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BETWEEN ARZELÁ AND WHITNEY CONVERGENCE

Abstract

A stronger form of the Arzelá convergence is defined and it is compared to other types of convergence.

Throughout the article, X will denote a topological space in which no separation axioms are assumed if none are explicitly stated. Thus, just as in [3] and [10], compactness, paracompactness (also countable compactness and countable paracompactness) are presumed without the T_2 axiom and pseudocompact spaces need not be $T_{3\frac{1}{2}}$. For any subset A of the space X its closure will be denoted by $\mathrm{cl}\,(A)$. In a metric space (Y,ρ) the open ball with center at y and radius r will be denoted by $B(y,r)$. Furthermore, $\mathcal{F}(X,Y)$ and $\mathcal{C}(X,Y)$ will denote the classes of all functions and all continuous functions from X to Y, respectively, and $\mathbb{R}^+$ will denote the set of all positive real numbers. This set will be endowed with the natural topology.

Definition 1. [1], [2] A net $\{f_j : j \in J\}$ of functions $f_j : X \longrightarrow Y$ is said to be convergent to a function $f : X \longrightarrow Y$ in the sense of Arzelá (or simply A-convergent) if this net pointwise converges to f and for every positive ε, every j_0 in J there exists a finite subset J_1 of J such that $j \geq j_0$ for $j \in J_1$ and

$$\min\{\varrho(f_j(x), f(x)) : j \in J_1\} < \varepsilon$$

for each x in X.

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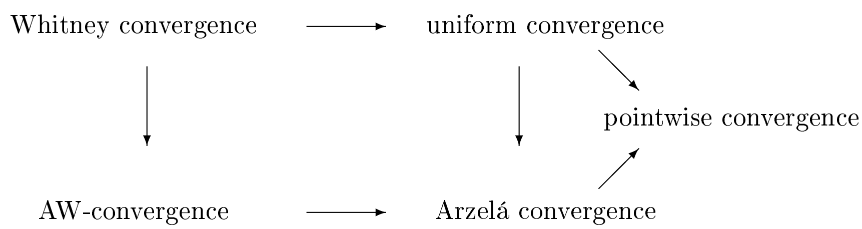
Definition 2. [4], [7], [8], [9] A net $\{f_j : j \in J\}$ of functions $f_j : X \longrightarrow Y$ is said to be convergent to a function $f : X \longrightarrow Y$ in the sense of Whitney if for each φ from $\mathcal{C}(X, \mathbb{R}^+)$ there exists $j_0 \in J$ such that $\varrho\,(f_j(x), f(x)) < \varphi(x)$ for each $x \in X$ and for each $j \in J$ such that $j \geq j_0$.

Definition 3. A net $\{f_j : j \in J\}$ of functions $f_j : X \longrightarrow Y$ is said to be convergent to a function $f : X \longrightarrow Y$ in the sense of Arzelá-Whitney (or simply AW-convergent) if this net pointwise converges to f and for every $\varphi \in \mathcal{C}(X, \mathbb{R}^+)$, every j_0 in J there exists a finite subset J_1 of J such that $j \geq j_0$ for $j \in J_1$ and

$$\min\{\varrho\,(f_j(x), f(x)) : j \in J_1\} < \varphi(x) \quad \text{if} \quad x \in X.$$

We have the following relations between the mentioned types of convergence.



None of the implications in this diagram is reversible. Moreover, AW-convergence and uniform convergence are independent.

Examples.
(1) Let the functions f_n for $n \in \mathbb{N}$ and a function f be given by $f_n(x) = \frac{1}{n}$ and $f(x) = 0$ for each $x \in \mathbb{R}^+$. The sequence $(f_n)_{n=1}^{\infty}$ is uniformly convergent to the function f, but it is not AW-convergent to this function. For instance, taking positive integers n, k, a continuous function φ given by $\varphi(x) = \frac{1}{x}$ and m greater than $n + k$ we have

$$\min\{|f_{n+i}(m) - f(m)| : i \in \{0, 1, \ldots, k\}\} = \frac{1}{n+k} > \varphi(m).$$

(2) Let the sequence $(g_n)_{n=1}^{\infty}$ and a function g be defined in $\mathbb{R}^+$ by $g(x) = 0$ for $x \in \mathbb{R}^+$ and

$$g_n(x) = \begin{cases} 0 & \text{if } x \in (0, n) \cup (n+2, \infty), \\ x - n & \text{if } x \in [n, n+1], \\ -x + n + 2 & \text{if } x \in (n+1, n+2]. \end{cases}$$

It is easy to see that the sequence $(g_n)_{n=1}^{\infty}$ is AW-convergent to the function g, but it is not uniformly convergent.

(3) Let the sequence $(h_n)_{n=1}^{\infty}$ and a function h be defined in $\mathbb{R}^+$ by $h(x) = 0$ for $x \in \mathbb{R}^+$, and

$$h_n(x) = \begin{cases} 0 & \text{if } x \in (0,n) \cup (n, n+2), \\ n & \text{if } x = n, \\ \frac{1}{n} & \text{if } x \in [n+2, \infty). \end{cases}$$

The sequence $(h_n)_{n=1}^{\infty}$ is A-convergent to the function h, but it is neither uniformly convergent nor AW-convergent.

A topological space X is called almost compact ([3]) if each open cover $\mathfrak{U}$ of X has a finite subfamily of sets $U_1, \ldots, U_n$ for which $\mathrm{cl}\left(\cup_{k=1}^{n} U_k\right) = X$. One can easily see that for regular spaces compactness and almost compactness coincide.

Theorem 1. *Let X be an almost compact space. If a net $\{f_j : j \in J\}$ of continuous functions $f_j : X \longrightarrow Y$ is pointwise convergent to a continuous function $f : X \longrightarrow Y$, then this net is AW-convergent to the function f.*

PROOF. Fix $j_0 \in J$ and $\varphi \in \mathcal{C}(X, \mathbb{R}^+)$. For each point $p \in X$ we can choose a neighborhood U_p of p such that $\frac{3}{4}\varphi(p) < \varphi(x)$ for $x \in U_p$. We put $W_p = B\left(f(p), \frac{1}{8} \cdot \varphi(p)\right)$. Thus

$$\mathcal{A} = \left\{U_p \cap f^{-1}(W_p) \cap f_j^{-1}(W_p) : p \in X \wedge j \geq j_0\right\}$$

is an open cover of X. By assumptions, we can select a finite subfamily

$$\left\{U_{p_k} \cap f^{-1}(W_{p_k}) \cap f_{j_k}^{-1}(W_{p_k}) : k \in \{1, \ldots, n\}\right\}$$

such that

$$\mathrm{cl}\left(\bigcup_{k=1}^{n}\left(U_{p_k} \cap f^{-1}(W_{p_k}) \cap f_{j_k}^{-1}(W_{p_k})\right)\right) = X.$$

Let x be in X. Then

$$x \in \mathrm{cl}\left(U_{p_k} \cap f^{-1}(W_{p_k}) \cap f_{j_k}^{-1}(W_{p_k})\right)$$

for some k in $\{1, \ldots, n\}$. Hence

$$\varphi(x) \in \varphi\left(\mathrm{cl}\left(U_{p_k}\right)\right) \subset \mathrm{cl}\left(\varphi\left(U_{p_k}\right)\right) \subset \left[\frac{3}{4} \cdot \varphi\left(p_k\right), \infty\right).$$

Consequently $\frac{3}{4} \cdot \varphi(p_k) \leq \varphi(x)$. Furthermore,

$$f(x) \in \operatorname{cl}(W_{p_k}) = \operatorname{cl}\left(B\left(f(p_k), \frac{1}{8} \cdot \varphi(p_k)\right)\right) \subset B\left(f(p_k), \frac{1}{4} \cdot \varphi(p_k)\right)$$

and analogously, $f_{j_k}(x) \in B\left(f(p_k), \frac{1}{4} \cdot \varphi(p_k)\right)$. Thus we infer that

$$\varrho(f(x), f_{j_k}(x)) < \frac{1}{2} \cdot \varphi(p_k) < \varphi(x).$$

Finally, letting $J_1 = \{j_1, \ldots, j_n\}$ we conclude that the net $\{f_j : j \in J\}$ is AW-convergent. $\square$

Theorem 2. *If X is a paracompact Hausdorff space, then the following conditions are equivalent:*

1. *X is a compact space,*
2. *for each metric space (Y, ρ) AW-convergence and pointwise convergence coincide in the class $\mathcal{C}(X, Y)$,*
3. *AW-convergence and pointwise convergence coincide in $\mathcal{C}(X, [0, 1])$.*

PROOF. The implication (1) $\Longrightarrow$ (2) is a consequence of Theorem 1. The implication (2) $\Longrightarrow$ (3) is evident.

To prove the implication (3) $\Longrightarrow$ (1), suppose that the space X is not compact. There exists an open cover $\mathfrak{U} = \{U_s : s \in S\}$, which has no finite subcover. Since X is a paracompact Hausdorff space, there exists a locally finite closed cover $\mathfrak{V} = \{M_s : s \in S\}$, for which $M_s \subset U_s$ if $s \in S$ (see [6] Lem. 5.1.6). Let $\leq$ be a well order in the set S and α be the order type of $(S, \leq)$. Thus the cover $\mathfrak{V}$ can be taken as a transfinite sequence

$$M_{s_0}, M_{s_1}, \ldots, M_{s_\xi}, \ldots, \qquad \xi < \alpha.$$

Now let

$$D_0 = M_{s_0},\ D_\xi = \bigcup_{\beta \leq \xi} M_{s_\beta},\ E_0 = X \setminus U_{s_0} \text{ and } E_\xi = X \setminus \bigcup_{\beta \leq \xi} U_{s_\beta}$$

when $\xi < \alpha$. Then $\mathfrak{D} = \{D_\xi : \xi < \alpha\}$ is a cover of X and the sets D_ξ and E_ξ are closed and disjoint for each $\xi < \alpha$. Moreover, if $\beta < \xi$, then $D_\beta \subset D_\xi$ and $E_\xi \subset E_\beta$. The space X is normal. Thus for each ξ less than α there exists a continuous function $f_\xi : X \longrightarrow [0, 1]$ such that $f_\xi(D_\xi) = \{1\}$ and $f_\xi(E_\xi) = \{0\}$. It is easy to see that the net $\{f_\xi : \xi < \alpha\}$ is pointwise convergent to the function f defined by $f(x) = 1$ if $x \in X$.

Take a finite sequence $\{f_{\xi_1}, f_{\xi_2}, \dots, f_{\xi_n}\}$, where $\xi_1 \le \xi_2 \le \dots \le \xi_n < \alpha$ and a continuous function φ given by $\varphi(x) = \frac{1}{2}$, $x \in X$. Since $E_{\xi_n} \subset E_{\xi_k}$, if $k \in \{1, 2, \dots, n\}$, then $f_{\xi_k}(x) = 0$ if $x \in E_{\xi_n}$ and $k \le n$. From this we infer that

$$\min\{|f_{\xi_k} - f(x)| : k \le n\} > \varphi(x) \text{ if } x \in E_{\xi_n}.$$

In this way we have proved that the net $\{f_\xi : \xi < \alpha\}$ is not AW-convergent to the function f. □

Theorem 3. *If X is pseudocompact, then for every metric space (Y, ρ) the AW-convergence in the class $\mathcal{F}(X, Y)$ is equivalent to A-convergence.*

PROOF. Let $\{f_j : j \in J\}$ be a net of functions from X into Y which is A-convergent to a function $f : X \longrightarrow Y$ and let φ be a function from the class $\mathcal{C}(X, \mathbb{R}^+)$. From the pseudocompactness of the space X we infer that

$$\inf\{\varphi(x) : x \in X\} = r > 0.$$

It follows from A-convergence, that for any j_0 from J there exists a finite subset J_1 of J such that $j \ge j_0$ for any $j \in J_1$ and

$$\inf\{\rho(f_j(x), f(x)) : j \in J_1\} < \frac{1}{2} \cdot r < \varphi(x)$$

for each $x \in X$. □

In the sequel we will apply the following result.

Lemma 1. [2; Th. 4] *For a topological space X the following conditions are equivalent:*

1. *every sequence $(f_n)_{n=1}^{\infty}$, where $f_n \in \mathcal{C}(X, \mathbb{R})$ which is pointwise convergent to a function from the class $\mathcal{C}(X, \mathbb{R})$ is also A-convergent;*
2. *X is pseudocompact.*

As an immediate consequence of Theorem 3 and Lemma 1 we get the following.

Corollary 1. *For a topological space X the following conditions are equivalent:*

1. *X is pseudocompact;*
2. *every sequence $(f_n)_{n=1}^{\infty}$, where $f_n \in \mathcal{C}(X, \mathbb{R})$ which is pointwise convergent to a continuous function $f : X \longrightarrow \mathbb{R}$ is also AW-convergent to the function f.*

Applying the above corollary and Theorem 1, we obtain this consequence.

Corollary 2. *Every almost compact space is pseudocompact.*

Theorem 4. *If X is a countably paracompact T_4 space, then the following conditions are equivalent:*

1. *X is countably compact;*
2. *for any metric space (Y,ρ), every sequence $(f_n)_{n=1}^{\infty}$ of functions from the class $\mathcal{C}(X,Y)$, which is pointwise convergent to a function f from the class $\mathcal{C}(X,Y)$, is also AW-convergent to the function f.*
3. *every sequence $(f_n)_{n=1}^{\infty}$ of continuous functions, where $f_n : X \longrightarrow [0,1]$, which is pointwise convergent to a continuous function, is also AW-convergent.*

PROOF. First we will prove the implication (1) $\Longrightarrow$ (2). Assume that X is countably compact. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from the class $\mathcal{C}(X,Y)$, which is pointwise convergent to a function f from the same class. For any positive integer n and any function φ from the class $\mathcal{C}(X,\mathbb{R}^+)$ we put

$$V_k = \{x \in X : \rho\left(f_k(x), f(x)\right) < \varphi(x)\}\,.$$

The family $\{V_k : k \geq n\}$ forms an open cover of X. Thus sets

$$V_n, V_{n+1}, \ldots, V_{n+m}$$

can be chosen in such a way that $\bigcup_{i=0}^{m} V_{n+i} = X$, from whence AW-convergence followed.

The implication (2) $\Longrightarrow$ (3) is evident.

Finally, suppose that X is not countably compact. Then there is an open cover $\{U_n : n \in \mathbb{N}\}$ of X which has no finite subcover. Without loss of generality we can assume that $U_n \not\subset U_k$ if $n \neq k$. Since X is a paracompact T_4 space, there exists a locally finite open cover $\{V_n : n \in \mathbb{N}\}$ such that $\mathrm{cl}\,(V_n) \subset U_n$ if $n \in \mathbb{N}$. Now let $D_n = \mathrm{cl}\,(\cup_{i=1}^{n} V_i)$ and $M_n = X \setminus \cup_{i=1}^{n} U_i$ for each positive integer n. Then the sets D_n and M_n are closed and satisfy

$$D_n \cap M_n = \varnothing,\ D_n \subset D_{n+1} \text{ if } n \in \mathbb{N} \text{ and } \cup_{n=1}^{\infty} D_n = X.$$

The normality of the space X implies that for each positive integer n there exists a continuous function $f_n : X \longrightarrow [0,1]$ such that $f_n(D_n) = \{1\}$ and $f_n(M_n) = \{0\}$. Let f be defined by $f(x) = 1$ if $x \in X$. It is not difficult (applying arguments similar to those in the proof of the implication (3) $\Longrightarrow$ (1) in Theorem 2) to prove that the sequence $(f_n)_{n=1}^{\infty}$ is pointwise convergent to f, but it is not AW-convergent to f. $\square$

Definition 4. [5] A sequence $(f_n)_{n=1}^{\infty}$ of functions from $\mathcal{F}(X,Y)$ is said to be locally A-convergent to a function $f : X \longrightarrow Y$ at a point $x_0 \in X$ if $f_n(x_0) \longrightarrow f(x_0)$ and for each positive ε and positive integer m there exist a neighborhood U of x_0 and a positive integer n such that

$$\min\{\rho(f_{m+k}(x), f(x)) : k \in \{0,1,\ldots,n\}\} < \varepsilon$$

for each x in U.

A sequence $(f_n)_{n=1}^{\infty}$ of functions from $\mathcal{F}(X,Y)$ is said to be locally A-convergent to a function $f : X \longrightarrow Y$ if it is A-convergent to f at each point x from the set X.

Evidently, every A-convergent sequence is also locally A-convergent, but the converse is false. For instance, let the functions $f_n : (0,1) \longrightarrow (0,1)$ and a function $f : (0,1) \longrightarrow (0,1)$ be given by $f_n(x) = x^n$ and $f(x) = 0$. Then the sequence $(f_n)_{n=1}^{\infty}$ is locally A-convergent to f but it is not A-convergent.

Using Corollary 1 we obtain the following.

Corollary 3. *Let X be a pseudocompact space and $f \in \mathcal{C}(X,\mathbb{R})$, $f_n \in \mathcal{C}(X,\mathbb{R})$ for any positive integer n. Then the following conditions are equivalent:*

1. *the sequence $(f_n)_{n=1}^{\infty}$ is pointwise convergent to f;*
2. *the sequence $(f_n)_{n=1}^{\infty}$ is locally A-convergent to f;*
3. *the sequence $(f_n)_{n=1}^{\infty}$ is A-convergent to f;*
4. *the sequence $(f_n)_{n=1}^{\infty}$ is AW-convergent to f.*

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