moreover g is Henstock integrable on P', because it is measurable and bounded. Thus by Theorem 2.4, g is H_1-integrable on P', and in consequence of Theorem 2.1, f is H_1-integrable on P'.

Let $(I_n)_{n=1}^\infty$ be a sequence of the closures of all intervals contiguous to P' in J. Since F is a VB_*-function on P', the series $\sum_{n=1}^\infty \operatorname{osc}(F, I_n)$ is convergent. Therefore by Theorem 2.3, f is H_1-integrable on $J \setminus P'$. Thus f is H_1-integrable on J. But $J \cap P \neq \emptyset$, an impossibility. Therefore f is H_1-integrable on $\langle a,b\rangle$. □

Lemma 3.4. *Let $f\colon \langle a,b\rangle \to \mathbb{R}$. Then condition* (1) *is equivalent to the following one*:

*there exists a $B \in \mathfrak{I}$ such that the restriction $f{\restriction}(\langle a,b\rangle \setminus B)$ is Baire*1 in its domain;* (3)

i.e., there exists a $B \in \mathfrak{I}$ *with the property that for every closed set* $P \subset \langle a, b\rangle$ *with* $P \setminus B \neq \emptyset$*, we can find an open interval* J *with* $P \cap J \setminus B \neq \emptyset$ *such that the restriction* $f\restriction(P \cap J \setminus B)$ *is continuous.*

PROOF. ($\Rightarrow$). First assume that the function f fulfills condition (1). Denote by Ω the first uncountable ordinal. We proceed by transfinite induction.

1. Let $U_0 = \emptyset$.

2. Let $\alpha < \Omega$ and assume we have already defined an open set U_α and a $B_\alpha \in \mathfrak{I}$. If $U_\alpha = \mathbb{R}$, then we let $U_{\alpha+1} = \mathbb{R}$ and $B_{\alpha+1} = \emptyset$. Otherwise let $\mathfrak{J}_\alpha$ be the family of all open intervals J with rational endpoints such that $J \setminus U_\alpha \neq \emptyset$ and the restriction $f\restriction\big((J \setminus U_\alpha) \setminus A_J\big)$ is continuous for some $A_J \in \mathfrak{I}$. Define

$$U_{\alpha+1} = U_\alpha \cup \bigcup\nolimits_{J\in\mathfrak{J}_\alpha} J \text{ and } B_{\alpha+1} = \bigcup\nolimits_{J\in\mathfrak{J}_\alpha} A_J.$$

   Then clearly the set $U_{\alpha+1}$ is open and $B_{\alpha+1} \in \mathfrak{I}$. Observe that by (1), $\mathfrak{J}_\alpha \neq \emptyset$, so $U_\alpha \neq U_{\alpha+1}$.

3. If $\alpha < \Omega$ is a limit ordinal, then let $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$ and $B_\alpha = \emptyset$.

In this manner we defined a transfinite ascending sequence $\{U_\alpha \colon \alpha < \Omega\}$ of open subsets of $\mathbb{R}$. By the Cantor–Baire stationary principle [5, Theorem 2, p. 146], there exists an $\alpha_0 < \Omega$ such that $U_{\alpha_0} = U_{\alpha_0+1}$, whence $U_{\alpha_0} = \mathbb{R}$.

Put $B = \bigcup_{\alpha<\alpha_0} B_\alpha$. Then $B \in \mathfrak{I}$, since $\alpha_0 < \Omega$. Let $P \subset \langle a, b\rangle$ be a closed set with $P \setminus B \neq \emptyset$, and let

$$\alpha_1 = \min\big\{\alpha < \Omega \colon U_\alpha \cap P \setminus B \neq \emptyset\big\}.$$

Clearly $\alpha_1 > 0$ and α_1 is not a limit ordinal. So, $\alpha_1 = \alpha_2 + 1$ for some $\alpha_2 < \Omega$. Then $P \cap U_{\alpha_2} \setminus B = \emptyset$, so we can choose a $J \in \mathfrak{J}_{\alpha_2}$ with $P \cap J \setminus B \neq \emptyset$. By definition, the restriction $f\restriction\big((J \setminus U_{\alpha_2}) \setminus A_J\big)$ is continuous. Hence the restriction $f\restriction(P \cap J \setminus B)$ is continuous, too.

($\Leftarrow$). Now assume that the function f fulfills condition (3). Let $P \subset \langle a, b\rangle$ be a nonvoid closed set. If $P \setminus B \neq \emptyset$, then by (3), there is an open interval J with $P \cap J \neq \emptyset$ such that the restriction $f\restriction(P \cap J \setminus B)$ is continuous. Otherwise pick an arbitrary $x \in P$, and define $J = (a-1, b+1)$ and $A = B \setminus \{x\}$. Then evidently J is an open interval with $P \cap J \neq \emptyset$, $A \in \mathfrak{I}$, and the restriction $f\restriction(P \cap J \setminus A) = f\restriction\{x\}$ is continuous. □

From Theorem 3.3 and Lemma 3.4 we immediately get the following corollary.

Corollary 3.5. *Let $f\colon \langle a,b\rangle \to \mathbb{R}$. The following are equivalent:*

a) *the function f is H_1-integrable on $\langle a,b\rangle$,*

b) *the function f is Henstock integrable on $\langle a,b\rangle$ and it fulfills condition* (3).

The following corollary gives an affirmative answer to the question asked in [8, Problem 6.1].

Corollary 3.6. *Let $f\colon \langle a,b\rangle \to \mathbb{R}$ and $E_1, E_2 \subset \langle a,b\rangle$. Assume that the function f is Henstock integrable on $E_1 \cup E_2$, and that it is H_1-integrable both on E_1 and on E_2. Then f is H_1-integrable on $E_1 \cup E_2$.*

Proof. By Corollary 3.5, for $i \in \{1,2\}$, there is a $B_i \in \mathfrak{I}$ such that the restriction $(f\chi_{E_i})\restriction(\langle a,b\rangle \setminus B_i)$ is Baire*1 in its domain. Let $B = B_1 \cup B_2$. Then $B \in \mathfrak{I}$ and the function $(f\chi_{E_i})\restriction(\langle a,b\rangle \setminus B)$ is Baire*1 in $\langle a,b\rangle \setminus B$ for $i \in \{1,2\}$, whence $(f\chi_{E_1\cup E_2})\restriction(\langle a,b\rangle \setminus B)$ is Baire*1 in its domain, too. Actually, suppose first that $f \geq 0$. Thus $f\chi_{E_1\cup E_2} = \max\{f\chi_{E_1}, f\chi_{E_2}\}$, so $(f\chi_{E_1\cup E_2})\restriction(\langle a,b\rangle \setminus B)$ is Baire*1 in its domain. In the general case we write $f = f^+ - f^-$ where $f^+, f^- \geq 0$. Using Corollary 3.5 again, we conclude that the function $f\chi_{E_1\cup E_2}$ is H_1-integrable on $\langle a,b\rangle$; i.e., that the function f is H_1-integrable on $E_1 \cup E_2$. □

In [6] we proved that the uniform convergence theorem does not hold for the H_1-integral; i.e., that there is a uniformly convergent sequence of H_1-integrable functions, whose limit is not H_1-integrable. So, it is natural to ask for the characterization of the limits of uniformly convergent sequences of H_1-integrable functions. The answer to this question is the following theorem.

Theorem 3.7. *Let $f\colon \langle a,b\rangle \to \mathbb{R}$. The following are equivalent:*

a) *the function f is the limit of some uniformly convergent sequence of H_1-integrable functions,*

b) *the function f is Henstock integrable on $\langle a,b\rangle$ and*

there exists a $B \in \mathfrak{I}$ such that the restriction $f\restriction(\langle a,b\rangle \setminus B)$ is Baire one in its domain. (4)

Proof. a) ⇒ b). Assume that $f = \lim f_n$, where $(f_n)_{n=1}^{\infty}$ is a uniformly convergent sequence of H_1-integrable functions. Then f is Henstock integrable on $\langle a,b\rangle$. For each n, let $B_n \in \mathfrak{I}$ be such that the restriction $f_n\restriction(\langle a,b\rangle \setminus B_n)$ is Baire*1 in its domain. (Cf. Corollary 3.5.) Define $B = \bigcup_{n=1}^{\infty} B_n$. Then $B \in \mathfrak{I}$ and the restriction $f\restriction(\langle a,b\rangle \setminus B)$ is the limit of the sequence $\big(f_n\restriction(\langle a,b\rangle \setminus$

$B_n))_{n=1}^{\infty}$, which is uniformly convergent. Hence $f\restriction(\langle a,b\rangle \setminus B)$ is Baire one in its domain.

b) $\Rightarrow$ a). Assume that f is a Henstock integrable function which fulfills (4). By [5, p. 294–295], there is a uniformly convergent sequence $(f_n)_{n=1}^{\infty}$ of Baire*1 functions defined on $\langle a,b\rangle \setminus B$, whose limit is $f\restriction(\langle a,b\rangle \setminus B)$; we may assume that $|f - f_n| < 1$ on $\langle a,b\rangle \setminus B$. For each n, extend the function f_n to $\langle a,b\rangle$ setting $f_n = f$ on B. Then $f - f_n$ is a bounded measurable function, so it is Lebesgue integrable. By Corollary 3.5, each function f_n is H_1-integrable. Clearly the sequence $(f_n)_{n=1}^{\infty}$ is uniformly convergent to f. □

The next theorem is the answer to [8, Problem 4.8]. Recall that a set D is *ambiguous*, if it is both an $\mathcal{F}_\sigma$ and a $\mathcal{G}_\delta$ set.

Theorem 3.8. *Let $E \subset \langle a,b\rangle$. The following are equivalent*:

a) *the function χ_E is H_1-integrable on $\langle a,b\rangle$,*

b) *there is an ambiguous set D such that $E \triangle D \in \mathfrak{I}$.*

PROOF. a) $\Rightarrow$ b). Assume that the function χ_E is H_1-integrable on $\langle a,b\rangle$. By Corollary 3.5, the function χ_E fulfills condition (3). So, there is a $B \in \mathfrak{I}$ and an ascending sequence $(S_n)_{n=1}^{\infty}$ of closed subsets of $\langle a,b\rangle$ such that $\bigcup_{n=1}^{\infty} S_n \supset \langle a,b\rangle \setminus B$ and for each n, the restriction $\chi_E\restriction(S_n \setminus B)$ is continuous. (See, e.g., [4, Theorem 5].) Without loss of generality we may assume that B is an $\mathcal{F}_\sigma$ set. Let $B = \bigcup_{n=1}^{\infty} B_n$, where $(B_n)_{n=1}^{\infty}$ is an ascending sequence of closed sets. Set $P_0 = \emptyset$, and for each n put $P_n = B_n \cup \operatorname{cl}(S_n \setminus B)$. Observe that $(P_n)_{n=1}^{\infty}$ is an ascending sequence of closed sets and

$$\langle a,b\rangle \supset \bigcup_{n=1}^{\infty} P_n \supset B \cup \bigcup_{n=1}^{\infty} (S_n \setminus B) = B \cup \bigcup_{n=1}^{\infty} S_n = \langle a,b\rangle.$$

Notice that for each n, both

$$\begin{aligned} D_n &= \operatorname{cl}(S_n \cap E \setminus B) \setminus P_{n-1}, \text{ and} \\ T_n &= \big(B_n \cup \operatorname{cl}(S_n \setminus (E \cup B))\big) \setminus (D_n \cup P_{n-1}) \end{aligned}$$

are $\mathcal{F}_\sigma$ sets, and

$$\begin{aligned} D_n \cup T_n &= \big(\operatorname{cl}(S_n \cap E \setminus B) \cup B_n \cup \operatorname{cl}((S_n \setminus E) \setminus B)\big) \setminus P_{n-1} \\ &= \big(B_n \cup \operatorname{cl}(S_n \setminus B)\big) \setminus P_{n-1} = P_n \setminus P_{n-1}. \end{aligned}$$

Let $D = \bigcup_{n=1}^{\infty} D_n$. Then D is an $\mathcal{F}_\sigma$ set. Since

$$\langle a, b\rangle \setminus D = \bigcup_{n=1}^{\infty} (P_n \setminus P_{n-1}) \setminus D = \bigcup_{n=1}^{\infty} \big((P_n \setminus P_{n-1}) \setminus D_n\big) = \bigcup_{n=1}^{\infty} T_n,$$

D is a $\mathcal{G}_\delta$ set, too. Thus D is ambiguous. We will show that $E \vartriangle D \subset B$.

Suppose this is not the case. First suppose that there is an $x \in (E \setminus D) \setminus B$. Fix an n with $x \in T_n$. Since the restriction $\chi_E \restriction (S_n \setminus B)$ is continuous at x and $x \in E$, there is an open interval $I \ni x$ such that $I \cap S_n \setminus B \subset E$. Consequently, $x \notin \operatorname{cl}\big(S_n \setminus (E \cup B)\big) \supset T_n \setminus B$, an impossibility.

Now suppose that there is an $x \in (D \setminus E) \setminus B$. Fix an n with $x \in D_n$. Since the restriction $\chi_E \restriction (S_n \setminus B)$ is continuous at x and $x \notin E$, there is an open interval $I \ni x$ such that $I \cap S_n \setminus B \subset \mathbb{R} \setminus E$. Consequently, $x \notin \operatorname{cl}(S_n \cap E \setminus B) \supset D_n$, an impossibility.

b) $\Rightarrow$ a). Assume that $E \vartriangle D \in \mathfrak{I}$ for some ambiguous set D. By Theorem 2.1 it is enough to prove that χ_D is H_1-integrable. Let $D = \bigcup_{n=1}^{\infty} D_n$, where all D_n's are closed, and consider arbitrary closed subset $P \subset \langle a, b\rangle$. We have

$$P = (P \setminus D) \cup \bigcup_{n=1}^{\infty} (P \cap D_n).$$

Since $P \setminus D$ is an $\mathcal{F}_\sigma$ set, by the Baire Category Theorem there is an open interval I, such that the portion $P \cap I$ is contained in $P \setminus D$ or in some $P \cap D_n$. In both cases $\chi_D \restriction (P \cap I)$ is continuous. Thus χ_D is Baire*1. By Corollary 3.5, the function χ_D is H_1-integrable. □

4 Examples

The following example proves that not every H_1-integrable function is $\mathfrak{I}$-almost everywhere equal to some Baire*1 function.

Example 4.1. *There is a bounded, almost everywhere continuous, approximately continuous function f such that* $\operatorname{int}\big\{x \in \mathbb{R} \colon f(x) \neq g(x)\big\} \neq \emptyset$ *for each Baire*1 function g.*

Construction. Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be a bounded approximately continuous function, which is continuous on $\mathbb{R} \setminus \{0\}$, such that

$$\limsup_{t \to 0} \varphi(t) - \varphi(0) = 2. \tag{5}$$

(See, e.g., [9].) Arrange all rationals in a sequence, $(q_n)_{n=1}^{\infty}$. For each n define $\varphi_n(x) = \varphi(x - q_n)/2^n$. Put $f = \sum_{n=1}^{\infty} \varphi_n$. This series is uniformly convergent, so function f is bounded, almost everywhere continuous, and approximately continuous.

Let g be a Baire*1 function. There is an open interval I_0 such that the restriction $g\restriction I_0$ is continuous. Choose an n with $q_n \in I_0$. The function $f - \varphi_n - g$ is continuous at q_n; so there is an open interval $I_1 \subset I_0$ such that $q_n \in I_1$ and

$$\bigl|f - \varphi_n - g - (f - \varphi_n - g)(q_n)\bigr| < 2^{-n} \text{ on } I_1.$$

By (5), there is an open interval $I_2 \subset I_1$ such that

$$\bigl|\varphi_n + (f - \varphi_n - g)(q_n)\bigr| > 2^{-n} \text{ on } I_2.$$

(Recall that φ_n is continuous on $\mathbb{R} \setminus \{q_n\}$.) Then for each $t \in I_2$,

$$\begin{aligned}\bigl|g(t) - f(t)\bigr| \geq &\bigl|\varphi_n(t) + (f - \varphi_n - g)(q_n)\bigr| \\ &- \bigl|(f - \varphi_n - g)(t) - (f - \varphi_n - g)(q_n)\bigr| > 0.\end{aligned}$$

Hence $I_2 \subset \operatorname{int}\bigl\{x \in \mathbb{R} \colon f(x) \neq g(x)\bigr\}$. □P. Sworowski asked whether each Henstock integrable Baire one function is H_1-integrable [8, Problem 6.2], and whether each derivative is H_1-integrable [8, Problem 6.3]. Since every bounded, approximately continuous function is a derivative, Example 4.2 proves that the answer to both these questions is negative.

Example 4.2. *There is a bounded approximately continuous function f such that $\mu\bigl(\{x \colon f(x) \neq g(x)\}\bigr) > 0$ for each H_1-integrable function g.*

CONSTRUCTION. Let E be a dense $\mathcal{G}_\delta$ set of measure zero. There is an upper semicontinuous, approximately continuous function $f \colon \langle 0,1\rangle \to \langle 0,1\rangle$ such that

$$E = \bigl\{x \in \langle 0,1\rangle \colon f(x) = 0\bigr\}.$$

(See, e.g., [9, Lemma 12].)

Let g be an H_1-integrable function. By (1), there are an open interval $J \subset \langle 0,1\rangle$ and an $A \in \mathfrak{I}$ such that the restriction $g\restriction(J \setminus A)$ is continuous. Then $J \cap E \setminus A$ is a dense $\mathcal{G}_\delta$ subset of J. If $g = 0$ on $J \cap E \setminus A$, then $g = 0$ on $J \setminus A$. Hence

$$\mu\bigl(\{x \in \langle 0,1\rangle \colon f(x) \neq g(x)\}\bigr) \geq \mu\bigl(J \setminus (A \cup E)\bigr) = \mu(J) > 0.$$

So, suppose that there is an $x \in J \cap E \setminus A$ with $g(x) \neq 0$. Then f is continuous at x. Let $I \subset J$ be an open interval such that

$$|g(t)| > |g(x)|/2 \text{ and } |f(t)| < |g(x)|/2$$

for each $t \in I \setminus A$. Then

$$\mu\big(\{x \in \langle 0,1\rangle \colon f(x) \neq g(x)\}\big) \geq \mu(I) > 0. \qquad \square$$

Moreover, the above example shows falseness of [3, Theorem 10]. There are Henstock integrable functions equal almost everywhere to no H_1-integrable function.

Problem 4.3. *Characterize H_1-primitives.*

On the other hand, we have the following simple proposition.

Proposition 4.4. *If a function $f \colon \langle a,b\rangle \to \mathbb{R}$ is the limit of some uniformly convergent sequence of H_1-integrable functions (so, in particular if it is a derivative), then f can be written as the sum of an H_1-integrable function and a Lebesgue integrable one.*

Proof. By assumption, there is an H_1-integrable function $g \colon \langle a,b\rangle \to \mathbb{R}$ such that $|f-g| < 1$ on $\langle a,b\rangle$. Then $f-g$ is a bounded measurable function, whence it is Lebesgue integrable. $\square$

The following problems are open.

Problem 4.5. *Characterize sums of Lebesgue integrable and H_1-integrable functions. In particular, can every Henstock integrable function be written as the sum of an H_1-integrable function and a Lebesgue integrable one?*

Problem 4.6. *Characterize limits of pointwise convergent sequences of H_1-integrable functions.*

Example 4.7. *There is a bounded Baire two function f with the property that $\mu\big(\{x \colon f(x) \neq g(x)\}\big) > 0$ for each function g, which is the limit of some uniformly convergent sequence of H_1-integrable functions.*

Proof. [Construction] Let E be a metrically dense $\mathcal{F}_\sigma$ set, whose complement is also metrically dense in $\langle 0,1\rangle$; i.e., $\mu(J \cap E) > 0$ and $\mu(J \setminus E) > 0$ for each nondegenerate interval $J \subset \langle 0,1\rangle$. Define $f = \chi_E$. Then clearly f is a bounded Baire two function.

Let $h \colon \langle 0,1\rangle \to \mathbb{R}$ be such that

$$\mu\big(\{x \in \langle 0,1\rangle \colon |f(x) - h(x)| < 1/4\}\big) = 0. \qquad (6)$$

We will show that h is not H_1-integrable. Let J be an open interval and $A \in \mathcal{J}$. Define

$$S = \{x \in \langle 0, 1\rangle : h(x) > 3/4\} \text{ and } T = \{x \in \langle 0, 1\rangle : h(x) < 1/4\}.$$

By (6), both S and T are metrically dense. So, both $S \cap J \setminus A$ and $T \cap J \setminus A$ are dense in J, and the restriction $h \restriction (J \setminus A)$ is not continuous. Consequently, the function h does not fulfill condition (1), whence it is not H_1-integrable. □

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