Real Analysis Exchange
Vol. 29(1), 2003/2004, pp. 335–340

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A NOTE ON THE LEBESGUE DIFFERENTIATION THEOREM IN SPACES OF HOMOGENEOUS TYPE

Abstract

We prove that the Lebesgue differentiation theorem holds in the general setting of spaces of homogeneous type if the balls are subspaces of homogeneous type.

The theory of differentiation of an integral is an important tool in the classical theory of harmonic analysis for maximal functions, singular integrals and weighted norm inequalities on $\mathbb{R}^n$. One of the most popular abstract setting for the above theories is the case of the spaces of homogeneous type which are, in the sense of Coiffman and Weiss, quasimetric spaces with a Borel measure satisfying a doubling condition on balls (see the definitions below). In this context the $(1,1)$-weak type of the maximal operator of Hardy-Littlewood allows proving that almost every point is a Lebesgue point of f assuming, for instance, that the measure is regular. A weaker condition than the regularity of the underlying measure is usually assumed (see [2] for example). The set of continuous functions is dense in L^1. This is probably the most general situation in which the Lebesgue differentiation theorem is known to hold in the setting of spaces of homogeneous type. On the other hand, when we are facing a particular case, it may occur that the regularity of the underlying measure or the density condition of the continuous functions in L^1 are not easy to verify.

The aim of this note is to show that a condition of a geometrical nature together with a slight generalization of a standard result in the theory of probability measures on metric spaces imply the validity of the Lebesgue differentiation theorem in the general setting of the spaces of homogeneous

Key Words: Differentiation of Integrals, Spaces of Homogeneous Type
Mathematical Reviews subject classification: 58C99
Received by the editors April 1, 2003
Communicated by: B. S. Thomson

*The author was supported in part by Conicet type without asking for the regularity of the measure or the density of the continuous functions in L^1. The geometrical condition is that the balls are subspaces of homogeneous type. At first sight this could be a serious restriction because even in a rather simple case such as the one shown in [1] the balls are not subspaces of homogeneous type. But this difficulty may be overcame by recalling a result in [5] which says that given a space of homogeneous type we can always find a quasidistance equivalent to the original one such that the balls defined by this new quasidistance are subspaces of homogeneous type.

The key result for the approach given here is a result of Macías and Segovia proved in [4].

Now we give some standard definitions and notation. Let X be a set. A nonnegative symmetric function d defined on $X \times X$ is called a *quasidistance on* X if and only if there exists a constant $K \geq 1$ such that for all x, y and $z \in X$ the following conditions hold:

i) $d(x, y) = 0$ if and only if $x = y$;

ii) $d(x, y) = d(y, x)$;

iii) $d(x, y) \leq K\left(d(x, z) + d(z, y)\right)$.

Inequality iii) is often called *quasitriangular inequality* and K is often called the *quasitriangular constant* of d. Of course, d is called a metric when $K = 1$.

A pair (X, d) is called a *quasimetric space* if X is a set and d is a quasidistance on X.

Let $x \in X$ and let r be a positive real number. The set $\{y \in X \,/\, d(x, y) < r\}$ is called *an open ball of radius* r *centered in* x and will be denoted by $B(x, r)$. We shall say that a set $E \subset X$ is *open* if for every $x \in E$ there exists a number $r > 0$ such that $B(x, r) \subset E$. Unlike the metric case; i.e., $K = 1$, the open balls defined by a quasidistance may not be open sets.

We shall say that a set $E \subset X$ is *bounded* if there exist $x \in X$ and $r > 0$ such that $E \subset B(x, r)$.

A triple (X, d, μ) is called a *measurable quasimetric space* if (X, d) is a quasi metric space and μ is a positive measure defined on a σ-algebra of subsets of X containing the balls. Given a measurable set $E \subset X$ we shall denote the restriction of d and μ to E with the symbols d_E and μ_E respectively.

Following Coiffman and Weiss in [3] we shall say that a measurable quasimetric space (X, d, μ) is a space of homogeneous type if μ is a Borel measure satisfying the following so-called doubling condition. There exists a positive constant C such that for all $x \in X$ and all $r > 0$ we have

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r)) < \infty.$$

Notice that we do not require (as is done in [3]) that the open balls are open sets.

A set $E \subset X$ will be called a subspace of homogeneous type if the triple (E, d_E, μ_E) is a space of homogeneous type.

Let (X, d, μ) be a measurable quasimetric space and let $f \in L^1_{loc}(X)$. A point $x \in X$ is said to be a *Lebesgue point of* f if

$$\lim_{r\to 0^+} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, d\mu(y) = f(x).$$

In this work we shall make use of some known results. We just give the statement of them as follow:

a) In a space of homogeneous type a subset is bounded if and only if it has finite measure.

b) Let (X, d) be a quasimetric space. There exist positive constants c_1 and c_2 and a quasidistance d' such that d is equivalent to d' in the sense that $c_1 d' \leq d \leq c_2 d'$ and such that the d'-balls are open sets in the topology induced by both d and d'. Furthermore $d' = \rho^\alpha$ where ρ is a metric on X and α is a positive constant depending on d.

c) Let (X, d, μ) be a space of homogeneous type and let $f \in L^1_{\text{loc}}(X)$. If μ is regular, then almost every point is a Lebesgue point of f.

We are now in a position to give the statement of the main result of this note.

Theorem. *Let (X, d, μ) be a space of homogeneous type such that the balls are subspaces of homogeneous type and let $f \in L^1_{loc}(X)$. Then almost every point of X is a Lebesgue point of f.*

For the proof of this theorem we will need the following lemma which is a slight generalization of a classical result in the theory of probability measures on metric spaces.

Lemma. *Let (X, d, μ) be a measurable quasimetric space such that $\mu(X) < \infty$. Then the σ-algebra of the μ-regular sets contains the σ-algebra of the Borel sets*

Proof. Let $\mathcal{S}$ be the σ-algebra of the μ-regular sets. It is enough to show that $\mathcal{S}$ contains the closed sets. Let C be a subset of X and consider the function $x \mapsto d(x, C)$. Since we are in a quasimetric space, this function need not be continuous in the underlying topology given by d. To overcome this difficulty we use a result proved in [4] which says that for all x, $y \in X$, there exist positive constants c_1, c_2 and α and a metric ρ on X such that

$$c_1\rho(x,y)^\alpha \leq d(x,y) \leq c_2\rho(x,y)^\alpha.$$

The function $x \mapsto \rho(x, C)$ is continuous in both the topology given by ρ and d. Let C be a closed subset of X. Since C is closed, we only need to show that for any $\epsilon > 0$ there exists an open set V such that $C \subset V$ and $\mu(V - C) < \epsilon$. To see this, notice that

$$C \subset \{x \in X \,/ d(x,C) < n^{-1}\} \subset \{x \in X \,/ \rho(x,C) < 2(c_1 n)^{-1/\alpha}\} = V_n,$$

for all $n \in \mathbb{N}$. We have that each V_n is open in both the topology given by ρ and d. Since C is a closed set in the topology given by ρ, we have that

$$C = \bigcap_{n \geq 1} V_n,$$

with $V_{n+1} \subset V_n$ for all $n \geq 1$. Therefore, $\mu(V_n) \to \mu(C)$ as $n \to \infty$ and this implies that for any $\epsilon > 0$ we can find a V_n satisfying that $\mu(V_n - C) < \epsilon$ and the lemma follows. □

Now we prove the theorem. Let E be the Borel set consisting of all points of X which are not Lebesgue points of f and suppose that $\mu(E) > 0$. Let d' be a quasidistance equivalent to d such that the d'-balls are open sets in the topology induced by d. Let $z \in X$ be an arbitrary point. Then

$$X = \bigcup_{n \in \mathbb{N}} B'(z, n),$$

where B' means d'-balls. Therefore, there exists an index n_0 such that

$$\mu\left(E \cap B'(z, n_0)\right) > 0.$$

Since d is equivalent to d', there exists a d-ball B such that $B'(z, n_0) \subset B$.

Recall that a space of homogeneous type is bounded if and only if it has finite measure. By hypothesis, the ball B is a subspace of homogeneous type so that (B, d_{B}, μ_{B}) is a space of homogeneous type of finite measure. Now the above lemma implies that μ_B is a regular Borel measure. But in this context, regularity implies that almost every point of B is a Lebesgue point of f.

On the other hand given $x \in E \cap B'(z, n_0)$ there exists a positive number r_x such that for all $r < r_x$ we have $B(x, r) \subset B'(z, n_0) \subset B$, because the

d'-ball $B'(z, n_0)$ is open in the topology induced by d. But then

$$\lim_{r\to 0^+} \frac{1}{\mu(B\cap B(x,r))} \int_{B\cap B(x,r)} f(y)\, d\mu(y)$$
$$= \lim_{\substack{r\to 0^+ \\ r<r_x}} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)\, d\mu(y) \neq f(x),$$

so that every point of $E\cap B'(z, n_0)$, which is a subset of B of positive measure, is not a Lebesgue point of f in B and this is a contradiction. Therefore $\mu(E) = 0$ and the theorem follows. □

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