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SOME TYPES OF CONVERGENCE OF SEQUENCES OF REAL VALUED FUNCTIONS

Abstract

Using the notions of uniform equal and uniform discrete convergence for sequences of real valued functions, the classes of functions which are uniform equal limits and uniform discrete limits of sequences of real valued functions belonging to certain class are studied. Also, new types of convergence of sequences of real valued functions, called α-uniform equal, α-strong uniform equal and α-equal are defined and studied. Using α-uniform equal convergence, a characterization of compact metric space is obtained.

1 Introduction

In recent papers [8] and [9], Papanastassiou has defined and studied the notions of uniform equal convergence and uniform discrete convergence for sequences of real valued functions. Using these convergences the author has obtained some results in measure theory. It is observed that uniform discrete convergence is stronger than uniform equal convergence as well as discrete convergence defined in [4]. On the other hand, uniform equal convergence is weaker than uniform convergence and stronger than equal convergence defined by Császár and Laczkovich in [4]. In the present paper we study the properties of classes of functions which are uniform equal limits and uniform discrete limits of sequences of functions belonging to a particular class. We also define and study

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α-uniform equal convergence, α-strong uniform equal convergence and α-equal convergence which are stronger than α- convergence (known as continuous convergence [10]) and obtain applications of our results in metric spaces.

Section 2 contains notation and terminology used in subsequent sections. In Section 3, we study the classes $\Phi^{u.e.}$ and $\Phi^{u.d.}$ consisting respectively of the real-valued functions on a non-empty set X which are uniform equal limits and uniform discrete limits of sequences of functions in a particular class Φ (e.g. Theorems 3.6 and 3.8).

The notion of α-convergence (known as continuous convergence) turned out to be useful for characterizing compactness in metric spaces ([7], Theorem 3.2, p. 129). We recall the definition of α-convergence.

Let X be a metric space and f, f_n, $n \in \mathbb{N}$ be real-valued functions defined on X. Then (f_n) **α-converges** to f (written as $f_n \xrightarrow{\alpha} f$) if for any $x \in X$ and for any sequence (x_n) of points of X converging to x, $(f_n(x_n))$ converges to $f(x)$. It is clear from the definition that this convergence is stronger than pointwise convergence. On the other hand, if the limit function f is continuous, then this convergence is weaker than uniform convergence.

In Section 4, we define a new type of convergence called **α-uniform equal convergence** (α-u.e. for short) which turns out to be stronger than uniform equal convergence as well as α-convergence. We investigate properties of this convergence and obtain a necessary and sufficient condition for a metric space to be compact in terms of it. We also define and study a notion, stronger than α-u.e. convergence, called **α-strong uniform equal convergence** (α-s.u.e. for short).

In Section 5, we introduce the notion of **α-equal convergence** which is weaker than α-uniform equal convergence and stronger than equal as well as α-convergence. We study properties of this convergence. We do the comparative study of all these types of convergences. We end the section with some open problems concerning these convergences.

2 Notation and Terminology

By $\mathbb{N}$, we mean the set of all natural numbers and by $\mathbb{R}$, we mean the set of all real numbers. If Γ is a set, then $|\Gamma|$ denotes the cardinality of Γ. If $x \in \mathbb{R}$, then $[x]$ denotes the integer part of x.

Let X be a non-empty set. By a function on X, we mean a real valued function on X. Let Φ be an arbitrary class of functions defined on X. Then we have the following definitions.

Definition 2.1. A sequence of functions (f_n) in Φ is said to converge **uniformly equally** to a function f in Φ (written as $f_n \xrightarrow{u.e.} f$) if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and a natural number n_0 such that the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}$ is at most n_0, for each $x \in X$ [8].

Definition 2.2. A sequence of functions (f_n) in Φ is said to converge **uniformly discretely** to a function f in Φ (written as $f_n \xrightarrow{u.d.} f$) if there exists a natural number n_0 such that the cardinality of the set $\{n \in \mathbb{N} : |f_n(x) - f(x)| > 0\}$ is at most n_0, for each $x \in X$ (see [9]).

We denote by $\Phi^{u.e.}$, the set of all functions on X which are uniform equal limits of sequences of functions in Φ. Similarly $\Phi^{u.d.}$ denotes the set of all functions which are uniform discrete limits of sequences of functions in Φ.

Note 2.3. One can observe that if $f \in \Phi^{u.e.}$, then for any sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive reals converging to zero, there exists a sequence of functions in Φ which converges uniformly equally to f with witnessing sequence $(\lambda_n)_{n\in\mathbb{N}}$.

Definition 2.4. A sequence of functions (f_n) in Φ is said to **converge equally** to f (written as $f_n \xrightarrow{e.} f$) if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero such that, for each $x \in X$, there exists a natural number $n(x)$ satisfying $|f_n(x) - f(x)| < \varepsilon_n$, for each $n \geq n(x)$. Also, (f_n) is said to converge **discretely** to a function f in Φ (written as $f_n \xrightarrow{d.} f$) if, for every $x \in X$, there exists $n(x) \in \mathbb{N}$ such that $f(x) = f_n(x)$ for all $n \geq n(x)$ [4].

Note 2.5. For a sequence of funtions in Φ, it is clear that we have the implications: uniform convergence implies uniform equal convergence, and uniform equal convergence implies equal convergence. On the other hand uniform discrete convergence implies both discrete and uniform equal convergence. The symbol $f_n \overset{\beta}{\not\to} f$ means that (f_n) does not converge to f in the respective β-convergence.

Example 2.6. The following four examples show that the converse of each of the above implications fail.

(i) Let $f_n(x) = x^n$ for $x \in [0, 1)$. Then the sequence (f_n) converges equally to the zero function on $[0, 1)$ but not uniformly equally. (Refer to Example 4.9)

(ii) Let f_n be the piecewise linear function supported on $[n-1, n+1+\frac{1}{n}]$ and given by

$$f_n(x) = \begin{cases} x + 1 - n & \text{for } x \in [n-1, n] \\ 1 & \text{for } x \in [n, n+\frac{1}{n}] \\ n + 1 + \frac{1}{n} - x & \text{for } x \in [n+\frac{1}{n}, n+1+\frac{1}{n}]. \end{cases}$$

Then the sequence (f_n) of continuous functions satisfies $\|f_n\|_\infty = 1$ for each $n \in \mathbb{N}$, and therefore does not converge uniformly to the zero function. On the other hand, it converges uniformly equally to the zero function. In fact, if $(\varepsilon_n)_{n\in\mathbb{N}}$ is a null sequence of positive reals, then we have

$$|\{n \in \mathbb{N} : f_n(x) \geq \varepsilon_n| \leq 3 \text{ for each } x \in \mathbb{R}.$$

(iii) Let $0 < \delta < 1$ and $f_n(x) = x^n$ for $x \in [0, \delta]$. Then the sequence (f_n) converges uniformly equally to the zero function on $[0, \delta]$ but not uniformly discretely, since the set $\{n \in \mathbb{N} : \delta^n > 0\}$ is unbounded [9].

(iv) Let

$$f_n(x) = \begin{cases} 0 & \text{for } x \in (-\infty, n-1] \\ x - n + 1 & \text{for } x \in [n-1, n] \\ 1 & \text{for } x \in [n, +\infty). \end{cases}$$

Then the sequence (f_n) converges discretely to the zero function on $\mathbb{R}$ but not uniformly discretely [9].

For the function class Φ on X, we have the following definitions [5].

Definition 2.7. (a) Φ is called a **lattice** if Φ contains all constants and $f, g \in \Phi$ implies $\max(f, g) \in \Phi$ and $\min(f, g) \in \Phi$.
(b) Φ is called a **translation lattice** if it is a lattice and $f \in \Phi$, $c \in \mathbb{R}$ implies $f + c \in \Phi$.
(c) Φ is called a **congruence lattice** if it is a translation lattice and $f \in \Phi \Rightarrow -f \in \Phi$.
(d) Φ is called a **weakly affine lattice** if it is a congruence **lattice** and there is a set $C \subset (0, \infty)$ such that C is not bounded and $f \in \Phi$, $c \in C$ implies $cf \in \Phi$.
(e) Φ is called an **affine lattice** if it is a congruence lattice and $f \in \Phi$, $c \in \mathbb{R}$ implies $cf \in \Phi$.
(f) Φ is called a **subtractive lattice** if it is a lattice and $f, g \in \Phi$ implies $(f - g) \in \Phi$.
(g) Φ is called an **ordinary class** if it is a subtractive lattice, $f, g \in \Phi$ implies $f \cdot g \in \Phi$ and $f \in \Phi$, $f(x) \neq 0$, for all $x \in X$ implies $1/f \in \Phi$.

3 On the Classes of Uniform Equal and Uniform Discrete Limits

We first observe the following equivalent condition for the uniform equal convergence.

Theorem 3.1. *Let $f_n, f : X \to \mathbb{R}$, $n \in \mathbb{N}$. Then $f_n \stackrel{u.e.}{\to} f$ if and only if there exists an unbounded sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive integers such that $\rho_n|f_n - f| \stackrel{u.e.}{\to} 0$.*

Proof. Suppose $f_n \stackrel{u.e}{\to} f$. Then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_0, \text{ for each } x \in X.$$

Note that

$$|\{n \in \mathbb{N} : \rho_n|f_n(x) - f(x)| \geq \sqrt{\varepsilon_n}\}| \leq n_0, \text{ for each } x \in X,$$

where $(\rho_n) = \left(\left[\frac{1}{\sqrt{\varepsilon_n}}\right]\right)$, is an unbounded sequence of positive integers and hence $\rho_n|f_n - f| \stackrel{u.e.}{\to} 0$.

Conversely, if $\rho_n|f_n - f| \stackrel{u.e.}{\to} 0$, where (ρ_n) is an unbounded sequence of positive integers, then there exists a sequence (λ_n) of positive reals converging to zero and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : \rho_n|f_n(x) - f(x)| \geq \lambda_n\}| \leq n_0, \text{ for each } x \in X.$$

For $(\theta_n) = \left(\frac{\lambda_n}{\rho_n}\right) \to 0$, we have

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \theta_n\}| \ \leq n_0, \quad \text{for each } x \in X$$

and hence $f_n \stackrel{u.e.}{\to} f$ with witnessing sequence (θ_n). □

The following result describes some properties the class $\Phi^{u.e.}$ must have if the class Φ has the properties.

Theorem 3.2. *Let Φ be a class of functions on X. If Φ is a lattice, a translation latice, a congruence lattice, a weakly affine lattice, an affine lattice or a subtractive lattice, then so is $\Phi^{u.e.}$.*

Proof. Suppose Φ is a lattice. Since Φ contains constant functions, $\Phi^{u.e}$ contains constant functions. By definition it follows that if $f_n \stackrel{u.e.}{\to} f$, then

$|f_n| \overset{u.e.}{\to} |f|$. Moreover, as observed in [8], if $f_n \overset{u.e.}{\to} f$, $g_n \overset{u.e.}{\to} g$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f_n + \beta g_n \overset{u.e.}{\to} \alpha f + \beta g$. Hence if $f, g \in \Phi^{u.e.}$, $f_n \overset{u.e.}{\to} f$ and $g_n \overset{u.e.}{\to} g$, then

$$\left(\frac{f_n + g_n}{2}\right) + \frac{|f_n - g_n|}{2} \overset{u.e.}{\to} \frac{f+g}{2} + \frac{|f-g|}{2} = \max(f,g)$$

which implies that $\max(f,g) \in \Phi^{u.e.}$. Similarly $\min(f,g) \in \Phi^{u.e.}$. Thus $\Phi^{u.e.}$ is a lattice. □

It is easy to observe that if Φ is a translation, a congruence, a weakly affine, an affine or a subtractive lattice, then so is $\Phi^{u.e.}$.

We first observe the following Lemmas.

Lemma 3.3. *Let $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$. If $f_n \overset{u.e.}{\to} 0$, then $f_n^2 \overset{u.e.}{\to} 0$.*

PROOF. If $(\lambda_n)_{n\in\mathbb{N}}$ is witnessing sequence for uniform equal convergence of (f_n) to zero, then (λ_n^2) is witnessing sequence for uniform equal convergence of (f_n^2) to zero. □

Lemma 3.4. *Let $f_n, f : X \to \mathbb{R}$ $n \in \mathbb{N}$. If f is bounded and $f_n \overset{u.e.}{\to} f$. Then $f_n \cdot f \overset{u.e.}{\to} f^2$.*

PROOF. Let M be a positive real number such that $|f(x)| \le M$, for each $x \in X$. Since $f_n \overset{u.e.}{\to} f$, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \ge \varepsilon_n\}| \le n_0, \text{ for each } x \in X.$$

Hence

$$|\{n \in \mathbb{N} : |(f_n \cdot f)(x) - f^2(x)| \ge \varepsilon_n \cdot M\}| \le n_0, \text{ for each } x \in X.$$

Thus $f_n \cdot f \overset{u.e.}{\to} f^2$. □

Using the above two lemmas, we obtain the following result regarding product of uniform equal limits.

Theorem 3.5. *Let $f, g : X \to \mathbb{R}$ be bounded functions and $f_n, g_n : X \to \mathbb{R}$, $n \in \mathbb{N}$ be such that $f_n \overset{u.e.}{\to} f$ and $g_n \overset{u.e.}{\to} g$. Then $f_n \cdot g_n \overset{u.e.}{\to} f \cdot g$.*

PROOF. Since $f_n \overset{u.e.}{\to} f$ and $g_n \overset{u.e.}{\to} g$, $f_n + g_n \overset{u.e.}{\to} f + g$ and $f_n - g_n \overset{u.e.}{\to} f - g$. Now using Lemmas 3.3 and 3.4, we get

$$f_n \cdot g_n = \frac{(f_n + g_n)^2 - (f_n - g_n)^2}{4} \overset{u.e.}{\to} \frac{(f+g)^2 - (f-g)^2}{4} = f \cdot g. \qquad \square$$

Theorem 3.6. *Let Φ be an ordinary class of functions on X. Let $f \in \Phi^{u.e.}$ be bounded and such that $f(x) \neq 0$ for each $x \in X$. If $\frac{1}{f}$ is bounded on X, then $\frac{1}{f} \in \Phi^{u.e.}$.*

Proof. Let λ be such that $f^2(x) > \lambda > 0$ for each $x \in X$. Since $f \in \Phi^{u.e.}$ and f is bounded, by Theorem 3.5, $f^2 \in \Phi^{u.e.}$ and hence by Note 2.3, there exists $f_n \in \Phi$, $n \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \leq n_0, \text{ for each } x \in X.$$

Let $g_n(x) = \max\{f_n(x), \frac{1}{n}\}$, $x \in X$ and $n \in \mathbb{N}$. Then $g_n \in \Phi$ for each $n \in \mathbb{N}$. Note that

$$|\{n \in \mathbb{N} : g_n(x) = f_n(x) \text{ and } |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \leq n_0$$

and

$$|\{n \in \mathbb{N} : g_n(x) = \frac{1}{n} \text{ and } |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \leq n^* + n_0,$$

where $n^* = \left[\frac{1}{\lambda}\right] + 1$.

Using the fact that,

$$\begin{aligned}\{n \in \mathbb{N} : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\} \\ &= \{n \in \mathbb{N} : g_n(x) = f_n(x) \text{ and } |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\} \\ &\cup \{n \in \mathbb{N} : g_n(x) = \frac{1}{n} \text{ and } |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\},\end{aligned}$$

we get,

$$|\{n \in \mathbb{N} : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \leq n_0 + (n_0 + n^*) \equiv n_1, \text{ for each } x \in X.$$

Therefore

$$\begin{aligned}|\{n \in \mathbb{N} : |\frac{1}{g_n(x)} - \frac{1}{f^2(x)}| \geq \frac{1}{n^3} \cdot n \cdot \frac{1}{\lambda}\} \\ &= |\{n \in \mathbb{N} : \frac{|g_n(x) - f^2(x)|}{|g_n(x)||f^2(x)|} \geq \frac{1}{n^3} \cdot n \cdot \frac{1}{\lambda}\}| \\ &\leq |\{n \in \mathbb{N} : |g_n(x) - f^2(x)| \geq \frac{1}{n^3}\}| \leq n_1, \text{ for each } x \in X.\end{aligned}$$

Thus $f^{-2} \in \Phi^{u.e}$ and so $f \cdot f^{-2} = \frac{1}{f} \in \Phi^{u.e.}$. $\square$

Now, we study the properties of class $\Phi^{u.d.}$ of uniform discrete limits of sequence of functions in a certain function class Φ.

The following result follows from definition.

Theorem 3.7. *If Φ is a lattice, a translation lattice, a congruence lattice, a weakly affine lattice, an affine lattice or a subtractive lattice, then so is $\Phi^{u.d.}$*

We have the following result for a function class Φ which is an ordinary class.

Theorem 3.8. *Let Φ be an ordinary class of functions on X. Then $f, g \in \Phi^{u.d.}$ implies $f \cdot g \in \Phi^{u.d.}$. Also, if $f \in \Phi^{u.d.}$ is such that $f(x) \neq 0$ for each $x \in X$ and $\frac{1}{f}$ bounded on X then $\frac{1}{f} \in \Phi^{u.d.}$.*

Proof. Let $f, g \in \Phi^{u.d.}$. Then there exist sequences (f_n) and (g_n) in Φ such that $f_n \overset{u.d.}{\to} f$, $g_n \overset{u.d.}{\to} g$. It follows from definition that $f_n \cdot g_n \overset{u.d.}{\to} f \cdot g$. Let f satisfy the assumptions. Choose λ such that $f^2(x) > \lambda > 0$ for each $x \in X$. We first show that $f^{-2} \in \Phi^{u.d.}$. Let $f_n \in \Phi$, $n \in \mathbb{N}$, be such that $f_n \overset{u.d.}{\to} f$. Since Φ is an ordinary class, $f_n^2 \in \Phi$, $n \in \mathbb{N}$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive reals converging to zero and $g_n = \max\{f_n^2, \varepsilon_n\}$. Then $g_n \in \Phi$. Since $f_n \overset{u.d.}{\to} f$, therefore there exists $n_0 \in \mathbb{N}$ satisfying $|\{n \in \mathbb{N} : f_n(x) \neq f(x)\}| \leq n_0$, for each $x \in X$. Hence, $|\{n \in \mathbb{N} : g_n(x) \neq \max\{f^2(x), \varepsilon_n\}\}| \leq n_0$, for each $x \in X$, which implies

$$|\{n \in \mathbb{N} : \frac{1}{g_n(x)} \neq \frac{1}{\max\{f^2(x), \varepsilon_n\}}\}| \leq n_0, \text{ for each } x \in X. \tag{1}$$

Since $(\varepsilon_n)_{n \in \mathbb{N}}$ converges to zero, there exists $n^* \in \mathbb{N}$ satisfying $\varepsilon_n < \lambda$, for all $n \geq n^*$. Hence,

$$|\{n \in \mathbb{N} : \frac{1}{\max\{f^2(x), \varepsilon_n\}} \neq \frac{1}{f^2(x)}\}| < n^*, \text{ for each } x \in X. \tag{2}$$

Now using equations (1) and (2) $|\{n \in \mathbb{N} : \frac{1}{g_n(x)} \neq \frac{1}{f^2(x)}\}| < n_0 + n^*$, for each $x \in X$. Hence $f^{-2} \in \Phi^{u.d.}$, consequently $f \cdot f^{-2} = f^{-1} \in \Phi^{u.d.}$. $\square$

4 α-Uniform Equal Convergence

The notion of α-convergence (known as continuous convergence) for sequences of real valued functions on a metric space turned out to be useful for characterizing compactness in metric spaces. It is known that if X is a metric space and $f, f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$ are such that $f_n \overset{\alpha}{\to} f$ (i.e. (f_n) α-converges to f), then f is continuous. Also, if X is a compact metric space, then $f_n \overset{\alpha}{\to} f$ implies $f_n \overset{u}{\to} f$, where u denotes uniform convergence (see [10]).

In [7], Holá and Šalát have obtained the following characterization of compact metric spaces.

Theorem 4.1. *A metric space (X, d) is compact if and only if for $f_n, f : X \to \mathbb{R}$, $n \in \mathbb{N}$ $f_n \overset{\alpha}{\to} f \Rightarrow f_n \overset{u}{\to} f$.*

We define here the notion of α-uniform equal convergence.

Definition 4.2. Let (X, d) be a metric space and $f, f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$. Then (f_n) converges α**-uniformly equally** to f (written as $f_n \overset{\alpha\text{-}u.e.}{\to} f$) if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and an $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \varepsilon_n\}| \leq n_0 \text{ for each } x \in X \text{ and } x_n \to x.$$

Remark 4.3. It is clear from this definition that α-$u.e.$ convergence implies both α-convergence and $u.e.$ convergence. However, the following examples show that the converse of each of the above implications fails.

Example 4.4. (i) Let f_n be the characteristic function of the interval $[n, n + \frac{1}{n}]$, $n \in \mathbb{N}$. Then $f_n \overset{u.e.}{\to} f \equiv 0$. For if $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive reals converging to zero then $|\{n \in \mathbb{N} : f_n(x) \geq \varepsilon_n\}| \leq 1$, for each $x \in \mathbb{R}$. Also $f_n \overset{\alpha}{\to} f$. For if $x_0 \in \mathbb{R}$ then there exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$ we have $x_0 < n$ and this implies $f_n(x_0) = 0$, for all $n \geq n^*$. If $x_n \to x_0$, then given $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n_0(\varepsilon)$ we have $x_n \in (x_0 - \varepsilon, x_0 + \varepsilon)$, but then $f_n(x_n) = 0$, for all $n \geq \max\{n^*, n_0(\varepsilon)\}$. Therefore $f_n(x_n) \to f(x_0) = 0$. Hence $f_n \overset{\alpha}{\to} f$.

Now we can observe that if

$$x_n = \begin{cases} n + \frac{1}{2n} & \text{if } n \leq m \\ x_0 - \frac{1}{n} & \text{if } n > m \end{cases}$$

where $m \in \mathbb{N}$ is fixed and $\varepsilon_n < 1$ for all $n \in \mathbb{N}$, then $|\{n \in \mathbb{N} : |f_n(x_n) - f(x_0)| \geq \varepsilon_n\}| = m$. Hence (f_n) does not converges α-uniformly equally to the zero function.

(ii) Let (f_n) be the sequence in example 2.6. (ii). Then as in the previous example, we see that (f_n) α- converges to the zero function but not α-uniformly equally.

In the above examples, in fact $f_n \overset{u.d.}{\to} 0$. Therefore $u.d.$-convergence need not imply α-$u.e.$-convergence. Moreover, the following example shows that α-$u.e.$ convergence also need not imply $u.d.$-convergence.

Example 4.5. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{n}$, $n \in \mathbb{N}$ and $f \equiv 0$ on $\mathbb{R}$. Then note that $f_n \overset{\alpha\text{-}u.e.}{\to} f$ but $f_n \overset{u.d.}{\not\to} f$.

Remark 4.6. (i) Let (X, d) be a metric space and $f_n, f : X \to \mathbb{R}$, $n = 1, 2, \ldots$ such that $f_n \overset{\alpha\text{-}u.e.}{\to} f$. Then $f_n \overset{\alpha}{\to} f$ and hence f is continuous even if the f_n are not (see [10]). Thus α-$u.e.$ convergence implies that the limit function is continuous.

(ii) In general, uniform convergence need not imply α-uniform equal convergence. For example if f is a discontinuous function from X to $\mathbb{R}$ and $f_n = f$, for all $n \in \mathbb{N}$, then $f_n \overset{u}{\to} f$ but since f is discontinuous, f_n does not converge α-uniformly equally to f. However, we have the following result.

Theorem 4.7. *Let X be a metric space and $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$. If the sequence (f_n) converges uniformly to the zero function, then the sequence (f_n) converges α-uniformly equally to the zero function.*

PROOF. Since $f_n \overset{u}{\to} 0$, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero and $n_0 \in \mathbb{N}$ such that $|f_n(x)| < \varepsilon_n$, for all $n \geq n_0$ and for each $x \in X$. This gives $|\{n \in \mathbb{N} : |f_n(x_n)| \geq \varepsilon_n\}| \leq n_0$ for every converging sequence (x_n) in X. Hence $f_n \overset{\alpha\text{-}u.e.}{\to} 0$. □

In the converse direction, we have the following result.

Theorem 4.8. *Let (X, d) be a compact metric space and $f_n, f : X \to \mathbb{R}$, $n \in \mathbb{N}$. Then $f_n \overset{\alpha\text{-}u.e.}{\to} f \Rightarrow f_n \overset{u}{\to} f$.*

PROOF. It follows from the fact that $f_n \overset{\alpha\text{-}u.e.}{\to} f \Rightarrow f_n \overset{\alpha}{\to} f \Rightarrow f_n \overset{u}{\to} f$, as X is a compact metric space (see [10]). □

The following example shows that α-convergence need not imply uniform equal convergence.

Example 4.9. Let $f_n : (0, 1) \to \mathbb{R}$ be defined by $f_n(x) = x^n$, $n \in \mathbb{N}$ and $f \equiv 0$. Then $f_n \overset{\alpha}{\to} f$. Let $0 < \delta < 1$ and $x_n \in (0, 1)$ be such that $x_n \to \delta$. If $\delta < \vartheta < 1$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $x_n < \vartheta$. But then $f_n(x_n) = x_n^n < \vartheta^n$. So $f_n(x_n) \to 0 = f(\delta)$, since $\vartheta^n \to 0$. Hence $f_n \overset{\alpha}{\to} 0$. However, $f_n \overset{u.e.}{\not\to} f$. For if $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence of positive reals converging to zero and $0 < \varepsilon < 1$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\varepsilon_n < \varepsilon$. Consequently we have

$$\{n \in \mathbb{N} : x^n \geq \varepsilon\} \cap [n_0, +\infty) \subset \{n \in \mathbb{N} : x^n \geq \varepsilon_n\} \cap [n_0, +\infty).$$

But the function $n_\varepsilon(x) = |\{n \in \mathbb{N} : x^n \geq \varepsilon\}|$, $x \in (0,1)$ is unbounded and therefore the function $n(x) = |\{n \in \mathbb{N} : x^n \geq \varepsilon_n\}|$, $x \in (0,1)$ is unbounded. Hence $f_n \overset{u.e.}{\not\to} f$ (see also [9]).

Note 4.10. **(i)** The previous example also shows that α-convergence need not imply α-uniform equal convergence. In addition we have that the sequence (f_n) converges equally to 0 on $(0,1)$, since $(0,1) = \cup_{k=2}^{\infty}[\frac{1}{k}, 1-\frac{1}{k}]$ and (f_n) converges uniformly to 0 on $[\frac{1}{k}, 1-\frac{1}{k}]$ for every $k \in \mathbb{N} - \{1\}$. So we have here a simple example which distinguishes α-convergence from α-uniformly equally convergence, and at the same time the equal convergence from uniformly equal convergence.

(ii) Examples in Remark 4.6 (ii) shows that uniform equal convergence need not imply α-convergence (since f being discontinuous, $f_n \overset{\alpha}{\not\to} f$). Also, the following example shows the same.

Example 4.11. Let

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{for } x \in [0, \frac{1}{n}] \\ 0 & \text{for } x \in (\frac{1}{n}, 1) \\ 1 & \text{for } x = 1. n \in \mathbb{N} \end{cases}$$

and $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} 1 & \text{for } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $f_n \xrightarrow{u} f$ and hence $f_n \xrightarrow{u.e.} f$ but f being discontinuous $f_n \overset{\alpha}{\not\to} f$.

We now obtain a characterization of compact metric spaces using α-uniform equal convergence.

Theorem 4.12. *A metric space (X, d) is compact if and only if the α-convergence of a sequence (f_n) of real valued functions defined on X to the zero function implies the α-uniform equal convergence of the sequence (f_n) to the zero function.*

Proof. If X is compact metric space, then by Theorem 4.1, $f_n \xrightarrow{\alpha} 0 \Rightarrow f_n \xrightarrow{u} 0$ and hence by Theorem 4.7 $f_n \xrightarrow{a\text{-}u.e.} 0$. Conversely, suppose (X, d) is not a compact metric space. We first recall the construction of maps f_p^* in Theorem 3.1 in [7]. Since X is not compact, there exists a sequence (x_k)

of distinct points of X such that there exists no convergent subsequence of (x_k). Since every point of the set $\{x_1, \ldots x_n, \ldots\}$ is an isolated point of the set $\{x_1, x_2, \ldots, x_n, \ldots\}$, there exist $\delta_k > 0$, $k = 1, 2, \ldots$ such that $\delta_k \to 0$ as $k \to \infty$ and the closed balls $B[x_k, \delta_k] = \{x \in X | d(x, x_k) \leq \delta_k\}$, $k = 1, 2, \ldots$ are pairwise disjoint. Then $H = \cup_{k=1}^{\infty} B[x_k, \delta_k]$ is a closed set. Define a sequence (f_p) of real valued functions on the set $\{x_1, x_2, \ldots, x_n, \ldots\}$ by $f_1(x_n) = 0$, $n = 1, 2, \ldots$ and for $p > 1$, $f_p(x_m) = (1 - 1/m)^{p-1}$ if $1 \leq m \leq p$ and $f_p(x_{p+j}) = f_p(x_p)$ for $j = 1, 2, \ldots$. Define for $p \in \mathbb{N}$, $f_p^*(x) = 0$ if $x \notin H$ and $f_p^*(x) = f_p(x_j) \cdot (\delta_j - d(x, x_j))/\delta_j$, if $x \in B[x_j, \delta_j]$ $(j = 1, 2, \ldots)$. Then as proved in [7], $f_p^* \xrightarrow{\alpha} 0$. However, the fact that $f_p^*(x_p) = (1 - \frac{1}{p})^{p-1}$ is decreasing and converges to e^{-1}, where e is Euler number, implies that (f_p^*) does not converge uniformly equally to the zero function. For if $(\varepsilon_n)_{n\in\mathbb{N}}$ is a null sequence of positive reals, then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ we have $\varepsilon_n < e^{-1}$. Let $\varepsilon \in (\varepsilon_{n_0}, e^{-1})$. Then for all $p \in \mathbb{N}$,

$$|\{n \in \mathbb{N} : f_n^*(x_p) \geq \varepsilon\}| \geq p - 2.$$

Also we have,

$$\{n \in \mathbb{N}, n \geq n_0 : f_n^*(x_p) \geq \varepsilon\} \subset \{n \in \mathbb{N}, n \geq n_0 : f_n^*(x_p) \geq \varepsilon_n\}.$$

Therefore for all $p \in \mathbb{N}$

$$|\{n \in \mathbb{N}, n \geq n_0 : f_n^*(x_p) \geq \varepsilon_n\}| \geq p - 2 - n_0.$$

Hence $f_p^* \overset{u.e.}{\not\to} 0$. Now, since α-$u.e.$ convergence implies $u.e.$ convergence, we have that the sequence (f_p^*) does not converges α-uniformly equally to the zero function. □

The following notion was introduced by A. Denjoy [6]. The series $\sum_{n=0}^{\infty} f_n$ of real-valued function converges pseudo-normally on a set X, if and only if there exists a convergent series $\sum_{n=0}^{\infty} \varepsilon_n$ of positive reals such that for every $x \in X$ there exists an index k_x such that $|f_k(x)| < \varepsilon_k$ for every $k > k_x$. In [3], the authors define a sequence (f_n), of real valued functions, to converge pseudo-normally to a real valued function f if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals with $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$ such that for each $x \in X$, there exists $n_0(x) \in \mathbb{N}$ satisfying $|f_n(x) - f(x)| \leq \varepsilon_n$ for all $n \geq n_0(x)$. The second-named author of the present paper introduced in [8] a stronger notion of convergence, called strong uniform equal, which is defined as follows.

Let $f_n, f : X \to \mathbb{R}$, $n = 1, 2, \ldots$. Then (f_n) converges to f **strongly uniformly equally** (written as $f_n \xrightarrow{s.u.e.} f$), if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals with $\sum_1^{\infty} \varepsilon_n < \infty$ and $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon_n\}| \leq n_0, \text{ for each } x \in X.$$

From the definition it follows that strong uniform equal convergence is stronger than uniform equal convergence as well as pseudo-normal convergence. However, the following examples show that they are not the same.

Example 4.13. **(i)** Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ be defined by $f_n(x) = \frac{1}{n}$, $x \in \mathbb{R}$. Then $f_n \overset{u.e.}{\to} 0$ but $f_n \overset{s.u.e.}{\not\to} 0$.

(ii) Let f_n be the characteristic function of the interval $[n, \infty)$, $n \in \mathbb{N}$. Then (f_n) converges pseudo-normally to $f \equiv 0$ on $\mathbb{R}$ but not strongly uniformly equally.

Here we define the notion of α-strong uniform equal convergence.

Definition 4.14. We say that a sequence of real valued functions (f_n) converges α-**strongly uniformly equally** to a function f (written as $f_n \overset{\alpha\text{-}s.u.e.}{\to} f$), if there exists a convergent series $\sum_{n=0}^{\infty} \varepsilon_n$ of positive reals and an index $n_0 \in \mathbb{N}$ such that

$$|\{n \in \mathbb{N} : |f_n(x_n) - f(x)| \geq \varepsilon_n\}| \leq n_0, \text{ for every } x \in X \text{ and } x_n \to x.$$

Remark 4.15. It is clear that the notion of α-strong uniform equal convergence is stronger than α-uniform equal convergence and strong uniform equal convergence both.

The following examples distinguish these types of convergence.

Example 4.16. **(i)** Let $f_n(x) = \frac{1}{n}$, $x \in \mathbb{R}$. Then $f_n \overset{\alpha\text{-}u.e.}{\to} f \equiv 0$ but $f_n \overset{\alpha\text{-}s.u.e.}{\not\to} f \equiv 0$ **(ii)** Let f_n be the characteristic function on $[n, n + \frac{1}{n}]$, $n \in \mathbb{N}$ and $f \equiv 0$ on $\mathbb{R}$. Then $f_n \overset{s.u.e.}{\to} 0$ but $f_n \overset{\alpha\text{-}s.u.e.}{\not\to} 0$.

We recall now the definition of strong uniform convergence defined in [8].

Let $f, f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$. Then (f_n) is said to converge **strongly uniformly** to f (written as $f_n \overset{s.u.}{\to} f$) on X, if and only if, there exists a sequence of positive reals (ε_n) with $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$ and an index $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon_n$, for all $n \geq n_0$ and for every $x \in X$.

Note that Example 4.16 **(i)** shows that $s.u.$ convergence is stronger than uniform convergence.

It is easy to observe the following result.

Theorem 4.17. *Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$ and $f : X \to \mathbb{R}$ be the zero function such that $f_n \overset{s.u.}{\to} f$. Then $f_n \overset{\alpha\text{-}s.u.e.}{\to} f$.*

5 α-Equal Convergence

We define the following convergence which is intermediate between α-$u.e.$ convergence and α convergence.

Definition 5.1. Let f, f_n, $n \in \mathbb{N}$ be functions on X. We say that the sequence (f_n) **converges α-equally** to f (written as $f_n \stackrel{\alpha\text{-}e.}{\to} f$) if there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals converging to zero such that for each $x \in X$ and sequence (x_n) of points of X such that $x_n \to x$, there exists a natural number $n_0 \equiv n_0(x, (x_n))$ satisfying $|f_n(x_n) - f(x)| < \varepsilon_n$ for all $n \geq n_0$.

Remark 5.2. It follows from the definition that:

(i) α-equal convergence implies equal convergence.

(ii) If $f_n \stackrel{\alpha\text{-}e.}{\to} f$, then $f_n \stackrel{\alpha}{\to} f$.

(iii) If $f_n \stackrel{\alpha\text{-}u.e.}{\longrightarrow} f$, then $f_n \stackrel{\alpha\text{-}e.}{\to} f$.

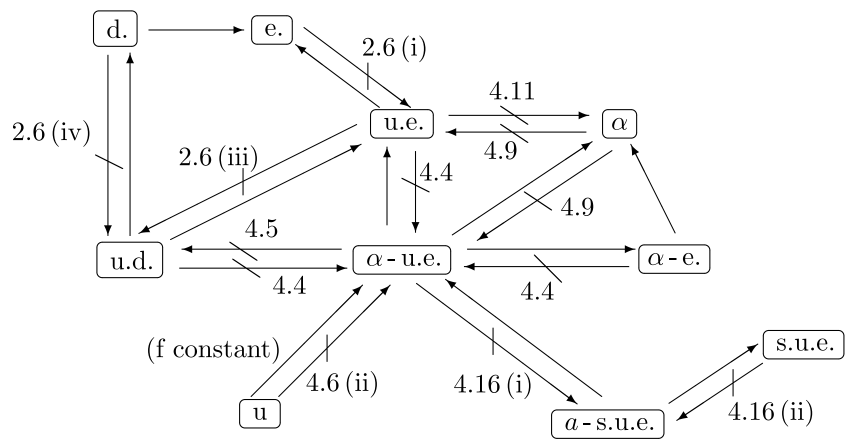
(iv) If $f_n \stackrel{\alpha\text{-}e.}{\to} f$, then f is continuous.

Example 5.3. We give some examples which distinguish all these types of convergence.

(i) Let $X = [0, 1]$ and $f_n(x) = x^n$, $n \in \mathbb{N}$ and f be defined by $f(x) = 0$, if $x \in [0, 1)$, $f(x) = 1$ if $x = 1$. Then $f_n \stackrel{e}{\to} f$ but (f_n) does not converge α-equally to f since f is not continuous.

(ii) In examples 4.4, $f_n \stackrel{\alpha\text{-}e.}{\to} f$ but $f_n \stackrel{\alpha\text{-}u.e.}{\not\to} f$.

Note 5.4. The following figure shows the relation between all the convergences discussed here. The numbers at the arrows refer to the examples showing that those implications are not true.



Note 5.5. We recall that a convergence structure is said to be an **$\mathcal{L}$-space** if every subsequence of a convergent sequence converges to the same limit and if every constant sequence converges to its common value (compare [1]). According to Remark 1.2 (ii) in [8] , the space ${}^{\mathbb{R}}X$ of all real valued functions defined on X with uniform equal convergence is an $\mathcal{L}$-space. Also, from the definitions, it follows that the space ${}^{\mathbb{R}}X$ with each of uniform discrete convergence, α-uniform equal convergence, α-equal convergence and α-convergence is an $\mathcal{L}$-space.

Open Problems (i) Let X be a topological space. We call X an α-**space** if whenever $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, continuous, are such that $f_n \overset{p.w.}{\to} 0$ then $f_n \overset{\alpha}{\to} 0$ and we call X a **weak** α-**space** if whenever $f_n \overset{p.w.}{\to} 0$, where p.w. denotes pointwise convergence, then there exists a subsequence say, (f_{n_k}) of (f_n) such that $f_{n_k} \overset{\alpha}{\to} 0$. Such kind of study has been done in [2]. Clearly an α-space is a weak α-space and $\mathbb{R}$ with cocountable topology τ_c is an α-space. (For if $f_n \overset{p.w.}{\to} 0$ and $x_n \to x_0$, $x_0 \in X$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ $x_n = x_0$, and therefore $f_n(x_n) = f_n(x_0) \to 0 = f(x_0)$.) It would be interesting to study α-spaces, and in a similar way, spaces in which $f_n \overset{p.w.}{\to} 0$ implies $f_n \overset{\alpha\text{-}e.}{\to} 0$ or $f_n \overset{p.w.}{\to} 0$ implies $f_n \overset{\alpha\text{-}u.e.}{\to} 0$.

(ii) Characterize the topological spaces in which $f_n \overset{\alpha}{\to} 0$ implies $f_n \overset{\alpha\text{-}e}{\to} 0$, where f_n, $n \in \mathbb{N}$ are continuous. In a similar way, characterize spaces in which $f_n \overset{\alpha}{\to} 0$ implies $f_n \overset{\alpha\text{-}u.e.}{\to} 0$. We recall that compact metric spaces are examples of such spaces, as on a compact metric space $f_n \overset{\alpha}{\to} 0$ implies $f_n \overset{\alpha\text{-}u.e.}{\to} 0$.

(iii) Find a characterization of α-convergence, α-e. and α-$u.e$. convergence analogous to the characterization of strong equal convergence obtained in [8].

(iv) Finally, (in the usual notations) find the conditions under which

(a) $\Phi^{\alpha\text{-}u.e.} = \Phi^{u.e.}$,

(b) $\Phi^{\alpha\text{-}u.e.} = \Phi^{\alpha}$,

(c) $\Phi^{\alpha\text{-}u.e.} = \Phi^{\alpha\text{-}s.u.e.}$,

(d) $\Phi^{u.d.} = \Phi^{u.e.}$,

(e) $\Phi^{u.e.} = \Phi^{e.}$

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