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NON-EXISTENCE OF CERTAIN TYPES OF LIFTINGS AND DENSITIES IN PRODUCT SPACES WITH σ-IDEALS

Abstract

We prove that if $(\Omega, \Sigma, \mathcal{I})$, $(\Theta, T, \mathcal{J})$ and $(\Omega \times \Theta, \Xi, \mathcal{K})$ are measurable spaces with σ-ideals satisfying some natural Fubini type conditions then there is no density on $(\Omega \times \Theta, \Xi, \mathcal{K})$ with density invariant sections.

1 Introduction

Throughout (Ω, Σ) and (Θ, T) are two measurable spaces and $\mathcal{I} \subset \Sigma$, $\mathcal{J} \subset T$ are arbitrary σ–ideals. Besides we consider a σ–algebra $\Xi \supseteq \Sigma \otimes T$ and a σ–ideal $\mathcal{K} \subset \Xi$. We assume that $\mathcal{I} \times \Theta \cup \Omega \times \mathcal{J} \subset \mathcal{K}$. We denote by $\Xi \widehat{\oplus} \mathcal{K}$ the σ-algebra $\Xi \oplus \mathcal{K}$ completed with respect to $\mathcal{K}$, i.e. $E \in \Xi \widehat{\oplus} \mathcal{K}$ if and only if there is $F \in \Xi$ and $K \in \mathcal{K}$ such that $E \triangle F \subset K$. In a similar way the σ–algebras $\Sigma \widehat{\oplus} \mathcal{I}$ and $T \widehat{\oplus} \mathcal{J}$ are defined. $A \in \Sigma \setminus \mathcal{I}$ is an $\mathcal{I}$-atom of Σ if A cannot be decomposed into two disjoint elements of $\Sigma \setminus \mathcal{I}$. Lower densities and liftings on $(\Omega, \Sigma, \mathcal{I})$ are defined exactly in the same way as densities and

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liftings for measure spaces (cf [5], chap. 28). The family of all (lower) densities on $(\Omega, \Sigma, \mathcal{I})$ is denoted by $\vartheta(\mathcal{I})$, and the family of liftings, by $\Lambda(\mathcal{I})$. Each density $\delta \in \vartheta(\mathcal{I})$ generates a collection of filters $\{\mathcal{F}(\omega) : \omega \in \Omega\}$ containing no elements of $\mathcal{I}$: $\mathcal{F}(\omega) = \{A \in \Sigma : \omega \in \delta(A)\}$. Similarly for the other ideals.
Other unexplained notations and terminology come from [3].

Definition 1. We say that the triplet $(\mathcal{I}, \mathcal{J}, \mathcal{K})$ of σ–ideals has the property (*), if $E \in \mathcal{K}$ yields always

$$\{\omega \in \Omega : E_\omega \notin \mathcal{J}\} \in \mathcal{I} \qquad \text{and} \qquad \{\theta \in \Theta : E^\theta \notin \mathcal{I}\} \in \mathcal{J}\,.$$

Definition 2. $\varphi \in \vartheta(\mathcal{K})$ is a product of $\delta \in \vartheta(\mathcal{I})$ and $\tau \in \vartheta(\mathcal{J})$ if $\varphi(A \times B) = \delta(A) \times \tau(B)$ for all $A \in \Sigma$ and $B \in T$. We write then $\varphi \in \delta \otimes \tau$.

We say that $\varphi \in \vartheta(\mathcal{K})$ has (δ, τ)-sub-invariant sections if for every (ω, θ) and every $E \in \Xi$ the equalities

$$[\varphi(E)]_\omega \supseteq \tau\left([\varphi(E)]_\omega\right) \quad \text{and} \quad [\varphi(E)]^\theta \supseteq \delta\left([\varphi(E)]^\theta\right)$$

are fulfilled. In case of equalities we have (δ, τ)-invariance. In a similar way liftings with lifting invariant sections are introduced.

The main problem investigated in this paper is the question of the existence of liftings and densities possessing invariant sections in the sense of the above definition. In Theorem 3.5 of [3] we proved that if the product measure space $(\Omega, \Sigma, \mu)\widehat{\otimes}(\Theta, T, \nu)$ admits a density possessing a density invariant sections, then one of the marginal measure spaces is atomic. We prove now that also in case of σ–ideals possessing the property (*) a characterization similar to that in [3] holds true.

2 Products of Liftings and Densities

We begin by proving an essential property of densities with density invariant sections.

Proposition 3. *Let $(\Omega, \Sigma, \mathcal{I})$, $(\Theta, T, \mathcal{J})$ and $(\Omega \times \Theta, \Xi, \mathcal{K})$ be spaces with the property (*). Assume that $\delta \in \vartheta(\mathcal{I})$ and $\tau \in \vartheta(\mathcal{J})$ are arbitrary and $\varphi \in \vartheta(\mathcal{K})$ has (δ, τ)-sub-invariant sections. Then $\varphi(A \times B) \supseteq \delta(A) \times \tau(B)$ for every $A \times B \in \Sigma \times T$. If φ has (δ, τ)-invariant sections, then $\varphi \in \delta \otimes \tau$ and is uniquely determined by the marginal densities. If δ, τ and φ are liftings then, (δ, τ)-sub-invariance of φ is equivalent to its (δ, τ)-invariance.*

PROOF. Assume that φ is the density possessing (δ,τ)-sub-invariant sections. It is sufficient to show that $\varphi(A\times\Theta)\supseteq\delta(A)\times\Theta$ and $\varphi(\Omega\times B)\supseteq\Omega\times\tau(B)$ for the appropriate A and B.

It follows from the (*) property, that given $A\in\Sigma$ there exists $M_A\in\mathcal{J}$ such that for all $\theta\notin M_A$

$$[\varphi(A\times\Theta)]^\theta\supseteq\delta\left([\varphi(A\times\Theta)]^\theta\right)=\delta\Big((A\times\Theta)^\theta\Big)=\delta(A)\,.$$

In particular, we get the inclusion $\varphi(A\times\Theta)\supseteq\delta(A)\times(\Theta\setminus M_A)$. Hence, if $\omega\in\delta(A)$, then $[\varphi(A\times\Theta)]_\omega\supseteq\tau(\Theta\setminus M_A)=\Theta$. Consequently,

$$\varphi(A\times\Theta)\supseteq\delta(A)\times\Theta\,. \tag{1}$$

Since the proof of the inclusion $\varphi(\Omega\times B)\supseteq\Omega\times\tau(B)$ is symmetric, we get the required inclusion $\varphi(A\times B)\supseteq\delta(A)\times\tau(B)$. Similar considerations yield the equality in case of (δ,τ)-invariant φ.

In case of liftings, we take also into account the inclusions

$$\varphi(A^c\times\Theta)\supseteq\delta(A^c)\times\Theta\qquad\text{and}\qquad\varphi(\Omega\times B^c)\supseteq\Omega\times\tau(B^c).$$

Combining them with (1) and $\varphi(\Omega\times B)\supseteq\Omega\times\tau(B)$, we get the required (δ,τ)-invariance of φ.

Suppose now that $\alpha,\beta\in\vartheta(\mathcal{K})$ are two densities possessing (δ,τ)-invariant sections. Since $\alpha(E)\triangle\beta(E)\in\mathcal{K}$ it follows from the (*) property of the ideals, that there exist sets $N_E\in\mathcal{I}$ and $M_E\in\mathcal{J}$ such that

$$[\alpha(E)]_\omega=[\beta(E)]_\omega\qquad\text{and}\qquad[\alpha(E)]^\theta=[\beta(E)]^\theta$$

for all $\omega\notin N_E$ and $\theta\notin M_E$, respectively. Hence

$$\alpha(E)\triangle\beta(E)\subseteq N_E\times M_E\,.$$

The (δ,τ)-invariance shows that also for $\omega\in N_E$, the equality $[\alpha(E)]_\omega=[\beta(E)]_\omega$ holds true. Consequently, $\alpha(E)=\beta(E)$. □

Definition 4. $\tau\in\vartheta(\mathcal{J})$ is said to be $\mathcal{J}$-continuous if $\bigcap_{n=1}^\infty\tau(B_n)=\emptyset$ for every decreasing sequence $\langle B_n\rangle$ of members of T such that $\bigcap_{n=1}^\infty B_n\in\mathcal{J}$.

One can easily see that if the Boolean algebra $T/\mathcal{J}$ is non–atomic and satisfies the countable chain condition (CCC), then τ cannot be $\mathcal{J}$-continuous. Hence, if τ is $\mathcal{J}$-continuous and $T/\mathcal{J}$ satisfies the countable chain condition (CCC), then $T/\mathcal{J}$ is purely atomic. If $T/\mathcal{J}$ has infinitely many atoms and

$Q_1, \ldots, Q_n, \ldots$ are corresponding $\mathcal{J}$-atoms of T, then the $\mathcal{J}$-continuity of τ is equivalent to

$$\bigcap_{n=1}^{\infty} \tau\left(\bigcup_{k=n}^{\infty} Q_k\right) = \emptyset\,.$$

The next result is a generalization of Theorem 3.5 of [3].

Theorem 5. *Assume that $(\Omega, \Sigma, \mathcal{I})$, $(\Theta, T, \mathcal{J})$ and $(\Omega \times \Theta, \Xi, \mathcal{K})$ are spaces satisfying (*). Assume moreover, that the marginal Boolean algebras satisfy (CCC). If there exist densities $\delta \in \vartheta(\mathcal{I})$, $\tau \in \vartheta(\mathcal{J})$ and $\varphi \in \vartheta(\mathcal{K})$ possessing (δ, τ)-sub-invariant sections, then either δ is $\mathcal{I}$-continuous or τ is $\mathcal{J}$-continuous.*

Proof. Assume that $\mathcal{I}$ and $\mathcal{J}$ are such that neither $\delta \in \vartheta(\mathcal{I})$ is $\mathcal{I}$–continuous nor $\tau \in \vartheta(\mathcal{J})$ is $\mathcal{J}$–continuous and $\varphi \in \vartheta(\mathcal{K})$ has (δ, τ)-sub-invariant sections. Let us fix (ω, θ) and decreasing sequences of sets $\widehat{A}_n = \delta(\widehat{A}_n)$ and $\widehat{B}_n = \tau(\widehat{B}_n)\,, n \in \mathbf{N}$, such that

$$\omega \in \bigcap_{n=1}^{\infty} \widehat{A}_n \quad \text{and} \quad \theta \in \bigcap_{n=1}^{\infty} \widehat{B}_n\,,$$

where $\bigcap_n \widehat{A}_n \in \mathcal{I}$ and $\bigcap_n \widehat{B}_n \in \mathcal{J}$. Assume for simplicity that $\widehat{A}_1 = \Omega$ and $\widehat{B}_1 = \Theta$. Now set

$$C_1 := \bigcap_{n=1}^{\infty} \widehat{A}_n \quad \text{and} \quad D_1 := \bigcap_{n=1}^{\infty} \widehat{B}_n$$

and for all $n \geq 1$

$$A_n := \widehat{A}_n \setminus C_1 \quad \text{and} \quad B_n := \widehat{B}_n \setminus D_1$$

$$C_{n+1} := A_n \setminus A_{n+1} \quad \text{and} \quad D_{n+1} := B_n \setminus B_{n+1}\,.$$

Notice that C_n's are pairwise disjoint and $\bigcup_n C_n = \Omega$. A similar property holds true also for $\langle D_n \rangle$.

We define two new sets in the product space by setting

$$U := \bigcup_{n=1}^{\infty} C_n \times B_n \qquad \text{and} \qquad V := \bigcup_{n=1}^{\infty} A_n \times D_n\,.$$

We have

$$U \cap V = \emptyset\,.$$

Let $\{\mathcal{F}(\omega) : \omega \in \Omega\}$ and $\{\mathcal{G}(\theta) : \theta \in \Theta\}$ be the collections of filters in Σ and T respectively, generated by δ and τ, respectively. Now if $\overline{\omega} \in \Omega$ then there is $n_{\overline{\omega}} \in \mathbf{N}$ such that $\overline{\omega} \in C_{n_{\overline{\omega}}}$ and so

$$U_{\overline{\omega}} = \Big(\bigcup_{n=1}^{\infty} C_n \times B_n\Big)_{\overline{\omega}} = \Big(C_{n_{\overline{\omega}}} \times B_{n_{\overline{\omega}}}\Big)_{\overline{\omega}} = B_{n_{\overline{\omega}}} \in \mathcal{G}(\theta).$$

Hence $U_{\overline{\omega}} \in \mathcal{G}(\theta)$, and similarly, $V^{\overline{\theta}} \in \mathcal{F}(\omega)$. By the assumption, we have for each $(\overline{\omega}, \overline{\theta})$ that

$$[\varphi(U)]_{\overline{\omega}} \supseteq \tau\Big([\varphi(U)]_{\overline{\omega}}\Big) \quad \text{and} \quad [\varphi(U)]^{\overline{\theta}} \supseteq \delta\Big([\varphi(U)]^{\overline{\theta}}\Big) .$$

Moreover, (*) yields the existence of a set $N_U \in \mathcal{I}$ such that

$$[\varphi(U)]_{\overline{\omega}} \triangle U_{\overline{\omega}} \in \mathcal{J} \quad \text{for all } \overline{\omega} \notin N_U .$$

It follows that

$$[\varphi(U)]_{\overline{\omega}} \supseteq \tau\left([\varphi(U)]_{\overline{\omega}}\right) = \tau\Big(U_{\overline{\omega}}\Big) \quad \text{for all } \overline{\omega} \notin N_U .$$

Now, since $U_{\overline{\omega}} \in \mathcal{G}(\theta)$ for all $\overline{\omega} \in \Omega$, we have

$$\theta \in \tau\Big(U_{\overline{\omega}}\Big) \subseteq [\varphi(U)]_{\overline{\omega}} \quad \text{for all } \overline{\omega} \notin N_U .$$

Consequently, $\theta \in [\varphi(U)]_{\overline{\omega}}$ for all $\overline{\omega} \notin N_U$ or equivalently, $\overline{\omega} \in [\varphi(U)]^{\theta}$ for all $\overline{\omega} \notin N_U$. Hence, $N_U^c \subseteq [\varphi(U)]^{\theta}$ and consequently,

$$\Omega = \delta(N_U^c) \subseteq \delta\left([\varphi(U)]^{\theta}\right) \subseteq [\varphi(U)]^{\theta} .$$

This yields $(\omega, \theta) \in \varphi(U)$. In a similar way we can see that $(\omega, \theta) \in \varphi(V)$. This is however impossible since $U \cap V = \emptyset$ and so $\varphi(U) \cap \varphi(V) = \emptyset$. This completes the whole proof. □

Corollary 6. *Let Ω, Θ be topological spaces possessing the following property: there are countable collections of non-empty open subsets of Ω and of Θ such that each open subset of Ω and each open subset of Θ contains a member of the corresponding collection (in particular Ω and Θ may be Polish spaces) . Let $\mathcal{I}, \mathcal{J}$ be the collections of 1st category subsets of Ω and Θ, respectively and Σ, T be the corresponding families of the sets with the Baire property. We assume that $\Sigma/\mathfrak{I}$ and $T/\mathfrak{J}$ satisfy CCC and both are non-atomic. Moreover, let $\mathcal{K}$ be the family of the 1st category sets in $\Omega \times \Theta$ and Ξ be the collection of the sets with the Baire property in the product space. Then, there are no densities $\delta \in \vartheta(\mathcal{I})$, $\tau \in \vartheta(\mathcal{J})$ and $\varphi \in \vartheta(\mathcal{K})$ possessing (δ, τ)-sub-invariant sections.*

Proof. According to the Kuratowski-Ulam Theorem (see for example [4], comments following Theorem 15.1) the triplet $(\mathcal{I},\mathcal{J},\mathcal{K})$ has the property $(*)$. Assume, if possible that there are densities $\delta \in \vartheta(\mathcal{I})$, $\tau \in \vartheta(\mathcal{J})$ and $\varphi \in \vartheta(\mathcal{K})$ possessing (δ,τ)-subinvariant sections. Since $\Sigma/\mathcal{I}$ and $T/\mathcal{J}$ are non-atomic it follows that neither δ is $\mathcal{I}$-continuous nor is τ $\mathcal{J}$-continuous. So the result follows from Theorem 5. □

If $\sigma \in \Lambda(\mathcal{J})$ is $\mathcal{J}$-continuous and $\langle Q_n\rangle$ are all the atoms of $T/\mathcal{J}$, then $\bigcup_{n=1}^{\infty}\sigma(Q_n) = \Theta$ and so we get

Theorem 7. *Let $(\Omega,\Sigma,\mathcal{I})$ and $(\Theta,T,\mathcal{J})$ be spaces with liftings $\rho \in \Lambda(\mathcal{I})$ and $\sigma \in \Lambda(\mathcal{J})$ with their Boolean algebras satisfying (CCC). Let $(\Omega\times\Theta,\Xi,\mathcal{K})$ be such that $(\mathcal{I},\mathcal{J},\mathcal{K})$ satisfies (*). If there exists a lifting $\pi \in \Lambda(\mathcal{K})$ possessing (ρ,σ)-invariant sections, then either $\Omega = \bigcup_n \rho(P_n)$ where P_n's are $\mathcal{I}$-atoms or $\Theta = \bigcup_n \sigma(Q_n)$, where Q_n's are $\mathcal{J}$-atoms.*

In case when Ξ is close to $\Sigma\otimes T$ we have also a reverse result.

Theorem 8. *Let $(\Omega,\Sigma,\mathcal{I})$ be a space possessing a lifting ρ and let $T/\mathcal{J}$ be an atomic algebra satisfying (CCC). Let $(\Omega\times\Theta,\Xi,\mathcal{K})$ be such that $(\mathcal{I},\mathcal{J},\mathcal{K})$ satisfies (*) and $\Sigma\otimes T$ is $\mathcal{K}$-dense in Ξ. Then there exist $\sigma \in \Lambda(\mathcal{J})$ and $\pi\in\Lambda(\mathcal{K})$ such that π has (ρ,σ)-invariant sections.*

Proof. Take $\sigma \in \Lambda(\mathcal{J})$ such that $\Theta = \bigcup_n \sigma(Q_n)$, where Q_n's are the $T/\mathcal{J}$-atoms, and an arbitrary $\rho\in\Lambda(\mathcal{I})$. If $E\in\Xi\oplus\mathcal{K}$, then for each n there is $A_n \in \Sigma\oplus\mathcal{I}$ with $E \stackrel{\mathcal{K}}{=} \bigcup_n A_n\times Q_n$. It is enough to set $\pi(E) = \bigcup_n \rho(A_n)\times\sigma(Q_n)$. According to Proposition 3 the lifting π is uniquely determined by ρ and σ. □

Corollary 9. *Let (Ω,Σ,μ) and (Θ,T,ν) be probability spaces with liftings $\rho\in\Lambda(\mu)$ and $\sigma\in\Lambda(\nu)$ and let $(\Omega\times\Theta,\Xi,\kappa)$ be a probability space such that $\Sigma\otimes T\subseteq\Xi$, $\Sigma\otimes T$ is κ-dense in Ξ and $\kappa|\Sigma\otimes T = \mu\otimes\nu$. Then a lifting $\pi\in\Lambda(\kappa)$ possessing (ρ,σ)-invariant sections exists if and only if, either $\Omega = \bigcup_n \rho(P_n)$ where P_n's are μ-atoms or $\Theta=\bigcup_n\sigma(Q_n)$ where Q_n's are ν-atoms.*

Notice that the assumptions of Corollary 9 are satisfied if μ and ν are τ-additive and κ is the τ-additive extension of $\mu\otimes\nu$. In particular μ, ν and κ may be Radon measures.

Question 10. We have proven in [3] that if (Ω,Σ,μ), (Θ,T,ν) are complete probability spaces and $(\Omega\times\Theta,\Xi,\kappa)$ is their completed product, then there are liftings $\rho\in\varLambda(\mu)$, $\sigma\in\varLambda(\nu)$ and $\pi\in\varLambda(\kappa)$ such that $\pi\in\rho\otimes\sigma$ and for every $E\in\Xi$ and every $\omega\in\Omega$ we have

$$[\pi(E)]_\omega = \sigma\Big([\pi(E)]_\omega\Big)\,.$$

We do not know if a similar result holds true in case of arbitrary σ-ideals satisfying $(*)$.

It is worth to mention at this place that in case of an arbitrary Baire space (no non-empty open set is of the first category in itself) always there is a lifting on the σ-algebra of sets with the Baire property, with respect to the ideal of the first category sets (cf [1] or [2]).

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