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ALGEBRAS WITH INNER MB-REPRESENTATION

Abstract

We investigate algebras of sets, and pairs $\langle\mathcal{A},\mathcal{I}\rangle$ consisting of an algebra $\mathcal{A}$ and an ideal $\mathcal{I}\subset\mathcal{A}$, that possess an inner MB-representation. We compare inner MB-representability of $\langle\mathcal{A},\mathcal{I}\rangle$ with several properties of $\langle\mathcal{A},\mathcal{I}\rangle$ considered by Baldwin. We show that $\mathcal{A}$ is inner MB-representable if and only if $\mathcal{A}=S(\mathcal{A}\setminus\mathcal{H}(\mathcal{A}))$, where $S(\cdot)$ is a Marczewski operation defined below and $\mathcal{H}$ consists of sets that are hereditarily in $\mathcal{A}$. We study the question of uniqueness of the ideal in that representation.

1 The Implications

Let X be a nonempty set and let $\mathcal{F}$ be a nonempty family of nonempty subsets of X. Following the idea of Burstin and Marczewski we define:

$$S(\mathcal{F})=\{A\subset X\colon (\forall P\in\mathcal{F})(\exists Q\in\mathcal{F})(Q\subset A\cap P \text{ or } Q\subset P\setminus A)\}$$

and

$$S_0(\mathcal{F})=\{A\subset X\colon (\forall P\in\mathcal{F})(\exists Q\in\mathcal{F})(Q\subset P\setminus A)\}.$$

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Then $S(\mathcal{F})$ is an algebra of subsets of X and $S_0(\mathcal{F})$ is an ideal on X. (See [BBRW].) For an ideal $\mathcal{I}$ on X an algebra $\mathcal{A}$ of subsets of X such that $\mathcal{I} \subset \mathcal{A}$ we say that

- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ (respectively, the algebra $\mathcal{A}$) *has inner MB-representation* provided there exists an $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{A} = S(\mathcal{F})$ and $\mathcal{I} = S_0(\mathcal{F})$ (respectively, $\mathcal{A} = S(\mathcal{F})$),
- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ *has density property* provided $\mathcal{I} = S_0(\mathcal{A} \setminus \mathcal{I})$,
- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ (respectively, the algebra $\mathcal{A}$) *is topological* provided there exists a topology τ on X such that $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ (respectively, $\mathcal{A} = S(\mathcal{F})$), where $\mathcal{F} = \tau \setminus \{\emptyset\}$,
- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ *has the hull property* provided for every $U \subset X$ there is a $V \in \mathcal{A}$ such that $U \subset V$ and for every $W \in \mathcal{A}$ if $U \subset W$, then $V \setminus W \in \mathcal{I}$,
- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ *is complete* provided the quotient algebra $\mathcal{A}/\mathcal{I}$ is complete,
- the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ *has the splitting property* provided for every $\mathcal{C} \subset \mathcal{D} \subset \mathcal{A}$, if $\mathcal{D}$ is an antichain (i.e., $A \cap B \in \mathcal{I}$ for every distinct $A, B \in \mathcal{D}$), then there exists a mapping $\mathcal{D} \ni D \mapsto I_D \in \mathcal{I}$ such that $C \setminus I_C$ and $D \setminus I_D$ are disjoint for every $C \in \mathcal{C}$ and $D \in \mathcal{D} \setminus \mathcal{C}$.

In the graph from Theorem 2 each of these properties is denoted, respectively, as: inner, dense, top, hull, comp, and split.

We start here with the following simple characterization of pairs with inner MB-representation. (Compare also [Wr, lemma 1].)

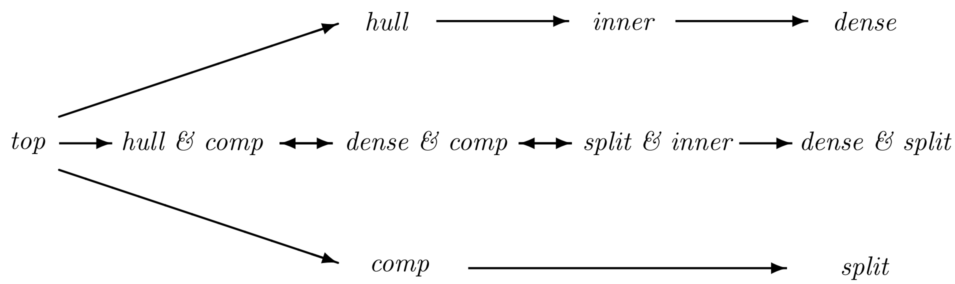
Proposition 1. *A pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has an inner MB-representation if and only if $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$.*

Proof. If $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$, then $\mathcal{A} \setminus \mathcal{I} \subset \mathcal{A} \setminus S_0(\mathcal{A} \setminus \mathcal{I})$, since we always have $\mathcal{F} \cap S_0(\mathcal{F}) = \emptyset$. So, $S_0(\mathcal{A} \setminus \mathcal{I}) \subset \mathcal{I}$. The other inclusion is obvious. Thus, $\langle \mathcal{A}, \mathcal{I} \rangle$ has an inner MB-representation.

Conversely, assume that $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ for some $\mathcal{F} \subset \mathcal{A}$. By [BBRW, prop. 1.2] to prove that $S(\mathcal{A} \setminus \mathcal{I}) = S(\mathcal{F})$ it is enough to show that the families $\mathcal{A} \setminus \mathcal{I}$ and $\mathcal{F}$ are *mutually coinitial*; that is, every element of each of these families contains an element from the other.

Clearly, $\mathcal{F} \subset \mathcal{A} \setminus S_0(\mathcal{F}) = \mathcal{A} \setminus \mathcal{I}$, so every element of $\mathcal{F}$ contains an element from $\mathcal{A} \setminus \mathcal{I}$. Conversely, if $A \in \mathcal{A} \setminus \mathcal{I}$, then there exists an $F \in \mathcal{F}$ with $F \subset A$, since $A \notin \mathcal{I} = S_0(\mathcal{F})$. □

Theorem 2. *We have the following implications between the properties of a pair $\langle \mathcal{A}, \mathcal{I} \rangle$.*



Diagram

Moreover, none of the implications can be reversed, with possible exception of "top $\Longrightarrow$ hull & comp."

PROOF. The facts that every topological pair is complete and has the hull property are well known and easy to see. Indeed, if $\langle \mathcal{A}, \mathcal{I} \rangle$ is a topological pair generated by a topology τ on X, then $\mathcal{I}$ consists of all nowhere dense sets (with respect to τ) and $\mathcal{A}$ consists of open sets (with respect to τ) modulo $\mathcal{I}$. (See [BR].) Then, for each $E \subset X$, the closure $\mathrm{cl}(E)$ plays a role of its hull. Since an open set U can be expressed as $U = V \setminus E$ where V is regular open and E is nowhere dense (see e.g. [O, thm. 4.5]), the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic to the Boolean algebra of regular open sets, which is complete (see e.g. [K]). Hence $\mathcal{A}/\mathcal{I}$ is complete.

The implication "inner $\Longrightarrow$ dense" results immediately from Proposition 1 and the definitions. All other implications follow from the following implications proved in Baldwin's paper [Ba]: "hull $\Longrightarrow$ inner," "comp $\Longrightarrow$ split," "split & inner $\Longrightarrow$ comp," and "dense & comp $\Longrightarrow$ hull."

The fact that the implications "top $\Longrightarrow$ hull" and "top $\Longrightarrow$ comp" cannot be reversed follows from Baldwin's examples from [Ba], where he shows that the properties hull and complete are independent of each other.

An example showing that "dense & split" does not imply "inner" is described in Example 3. This takes care of nonreversability of the implications "split & inner $\Longrightarrow$ dense & split," "inner $\Longrightarrow$ dense," and "comp $\Longrightarrow$ split."

Example 4 shows that the implications "hull $\Longrightarrow$ inner" cannot be reversed. □

The following example answers a question of Baldwin [Ba, question 2] whether every pair with density and splitting properties must be inner. Also, Baldwin had the example of a family with a splitting property which is not complete only under the assumption of the continuum hypothesis, while the example below is in ZFC.

Example 3. *If X is an infinite set, $\mathcal{A}$ is an algebra of subsets of X which are either finite or cofinite, and $\mathcal{I} = \{\emptyset\}$, then the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has density and splitting properties but is neither inner nor complete.*

Proof. The pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has density property since $S_0(\mathcal{A} \setminus \{\emptyset\}) = \{\emptyset\} = \mathcal{I}$. It does not have inner MB-representation by Proposition 1 and the fact that $S(\mathcal{A} \setminus \{\emptyset\}) = \mathcal{P}(X)$. The splitting property is satisfied trivially, since $\mathcal{I} = \{\emptyset\}$.

The pair $\langle \mathcal{A}, \mathcal{I} \rangle$ is not complete by the implications from Theorem 2. □

The following example answers a question of Baldwin [Ba, question 1] whether every pair with inner MB-representation must have a hull property. In what follows we use the standard set theoretic notation as in [Ci]. Let X be an infinite set of cardinality κ. We say that a family $\mathcal{F}_0 \subset [X]^\kappa$ is *almost disjoint* provided $|F_1 \cap F_2| < \kappa$ for every distinct $F_1, F_2 \in \mathcal{F}_0$.

Example 4. *There exists a maximal almost disjoint family $\mathcal{F}_0 \subset [X]^\kappa$ such that for $\mathcal{F} = \{F \triangle A \colon F \in \mathcal{F}_0 \ \&\ A \in [X]^{<\kappa}\}$ the pair $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ has inner MB-representation but neither is complete nor it has the hull property.*

Proof. In [BC, fact 4] it was proved that for every $\mathcal{F}$ as in the theorem the algebra $S(\mathcal{F})$ contains $\mathcal{F}$ (so it has inner MB-representation) and $S_0(\mathcal{F}) = [X]^{<\kappa}$.

Let $\{A, B\}$ be a partition of X into the sets of cardinality κ and let $\mathcal{G} \subset [X]^\kappa$ be a partition of X into κ many sets such that $|G \cap A| = |G \cap B| = \kappa$ for every $G \in \mathcal{G}$. Let $\mathcal{F}_0 \subset [X]^\kappa$ be a maximal almost disjoint family extending $\mathcal{G}$ such that for every $F \in \mathcal{F}_0$ either $F \subset A$ or $F \subset B$. Such an $\mathcal{F}_0$ exists by the Zorn lemma. It is easy to see that $\mathcal{F}_0$ is a maximal almost disjoint family in $[X]^\kappa$.

To see that $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$ does not have the hull property notice that $A \subset X$ does not have a hull. Indeed, take a $V \in S(\mathcal{F})$ containing A. Then for every $G \in \mathcal{G} \subset \mathcal{F}$ there is an $F_G \in \mathcal{F}$ contained in G such that F_G is either disjoint or contained in V. Thus, $F_G = G \setminus A_G$ for some $A_G \in [X]^{<\kappa}$, since elements of $\mathcal{F}_0$ are almost disjoint. This implies also that $F_G = G \setminus A_G$ must be a subset of V, since it cannot be disjoint with $V \supset A$. In other words, for every $G \in \mathcal{G}$ there exists an $x_G \in G \cap (V \setminus A)$. So, $Y = \{x_G \colon G \in \mathcal{G}\} \in [B]^\kappa$, and by the maximality, there exists an $F \in \mathcal{F}_0$ such that $|F \cap Y| = \kappa$. Then, for $W = V \setminus F \in S(\mathcal{F})$ we have $A \subset W \subset V$, while $V \setminus W = F \cap Y \notin [X]^{<\kappa} = S_0(\mathcal{F})$. Thus, there is no hull for A with respect to $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$. □

Problem 5. Is every complete pair $\langle \mathcal{A}, \mathcal{I} \rangle$ with the hull property topological?

2 Notes on Algebras with Inner MB-Representations

According to Proposition 1 if a pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has inner MB-representation, then it has a canonical one — by a family $\mathcal{F} = \mathcal{A} \setminus \mathcal{I}$. But what if we only consider inner MB-representability of an algebra $\mathcal{A}$? If $\mathcal{A}$ has an inner MB-representation, say $\mathcal{A} = S(\mathcal{F})$, then by Proposition 1 for $\mathcal{I} = S_0(\mathcal{F})$ we have $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$. Is there a canonical ideal $\mathcal{I}$ with this property? Is such an ideal unique?

To give a positive answer to the first of these questions we need the following fact. Note that, in general, $\mathcal{F}_2 \subset \mathcal{F}_1$ does not imply $S(\mathcal{F}_2) \subset S(\mathcal{F}_1)$. For instance, if $X = \{0, 1, 2\}$, $\mathcal{F}_2 = \{\{0\}\}$, and $\mathcal{F}_1 = \{\{0\}, \{1, 2\}\}$, then $\{2\} \in S(\mathcal{F}_2) \setminus S(\mathcal{F}_1)$.

Lemma 6. *If $\mathcal{I}_1 \subset \mathcal{I}_2$ are ideals contained in an algebra $\mathcal{A}$, then we have $S(\mathcal{A} \setminus \mathcal{I}_2) \subset S(\mathcal{A} \setminus \mathcal{I}_1)$.*

PROOF. Let $A \in S(\mathcal{A} \setminus \mathcal{I}_2)$. To show that $A \in S(\mathcal{A} \setminus \mathcal{I}_1)$ take a $P \in \mathcal{A} \setminus \mathcal{I}_1$. We need to find a $Q \in \mathcal{A} \setminus \mathcal{I}_1$ for which

$$\text{either } Q \subset P \cap A \text{ or } Q \subset P \setminus A. \tag{1}$$

If $P \in \mathcal{A} \setminus \mathcal{I}_2$, then clearly there is a $Q \in \mathcal{A} \setminus \mathcal{I}_2 \subset \mathcal{A} \setminus \mathcal{I}_1$ satisfying (1). So assume that $P \notin \mathcal{A} \setminus \mathcal{I}_2$. Then $P \in \mathcal{I}_2 \setminus \mathcal{I}_1$. So, $P \cap A$ and $P \setminus A$ belong to $\mathcal{I}_2$ and at least one of them does not belong to $\mathcal{I}_1$. This set can be taken as Q, since $\mathcal{I}_2 \setminus \mathcal{I}_1 \subset \mathcal{A} \setminus \mathcal{I}_1$. □

For an algebra $\mathcal{A}$ of subsets of X, the ideal of hereditary sets in $\mathcal{A}$ is defined as $\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{A} \colon \mathcal{P}(A) \subset \mathcal{A}\}$.

Proposition 7. *Let $\mathcal{I}$ be an ideal on a set X, let $\mathcal{A}$ be an algebra on X and assume that $\mathcal{I} \subset \mathcal{A} = S(\mathcal{A} \setminus \mathcal{I}) \neq \mathcal{P}(X)$. Then for every ideal $\mathcal{J}$ such that $\mathcal{I} \subset \mathcal{J} \subset \mathcal{H}(\mathcal{A})$ we have $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{J})$.*

PROOF. Notice that any ideal $\mathcal{J} \subset \mathcal{A}$ is a proper subset of $\mathcal{A}$ since $\mathcal{A} \neq \mathcal{P}(X)$. It is easy to see that for any such ideal we have $\mathcal{A} \subset S(\mathcal{A} \setminus \mathcal{J})$. Indeed, if $A \in \mathcal{A}$ and $P \in \mathcal{A} \setminus \mathcal{J}$, then either $Q = P \setminus A$ belongs to $\mathcal{A} \setminus \mathcal{J}$ or $Q = P \cap A$ belongs to $\mathcal{A} \setminus \mathcal{J}$. Now, by Lemma 6, we have

$$\mathcal{A} \subset S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) \subset S(\mathcal{A} \setminus \mathcal{J}) \subset S(\mathcal{A} \setminus \mathcal{I}) = \mathcal{A}.$$

This finishes the proof. □

The proposition implies immediately the following corollary, which shows, in particular, that the ideal $\mathcal{I} = \mathcal{H}(\mathcal{A})$ is canonical ideal in representation $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$.

Corollary 8. *An algebra $\mathcal{A} \neq \mathcal{P}(X)$ has an inner MB-representation if and only if $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$.*

Notice that Corollary 8 immediately implies [BBC, thm. 13], since conditions (I) and (II) from that theorem say that $\mathcal{H}(\mathcal{A}) = \mathcal{A} \cap [X]^{<\kappa}$ while (III) says that $S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) \setminus \mathcal{A} \neq \emptyset$. In particular, Corollary 8 implies easily that the following algebras do not have inner MB-representation:

- The algebra $\mathcal{B}$ of Borel subset of $\mathbb{R}$, since $S(\mathcal{B} \setminus \mathcal{H}(\mathcal{B})) = S(\mathcal{B} \setminus [\mathbb{R}]^{\leq\omega})$ is a classical Marczewski's algebra. (Compare [BBC, cor. 14].)
- The interval algebra $\mathcal{A}$ (i.e., generated by all intervals $[a, b)$, where $a, b \in \mathbb{R}$), since $\mathcal{H}(\mathcal{A}) = \{\emptyset\}$ and so $S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$ is an algebra of subsets of $\mathbb{R}$ with nowhere dense boundary. (Compare [BBC, prop. 12].)
- The algebra $\mathcal{A}$ generated by all open intervals (a, b) $(a, b \in \mathbb{R})$, since $\mathcal{H}(\mathcal{A}) = [\mathbb{R}]^{<\omega}$ and so $S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$ is an algebra of subsets of $\mathbb{R}$ with nowhere dense boundary.

Next, we will address the question of uniqueness of the ideal in the representation $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$. We will start with the following proposition.

Proposition 9. *Let $\mathcal{A}$ be an algebra, let $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$ be ideals, and $Y \in \mathcal{A}$.*

(a) *If every $P \subset Y$ from $\mathcal{A} \setminus \mathcal{J}$ contains a subset in $\mathcal{I} \setminus \mathcal{J}$, then $\mathcal{P}(Y) \subset S(\mathcal{A} \setminus \mathcal{J})$.*

(b) *If $\mathcal{I} \cap \mathcal{P}(Y) = \mathcal{J} \cap \mathcal{P}(Y)$, then $S(\mathcal{A} \setminus \mathcal{I}) \cap \mathcal{P}(Y) = S(\mathcal{A} \setminus \mathcal{J}) \cap \mathcal{P}(Y)$.*

Proof. (a): Let $A \in \mathcal{P}(Y)$ and take $P \in \mathcal{A} \setminus \mathcal{J}$. We need to find a $Q \in \mathcal{A} \setminus \mathcal{J}$ for which

$$\text{either } Q \subset P \cap A \text{ or } Q \subset P \setminus A.$$

If $P \in \mathcal{I} \setminus \mathcal{J}$, then either $P \cap A$ or $P \setminus A$ belongs to $\mathcal{I} \setminus \mathcal{J}$; so we may take this set as a Q. So, assume that $P \in \mathcal{A} \setminus \mathcal{I}$, then there is a $P_0 \in \mathcal{I} \setminus \mathcal{J}$ contained in P. Thus, as before, either $P_0 \cap A$ or $P_0 \setminus A$ belongs to $\mathcal{I} \setminus \mathcal{J}$ and we may take this set as a Q.

Part (b) is obvious. □

For an algebra $\mathcal{A} \subset \mathcal{P}(X)$ and the ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$ a set $Y \in \mathcal{A}$ will be called $\langle \mathcal{I}, \mathcal{J} \rangle$-*special* if $\mathcal{I} \cap \mathcal{P}(X \setminus Y) = \mathcal{J} \cap \mathcal{P}(X \setminus Y)$ and each set $P \subset Y$ such that $P \in \mathcal{A} \setminus \mathcal{J}$ has a subset in $\mathcal{I} \setminus \mathcal{J}$.

From Proposition 9 we easily derive the following corollary.

Corollary 10. *Let $\mathcal{A}$ be an algebra on X and let $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$ be ideals. If $Y \in \mathcal{A}$ is an $\langle \mathcal{I}, \mathcal{J} \rangle$-special set, then*

$$S(\mathcal{A} \setminus \mathcal{J}) = \{C \cup D \colon C \in \mathcal{P}(Y) \ \&\ D \in \mathcal{P}(X \setminus Y) \cap S(\mathcal{A} \setminus \mathcal{J})\}.$$

From Proposition 9 (a) applied to $Y = \mathbb{R}$ we immediately obtain the following facts.

- If $\mathcal{L}$ is the algebra of Lebesgue measurable subsets of $\mathbb{R}$, $\mathcal{N}$ is the ideal of measure zero sets, and $\mathcal{N}_0$ is the ideal generated by F_σ sets from $\mathcal{N}$, then $S(\mathcal{L} \setminus \mathcal{J}) = \mathcal{P}(\mathbb{R})$ for every ideal $\mathcal{J}$ contained either in $\mathcal{N}_0$ or in $\mathcal{N} \cap [\mathbb{R}]^{<2^\omega}$.

- If $\mathcal{B}$ is the algebra of subsets of $\mathbb{R}$ with the Baire property and $\mathcal{M}$ is the ideal of meager sets, then $S(\mathcal{B} \setminus \mathcal{J}) = \mathcal{P}(\mathbb{R})$ for every ideal $\mathcal{J}$ contained either in $\mathcal{N}_0$ or in $\mathcal{M} \cap [\mathbb{R}]^{<2^\omega}$.

From Corollary 10 we immediately see that, most of the time, $\mathcal{H}(\mathcal{A})$ is not the only ideal $\mathcal{I}$ for which $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$. The easiest way to see it is to notice the following conclusion from Corollary 10.

Corollary 11. *If $\mathcal{A}$ is an algebra on X, $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$ are ideals, $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{I})$ and there exists a $Y \in \mathcal{I}$ such that $\mathcal{I} \cap \mathcal{P}(X \setminus Y) = \mathcal{J} \cap \mathcal{P}(X \setminus Y)$, then $S(\mathcal{A} \setminus \mathcal{I}) = S(\mathcal{A} \setminus \mathcal{J})$.*

Finally we note that the existence of an $\langle \mathcal{I}, \mathcal{J} \rangle$-special set is by no means necessary for the conclusion of Corollary 11.

Example 12. *There exists an algebra $\mathcal{A}$ and an ideal $\mathcal{J} \subsetneq \mathcal{H}(\mathcal{A})$ for which $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) = S(\mathcal{A} \setminus \mathcal{J})$ while there is no $\langle \mathcal{H}(\mathcal{A}), \mathcal{J} \rangle$-special set $Y \in \mathcal{H}(\mathcal{A})$.*

PROOF. In the papers [R] and [NR] the authors investigated the family $\mathcal{D}$ of perfect subsets of $[\omega]^\omega$, where $[\omega]^\omega$ is endowed with the Ellentuck topology; that is, the topology generated by the sets $[x, y] = \{z \in [\omega]^\omega \colon x \subset z \subset y\}$, where $x \in [\omega]^{<\omega}$ and $y \in [\omega]^\omega$. A subset of $[\omega]^\omega$ is called a *chain* if it consists of sets incomparable with respect to inclusion. A chain is called a *Sorgenfrey chain* if its subspace topology is homeomorphic to the Sorgenfrey topology on $(0, 1]$. It is shown in [NR, thm. 3.4] that if $P \in \mathcal{D}$ does not contain a countable perfect set, then P contains a perfect uncountable Sorgenfrey chain.

Let $\mathcal{G}$ be the family of all perfect Sorgenfrey chains and let $\mathcal{A} = S(\mathcal{D})$. By [NR, thm. 3.5] and [R, cor. 1.10], we have $\mathcal{A} = S(\mathcal{D}) = S(\mathcal{G})$ and $\mathcal{J} = S_0(\mathcal{D}) \subsetneq S_0(\mathcal{G}) = \mathcal{H}(\mathcal{A})$. We will show that

(a) $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{J})$, and

(b) $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$, but

(c) there is no $\langle \mathcal{H}(\mathcal{A}), \mathcal{J} \rangle$-special set $Y \in \mathcal{H}(\mathcal{A})$.

To prove (a) observe that $\mathcal{D} \subset S(\mathcal{D})$ since, for any two perfect sets P and Q, at least one of the sets $P \cap Q$, $P \setminus Q$ has a perfect part. Now, from $\mathcal{D} \subset S(\mathcal{D})$ and $\mathcal{D} \cap S_0(\mathcal{D}) = \emptyset$ it follows that $\mathcal{D}$ and $\mathcal{A} \setminus \mathcal{J} = S(\mathcal{D}) \setminus S_0(\mathcal{D})$ are mutually coinitial which, by [BBRW, prop. 1.2], implies (a). The clause (b) results from (a) and Proposition 7.

To prove (c), by way of contradiction assume that there is a $\langle \mathcal{H}(\mathcal{A}), S_0(\mathcal{D}) \rangle$-special set $Y \in \mathcal{H}(\mathcal{A})$. Then $\mathcal{H}(\mathcal{A}) \cap \mathcal{P}([\omega]^\omega \setminus Y) = S_0(\mathcal{D}) \cap \mathcal{P}([\omega]^\omega \setminus Y)$. Since $\mathcal{H}(\mathcal{A}) = S_0(\mathcal{G})$, we have

$$S_0(\mathcal{G}) \cap \mathcal{P}([\omega]^\omega \setminus Y) = S_0(\mathcal{D}) \cap \mathcal{P}([\omega]^\omega \setminus Y). \tag{2}$$

Next observe that

(d) each set from $\mathcal{D} \cap \mathcal{P}([\omega]^\omega \setminus Y)$ contains a set from $\mathcal{G}$.

Indeed, let $D \in \mathcal{D} \cap \mathcal{P}([\omega]^\omega \setminus Y)$. Since $\mathcal{D} \subset S(\mathcal{D}) \setminus S_0(\mathcal{D})$, it follows from $S(\mathcal{D}) = S(\mathcal{G})$ and (2) that

$$D \in (S(\mathcal{D}) \setminus S_0(\mathcal{D})) \cap \mathcal{P}([\omega]^\omega \setminus Y) = (S(\mathcal{G}) \setminus S_0(\mathcal{G})) \cap \mathcal{P}([\omega]^\omega \setminus Y).$$

Hence by [BBRW, prop 1.1(4)], there is a $G \in \mathcal{G}$ such that $G \subset D$ as desired.

Since $\mathcal{G}$ consists of uncountable sets, from (d) we derive that no countable perfect set in $[\omega]^\omega$ is contained in $[\omega]^\omega \setminus Y$. From [NR] it follows that each nonempty open set in $[\omega]^\omega$ contains a set from $\mathcal{G}$. Thus Y, which is in $\mathcal{H}(\mathcal{A}) = S_0(\mathcal{G})$, has the empty interior. Consequently, $[\omega]^\omega \setminus Y$ is dense and so, by [R, thm. 1.5], it contains a countable perfect set Q. However, this contradicts the previous observation. □

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