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ON THE UNIQUENESS PROBLEM FOR FOURIER SERIES

Abstract

In this paper the relation between N. Wiener's theorem about characterization of irregular points for the Dirichlet problem and the uniqueness problem for Fourier series is established.

1 Introduction

Let us denote by $S_n(x,f),\ n=1,2,\dots$ the partial sums of the Fourier series of a function $f(x)\in L_1(-\pi,\pi)$, i.e.

$$S_n(x,f)=\sum_{k=-n}^{n}a_ke^{ikx}.$$

Definition 1. The subset $E\subset[-\pi,\pi]$ is said to be *a set of uniqueness* for a class of functions X, $\quad X\subset L_1(-\pi,\pi)$, if for each $f\in X$, whenever we have the condition

$$\lim_{n\to\infty}S_n(x,f)=0,\quad x\notin E, \tag{1}$$

it follows that $f(x)=0$ a.e. on $[-\pi,\pi]$.

Complete surveys of the classical results about the uniqueness problem can be found in [1] or [2]. For the convenience of the reader, we present some classical results, which qualitatively are close to the problem discussed here.

A. Zygmund [5] proved, that if $\epsilon_n>0,\quad n=0,1,2,\dots$ is a sequence for which

$$\lim_{n\to\infty}\epsilon_n=0,$$

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then there is a subset E, with measure arbitrary close to 2π such that E is a set of uniqueness for the class of trigonometric series (not necessary Fourier series) satisfying the conditions

$$|a_n| \leq \epsilon_{|n|}, \quad n = 0, \pm 1, \pm 2, \dots .$$

In 1973, this result was improved by J.-p. Kahane and Y. Katznelson [6] who showed that there is a set of uniqueness, E with $m(E) = 2\pi$.

The analog of A. Zygmund's original result, when the coefficient condition is replaced by

$$\sum_{n=-\infty}^{\infty} |a_n|^p < \infty,$$

$(1 < p < 2)$ was proved by Y. Katznelson in 1964, see [7]. And in 1975, L. Michele and p. Soardi [3] improved this last result by showing the existence of a set of uniqueness, E with $m(E) = 2\pi$.

For certain classes of functions there is a simple characterization of sets of uniqueness. For example, if $X = C[-\pi, \pi]$, the class of continuous functions on $[-\pi, \pi]$, then a set E is a set of uniqueness if and only if $[-\pi, \pi] \setminus E$ is dense in the interval $[-\pi, \pi]$. Sets of uniqueness for the class $X = L_2(-\pi, \pi)$ can also be characterized in a simple way; in this case, the sets of uniqueness are exactly the sets of measure zero.

For classes X, containing discontinuous functions, sets of uniqueness E have the property that their complements must be "spread" in some sense over the entire interval. In this paper we attempt to answer the following question:

> *In what sense must the subset $[-\pi, \pi] \setminus E$ be "spread" over the interval $[-\pi, \pi]$ in order that E is a set of uniqueness for a given class of functions, X?.*

To accomplish this, we relate our question with well known results of N. Wiener about the characterization of irregular points, see [4]. In Wiener's work, some classes of functions appear naturally, and although those classes are natural for potential theory, they are different from those which one usually investigates when investigating the uniqueness problem for trigonometric series.

2 Auxiliary Results

For completeness, we first give some well known definitions and results from Potential Theory, see [4, p. 169].

Definition 2. Let $0 < \sigma < 1$. For an arbitrary Borel set E let the $\sigma-$capacity of the set E be denoted by

$$C_\sigma(E) = \left(\inf_{\mu \prec E} \int_E \int_E \frac{d\mu(x) d\mu(y)}{|x-y|^\sigma} \right)^{-1},$$

where $\mu \prec E$ means that $d\mu$ is a probability measure with support in E.

Definition 3. Let $0 < \sigma < 1$. If $d\mu$ is an arbitrary nonnegative measure, define the $\alpha-$potential of $d\mu$ as

$$U_\alpha^\mu(x) = \int \frac{d\mu(y)}{|x-y|^{1-\alpha}}.$$

The following definition is classical in potential theory, see [4, p. 376].

Definition 4. The subset E is called $\alpha-$thin at the point x_0 if there is a nonnegative measure $d\mu$ for which

$$U_\alpha^\mu(x_0) < \liminf_{E \setminus \{x_0\} \ni x \to x_0} U_\alpha^\mu(x).$$

The following is a well known generalization of N. Wiener's Theorem, see [4, p. 353].

Theorem (N. Wiener) 1. *Let $0 < \alpha < 1$. Then the subset E is $\alpha-$thin at the point x_0 if and only if*

$$\sum_{n=1}^{\infty} 2^{n(1-\alpha)} C_{1-\alpha}(E_n(x_0)) < \infty,$$

where $E_n(x_0) = \{x \in E; \quad 2^{-1-n} < |x - x_0| < 2^{-n}\}$.

3 The Uniqueness for Fourier Series

We first prove the following lemma.

Lemma 1. *Let $d\mu$ be a measure on $[-\pi, \pi]$ (not necessary positive). Suppose that for almost every point $x_0 \in [-\pi, \pi]$ the subset $[-\pi, \pi] \setminus E$ is $\alpha-$thin at the point x_0. If $U_\alpha^\mu(x) = 0$ for every $x \notin E$, then $U_\alpha^\mu(x) = 0$ for every $x \in [-\pi, \pi]$.*

Proof. By Jordan's Theorem we put $d\mu = d\mu_+ - d\mu_-$, where $d\mu_\pm$ are nonnegative measures and we set $d|\mu| = d\mu_+ + d\mu_-$.

Then it follows directly from the definition that $F = \left\{x : U_\alpha^{|\mu|}(x) = +\infty\right\}$ has zero $1-\alpha$-capacity, i.e. $C_{1-\alpha}(F) = 0$. Thus, we have

$$U_\alpha^{\mu_+}(x) = U_\alpha^{\mu_-}(x), \text{ whenever } x \in [-\pi, \pi] \setminus E \cup F.$$

Now we want to prove that this equality is true at each point $x_0 \notin F$ where the set $[-\pi, \pi] \setminus E$ is α-thin. Assume that $U_\alpha^{\mu_+}(x_0) < U_\alpha^{\mu_-}(x_0)$ and fix $0 < A < \infty$ such that $U_\alpha^{\mu_+}(x_0) < A < U_\alpha^{\mu_-}(x_0)$. Since the function $U_\alpha^{\mu_-}(x)$ is continuous from below, it follows that $U_\alpha^{\mu_-}(x) > A$ in some neighborhood of x_0.

Since F has zero $1-\alpha$-capacity, the set $[-\pi, \pi] \setminus E \cup F$ is α-thin at the point x_0, and consequently, by N. Wiener's Theorem we have

$$U_\alpha^{\mu}(x_0) \geq \liminf_{E \cup F \setminus \{x_0\} \ni x \to x_0} U_\alpha^{\mu}(x).$$

Thus,

$$A \leq \liminf_{E \cup F \setminus \{x_0\} \ni x \to x_0} U_\alpha^{\mu_-}(x) = \liminf_{E \cup F \setminus \{x_0\} \ni x \to x_0} U_\alpha^{\mu_+}(x) \leq U_\alpha^{\mu_+}(x_0) < A.$$

This is contradiction. The case where $U_\alpha^{\mu_+}(x_0) > U_\alpha^{\mu_-}(x_0)$ is handled in a similar manner. But then, since both potentials $U_\alpha^{\mu_+}(x)$ and $U_\alpha^{\mu_-}(x)$ are continuous from below it follows that

$$U_\alpha^{\mu_+}(x) \equiv U_\alpha^{\mu_-}(x).$$

This completes the proof. $\square$

The following Theorem is the main result of this paper.

Theorem 1. *Let $0 < \alpha < 1$ and suppose*

$$\int_{-\pi}^{\pi} |f(x)|\, dx + \int_{-\pi}^{\pi} \int_0^{\pi} \frac{|f(x+t) - f(x)|}{t^{1+\alpha}}\, dx\, dt < \infty. \tag{1}$$

Let E be a subset for which

$$\sum_{n=1}^{\infty} 2^{n(1-\alpha)} C_{1-\alpha}(E_n(x_0)) = \infty,$$

where $E_n(x_0) = \{x \notin E;\quad 2^{-1-n} < |x - x_0| < 2^{-n}\}$ for almost all points $x_0 \in [-\pi, \pi]$. Suppose further that for each point $x \notin E$, $\lim_{n\to\infty} S_n(x, f) = 0$. Then, $f(x) = 0$ almost everywhere on $[-\pi, \pi]$.

Proof. It is easy to see that there is a function $f_0(x), -\infty < x < \infty$, with bounded support, which coincides with $f(x)$ for $-\pi < x < \pi$ and for which

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{|f_0(x+t) - f_0(x)|}{t^{1+\alpha}} \; dx \;\; dt < \infty.$$

If $0 < \delta < 1$, define

$$f_\delta(x) = \frac{1}{\delta} \int_{-\delta}^{\delta} \left(1 - \frac{|t|}{\delta}\right) f_0(x+t) \; dt.$$

If $0 < \delta$, then $f_\delta(x)$ is continuously differentiable and

$$\lim_{\delta \to 0^+} \int_{-\infty}^{\infty} |f_\delta(x) - f_0(x)| \; dx = 0.$$

For $0 < \delta < 1$, define the auxiliary functions

$$g_\delta(x) = \frac{\alpha}{2\pi} tan \left(\frac{\pi\alpha}{2}\right) \int_{-\infty}^{\infty} f_\delta(x) - f_\delta(t)|x-t|^{1+\alpha} \; dt, \quad -\infty < x < \infty.$$

It is known, see [9, p. 173], that

$$f_\delta(x) = \int_{-\infty}^{\infty} g_\delta(t)|x-t|^{1-\alpha} \; dt.$$

Thus, we have

$$\int_{-\infty}^{\infty} |g_\delta(x)| \; dx \leq \frac{\alpha}{2\pi} tan \left(\frac{\pi\alpha}{2}\right) \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{|f_\delta(x+t) - f_\delta(x)|}{t^{1+\alpha}} \; dx \; dt.$$

It follows from [2, p. 43–44] that there is a sequence $\{\delta_n\}$ tending to zero such that the measures $g_{\delta_n}(x) \; dx$ converge to a finite measure $d\mu(x)$ in the weak* topology.

Now consider $U_\alpha^\mu(x)$. For an arbitrary integer $n = 0, \pm 1, \ldots$ we have

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \; dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_0(x) e^{-inx} \; dx$$
$$= \lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{\delta_n}(x) e^{-inx} \; dx = \lim_{n \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} g_{\delta_n}(t) \left(\int_{-\pi}^{\pi} \frac{e^{-inx}}{|x-t|^{1-\alpha}} \; dx \right) dt$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} U_\alpha^\mu(x) e^{-inx} \; dx.$$

So, the functions $f(x), -\pi < x < \pi$, and $U^{\mu}_{\alpha}(x), -\pi < x < \pi$, have the same Fourier coefficients and consequently $U^{\mu}_{\alpha}(x) = f(x)$ almost everywhere on $[-\pi, \pi]$.

Finally, there is a number $0 < M < \infty$ such that for $0 < \alpha < 1$ the following inequality holds, see [1, p. 306],

$$\left| \sum_{n=1}^{m} \frac{\cos(nx)}{n^{\alpha}} \right| \leq \frac{M}{|x|^{1-\alpha}}, \quad m = 1, 2, \ldots .$$

It now follows from the Lebesgue Dominated Convergence Theorem, that at each point $x \notin E$, where the integral is defined, the function $U^{\mu}_{\alpha}(x)$ is absolutely continuous Hence, for $x \notin E$, we have

$$\lim_{n\to\infty} S_n(x, f) = \lim_{n\to\infty} S_n(x, U^{\mu}_{\alpha}) = U^{\mu}_{\alpha}(x).$$

It follows from Lemma 1 that $f(x) = 0$ a.e. on $[-\pi, \pi]$. This completes the proof of Theorem 1. □.

Remark 1. *For each $\sigma > 1 - \alpha$ there is a set E which satisfies the conditions of Lemma 1 and for which $C_{\sigma}(E) = 0$.*

PROOF. Actually, for arbitrary numbers $0 < t$ and $a \in [-\pi, \pi]$ we have $C_{1-\alpha}(E_t(a)) = t^{1-\alpha} C_{1-\alpha}(E)$, where $E_t(a) = \{a + tx : x \in E\}$. It was shown in [8, p. 38], that there is a set F for which $C_{1-\alpha}(F) > 0$, and $C_{\sigma}(F) = 0$. Then the set

$$E = \bigcup_{n=1}^{\infty} \left(\bigcup_{k=1}^{2^n} F_{2^{-n}}(k2^{-n}) \right)$$

satisfies the hypothesis Theorem 1, so that $C_{\sigma}(E) = 0$. □

Remark 2. *There is an extensive family of subsets E, with $0 < m(E) < 2\pi$ that are not sets of uniqueness for our classes. To see this, let $0 < \alpha < 1$ and set $E = [-\pi, \pi] \setminus \bigcup_{n=1}^{\infty} l_n$, where $l_n, \quad n = 1, 2, \ldots$ are disjoint open intervals satisfying $\sum_{n=1}^{\infty} |l_n|^{1-\alpha} < \infty$. Assuming that $m(E) > 0$, the characteristic function of $[-\pi, \pi] \setminus E$,*

$$f(x) = \begin{cases} 0 & x \notin E \\ 1 & x \in E \end{cases}$$

satisfies condition (1) of Theeorem 1, and for each point $x \notin E$ we have $\lim_{n\to\infty} S_n(x, f) = 0$.

Actually, if $0 < \alpha < 1$, then for the characteristic function $f_n(x)$ of the interval l_n we have

$$\int_{-\pi}^{\pi}\int_{0}^{\pi} \frac{|f(x+t)-f(x)|}{t^{1+\alpha}}\, dx\, dt \simeq |l_n|^{1-\alpha}.$$

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