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THE COMPOSITION OF TWO CONNECTED G_δ FUNCTIONS HAS A FIXED POINT

Abstract

We show that if $f, g\colon [0,1] \to [0,1]$ are both functions with connected G_δ graphs, then their composition has a fixed point. This is a generalization of the analogous result for Darboux Baire 1 functions.

1 Introduction

In the survey [6] Gibson and Natkaniec refer to the problem whether the composition of two derivatives (or DB_1 functions) with domain and range equal to the interval $\mathbb{I} = [0,1]$ has a fixed point. The question was answered in the positive by Csörnyei, O'Neil and Preiss ([3]) and independently by Elekes, Keleti and Prokaj ([4]). It is known that every derivative is DB_1 and every DB_1 function is a connected G_δ subset of $\mathbb{I}^2$ (see definitions below). We show that if $f, g\colon \mathbb{I} \to \mathbb{I}$ are both connected G_δ subsets of $\mathbb{I}^2$, then $f \circ g$ possesses a fixed point. Since Jones and Thomas constructed a function $f\colon \mathbb{I} \to \mathbb{I}$ with a connected G_δ graph which is Baire class 2 but not DB_1 ([7], see also [1] and [2]), this is a generalization of the result from [3] and [4].

Since this note have been written, the more general results have been proved in [11]. However, the proof presented here is considerably simpler, and self contained, while the proof presented in [11] depends heavily on the difficult results proved in [10].

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2 Preliminaries

Throughout this note for every $A \subset \mathbb{I}^2$ by A^{-1} we will denote the set:

$$\{\langle y, x\rangle \mid \langle x, y\rangle \in A\}.$$

By rectangle we will understand a rectangle whose sides are parallel to the usual coordinate axes. Topological notions like open, closed, etc., will be considered relatively to $\mathbb{I}^2$.

For every function f we will identify f with its graph. We will consider the following classes of functions from $\mathbb{I}$ into $\mathbb{I}$.

- D; we will say that f is Darboux ($f \in \mathrm{D}$) if $f(I)$ is an interval for every interval $I \subset \mathbb{I}$; equivalently, f has the intermediate value property;
- Conn; f is connected ($f \in \mathrm{Conn}$) if f is a connected subset of $\mathbb{I}^2$; we will also call such a function connectivity;
- B_1; f is Baire class 1 if f is the pointwise limit of a sequence of continuous functions; this is equivalent to the fact, that $f^{-1}(G)$ is an F_σ subset of $\mathbb{I}$ for every open set $G \subset \mathbb{I}$;
- B_2; f is Baire class 2 if f is the pointwise limit of a sequence of Baire 1 functions; this is equivalent to the fact, that $f^{-1}(G)$ is a $G_{\delta\sigma}$ subset of $\mathbb{I}$ for every open set $G \subset \mathbb{I}$;
- DB_1; f is Darboux Baire 1 ($f \in \mathrm{DB}_1$) if f is Darboux and Baire 1;
- G_δ; f is in the class G_δ if f is a G_δ subset of $\mathbb{I}^2$.

For properties of these classes of functions see e.g. the survey [5]. In particular, it is well-known that $\mathrm{Conn} \subset \mathrm{D}$ and $\mathrm{DB}_1 \subset \mathrm{B}_2 \cap \mathrm{Conn} \cap G_\delta$ (so, $\mathrm{Conn} = \mathrm{D}$ if we consider only Baire 1 functions). Moreover, if $f \in \mathrm{Conn}$ then $f \restriction I$ is connected for every interval $I \subset \mathbb{I}$. In the sequel we will also use the fact that every Darboux function is bilaterally dense in itself.

3 On Some Properties of Connectivity Functions

The following lemma is also a part of the paper [10].

Lemma 1. *Suppose $f\colon \mathbb{I} \to \mathbb{I}$ is a connectivity function, $A, B \subset f$, $A \cap B = \emptyset$ and $A \cup B$ is dense in f. If $\langle a, f(a)\rangle \in A$ and $\langle b, f(b)\rangle \in B$, then there exists $c \in [a, b]$ such that for every open neighbourhood U of $\langle c, f(c)\rangle$, $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$.*

PROOF. By $\mathrm{cl}_f(X)$ we will denote closure of the set X in the space f. Denote:

$$A_0 = A \cap ([a,b] \times \mathbb{I}) \text{ and } B_0 = B \cap ([a,b] \times \mathbb{I}).$$

Clearly $f \restriction [a,b] = \mathrm{cl}_f(A_0 \cup B_0) = \mathrm{cl}_f(A_0) \cup \mathrm{cl}_f(B_0)$. Since $\langle a, f(a)\rangle \in A_0$ and $\langle b, f(b)\rangle \in B_0$, $A_0 \neq \emptyset \neq B_0$. Since $f \restriction [a,b]$ is connected, $\mathrm{cl}_f(A_0) \cap \mathrm{cl}_f(B_0) \neq \emptyset$. Therefore as $\langle c, f(c)\rangle$ we can take any point from $\mathrm{cl}_f(A_0) \cap \mathrm{cl}_f(B_0)$. $\square$

The following technical lemma is a key part of the proof of Theorem 1. Note the symmetry between assumption and assertion of the Lemma 2.

Lemma 2. *Suppose $f\colon \mathbb{I} \to \mathbb{I}$ is a connectivity function, $g\colon \mathbb{I} \to \mathbb{I}$ is a Darboux function*[1]*, $f \cap g^{-1} = \emptyset$, $F \subset \mathbb{I}^2$ is an open neighbourhood of f, $[a,b] \times [c,d] \subset \mathbb{I}^2$ and there exist $x_1, x_2, y_1, y_2 \in \mathbb{I}$ such that (see Fig. 1):*

1. *$c < y_1 < y_2 < d$ and $x_1, x_2 \in (a,b)$;*
2. *$f(g(y_1)) > y_1$ and $f(g(y_2)) < y_2$;*
3. *$f(x_1) < y_1$ and $f(x_2) > y_2$;*
4. *either $x_1 < g(y_1)$ and $x_2 > g(y_2)$ or $x_1 > g(y_1)$ and $x_2 < g(y_2)$.*

Then there exist $[a',b'] \times [c',d'] \subset ([a,b] \times [c,d]) \cap F$ and $x_1', x_2', y_1', y_2' \in \mathbb{I}$ such that (see Fig. 1):

1. *$a' < x_1' < x_2' < b'$ and $y_1', y_2' \in (c',d')$;*
2. *$g(f(x_1')) > x_1'$ and $g(f(x_2')) < x_2'$;*
3. *$g(y_1') < x_1'$ and $g(y_2') > x_2'$;*
4. *either $y_1' < f(x_1')$ and $y_2' > f(x_2')$ or $y_1' > f(x_1')$ and $y_2' < f(x_2')$.*

PROOF. For every function $h\colon \mathbb{I} \to \mathbb{I}$ denote:

- $A(h) = \{\langle x, f(x)\rangle \in \mathbb{I}^2 \mid x < h(f(x))\}$;
- $B(h) = \{\langle x, f(x)\rangle \in \mathbb{I}^2 \mid x > h(f(x))\}$.

Clearly $A(h) \cap B(h) = \emptyset$. Moreover, if $f \cap h^{-1} = \emptyset$ then $f = A(h) \cup B(h)$.

Step 1. At the first step we will show that there exists $x' \in (a,b)$ such that $f(x') \in (c,d)$ and for every open set $U \ni \langle x', f(x')\rangle$, $A(g) \cap U \neq \emptyset$ and $B(g) \cap U \neq \emptyset$.

[1] In the proof we will use only the fact that g is bilaterally dense in itself, which is a consequence of Darboux property of g.

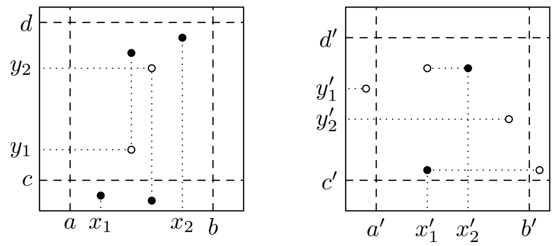

Figure 1: Configurations from the assumption and assertion of the Lemma 2. Points denoted by $\bullet$ are from f and points denoted by $\circ$ are from g^{-1}.

Define $g' \colon \mathbb{I} \to \mathbb{I}$:

$$g'(y) = \begin{cases} g(y) & \text{if } y \in (y_1, y_2); \\ g(y_1) & \text{if } y \leq y_1; \\ g(y_2) & \text{if } y \geq y_2. \end{cases}$$

Since $f(g(y_1)) > y_1$ and $f(g(y_2)) < y_2$, $f \cap {g'}^{-1} = \emptyset$.

From the fact that $f(x_1) < y_1$ and $f(x_2) > y_2$, and since $x_1 < g(y_1)$ and $x_2 > g(y_2)$ (or $x_1 > g(y_1)$, $x_2 < g(y_2)$), $\langle x_1, f(x_1)\rangle \in A(g')$ and $\langle x_2, f(x_2)\rangle \in B(g')$ (or $\langle x_1, f(x_1)\rangle \in B(g')$, $\langle x_2, f(x_2)\rangle \in A(g')$). So, by Lemma 1 there exists $x \in (\min\{x_1, x_2\}, \max\{x_1, x_2\})$ such that for every open neighbourhood V of $\langle x, f(x)\rangle$, $V \cap A(g') \neq \emptyset$ and $V \cap B(g') \neq \emptyset$. Since $g' \restriction [0, y_1)$ and $g' \restriction (y_2, 1]$ are constant and $f \cap {g'}^{-1} = \emptyset$, $\langle x, f(x)\rangle \in [\min\{x_1, x_2\}, \max\{x_1, x_2\}] \times [y_1, y_2] \subset \operatorname{int}(\,[a, b] \times [c, d]\,)$.

We will verify that x has all properties we need. First, observe that if $f(x) \in (y_1, y_2)$, then since $g' \restriction [y_1, y_2] = g \restriction [y_1, y_2]$, we can take $x' = x$.

Next, suppose $f(x) = y_1$ (the case $f(x) = y_2$ is analogous). Without loss of generality we can assume $x < g(y_1)$. Take any open rectangle $U \ni \langle x, f(x)\rangle$ such that $\langle g(y_1), y_1\rangle \notin U$. Since $U \cap (\mathbb{I} \times [0, y_1]) \cap f \subset A(g')$, there exists p such that $f(p) \in (y_1, y_2)$ and $\langle p, f(p)\rangle \in U \cap B(g')$. So, $\langle p, f(p)\rangle \in U \cap B(g)$. Since $\langle x, f(x)\rangle \in U \cap A(g)$, we can also take $x' = x$.

Step 2. Now find $a' < b'$ and $c' < d'$ such that $S = [a', b'] \times [c', d'] \subset \operatorname{int}(\,[a, b] \times [c, d]\,)$, $\langle x', f(x')\rangle \in \operatorname{int}(S)$, $S \subset F$ and $\langle g(f(x')), f(x')\rangle \notin S$.

Step 3. In the sequel we will assume $x' < g(f(x'))$, i.e. $\langle x', f(x')\rangle \in A(g)$ — the other situation is analogous. Then at least one of the following sets is dense in $\langle x', f(x')\rangle$:

$$\{\langle p, f(p)\rangle \in B(g) \mid p > x' \;\&\; f(p) > f(x')\}\,, \tag{Case 1}$$

$$\{\langle p, f(p)\rangle \in B(g) \mid p > x' \;\&\; f(p) < f(x')\}\,, \tag{Case 2}$$

$$\{\langle p, f(p)\rangle \in B(g) \mid p < x' \;\&\; f(p) > f(x')\}\,, \tag{Case 3}$$

$$\{\langle p, f(p)\rangle \in B(g) \mid p < x' \;\&\; f(p) < f(x')\}\,. \tag{Case 4}$$

Clearly it is enough to continue the proof only for the cases with the odd number. The situation in even cases is symmetric.

During the rest of the proof we will use the following observation: if $p < b'$ and $f(p) \neq f(x')$ then there exists $q \in (\min\{x', p\}, \max\{x', p\})$ such that $f(q) \in (\min\{f(x'), f(p)\}, \max\{f(x'), f(p)\})$ and $g(f(q)) > q$. Since (by the assumption made at the beginning of this step) $b' < g(f(x'))$ and g is bilaterally dense in $\langle f(x'), g(f(x'))\rangle$, this is a consequence of the intermediate value property of f.

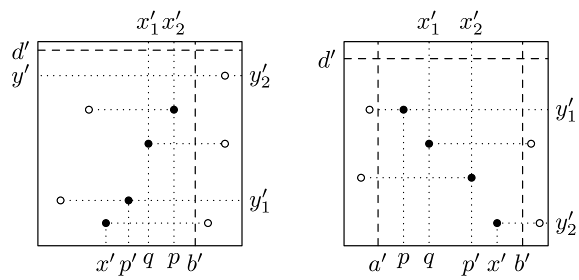

Figure 2: Configurations from the first and the third case of step 3 of Lemma's 2 proof. Labels at the bottom and left side of the diagrams are as during the progress of the proof, ones at the top and right are as in the assertion of the Lemma.

At the first case, since g is bilaterally dense in $\langle f(x'), g(f(x'))\rangle$ there exists $y' \in (f(x'), d')$ such that $g(y') > x'$. We can find $p \in (x', \min\{g(y'), b'\})$ such that $f(p) \in (f(x'), y')$ and $g(f(p)) < p$. As noted above we can also find $q \in (x', p)$ such that $f(q) \in (f(x'), f(p))$ and $g(f(q)) > q$. Using the assumption of the first case again, we can find $p' \in (x', q)$ such that $f(p') \in (f(x'), f(q))$ and $g(f(p')) < p'$ (see Fig. 2). Now if we take $x_1' = q$, $x_2' = p$, $y_1' = f(p')$ and $y_2' = y'$, we have the claim.

At the third case there exists $p \in (a', x')$ such that $f(p) \in (f(x'), d')$ and $g(f(p)) < p$. From the observation above we can also find $q \in (p, x')$ such that $f(q) \in (f(x'), f(p))$ and $g(f(q)) > q$. Finally, using the assumption of this case again, we can find $p' \in (q, x')$ such that $f(p') \in (f(x'), f(q))$ and $g(f(p')) < p'$ (see Fig. 2). So, if we take $x'_1 = q$, $x'_2 = p'$, $y'_1 = f(p)$ and $y'_2 = f(x')$, we have the claim. $\square$

4 Compositions of Connected G_δ Functions

In this section we will show that the composition of two connected G_δ functions has a fixed point. The proof is based on the following observation of Ciesielski. The composition of functions $f, g\colon \mathbb{I} \to \mathbb{I}$ possesses a fixed point if and only if $f \cap g^{-1} \neq \emptyset$.

Theorem 1. *If $u, v\colon \mathbb{I} \to \mathbb{I}$ are both connected G_δ functions, then $u \circ v$ possesses a fixed point.*

PROOF. The method used in this proof is similiar to that used in [3].

Suppose that $u \circ v$ has no fixed point, i.e. $u \cap v^{-1} = v \cap u^{-1} = \emptyset$. Since u and v are G_δ functions,

$$u = \bigcap_{n\in\mathbb{N}} U_n \text{ and } v^{-1} = \bigcap_{n\in\mathbb{N}} V_n,$$

where $\mathbb{I}^2 \supset U_0 \supset U_1 \supset \cdots$ and $\mathbb{I}^2 \supset V_0 \supset V_1 \supset \cdots$ are open sets (relatively to $\mathbb{I}^2$).

First we claim that we can assume:

$$u\left(v\left(\frac{1}{3}\right)\right) > \frac{1}{3},\ u\left(v\left(\frac{2}{3}\right)\right) < \frac{2}{3},\ u\left(\frac{1}{12}\right) < \frac{1}{3} \text{ and } u\left(\frac{11}{12}\right) > \frac{2}{3}.$$

To see this, consider the square $[-1, 2] \times [-1, 2]$. By extending u and v^{-1} as shown in Figure 3 and then rescaling we can define two new connectivity G_δ functions $\widetilde{u}, \widetilde{v}\colon \mathbb{I} \to \mathbb{I}$ with the required property such that $\widetilde{u} \circ \widetilde{v}$ has a fixed point if and only if the composition $u \circ v$ possesses it.

Note that

$$f = u,\ g = v,\ F = U_0,\ a = 0,\ b = 1,\ c = 0,\ d = 1,$$

$$x_1 = \frac{1}{12},\ x_2 = \frac{11}{12},\ y_1 = \frac{1}{3} \text{ and } y_2 = \frac{2}{3}$$

fulfill the assumption of Lemma 2. Next, if we take $a', b', c', d', x'_1, x'_2, y'_1$ and y'_2 as in this Lemma, then:

$$f = v,\ g = u,\ F = V_1^{-1},\ a = c',\ b = d',\ c = a',\ d = b',$$

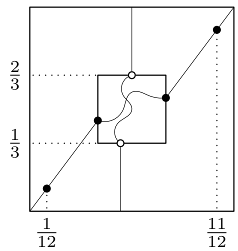

Figure 3: Example of extending and rescaling of u and v to $\widetilde{u}$ and $\widetilde{v}$ respectively as in the proof of Theorem 1. Points denoted by $\bullet$ are from $\widetilde{u}$ and points denoted by $\circ$ are from $\widetilde{v}^{-1}$.

$$x_1 = y_1',\ x_2 = y_2',\ y_1 = x_1' \text{ and } y_2 = x_2'$$

again fulfill the assumption of the Lemma. Hence we can consecutively use Lemma 2 to find a decreasing sequence of closed rectangles:

$$\mathbb{I}^2 \supset R_0 \supset R_1 \supset R_2 \supset \cdots$$

with $R_{2i} \subset U_i$ and $R_{2i+1} \subset V_i$ for each $i \in \mathbb{N}$. Since $\bigcap_{n\in\mathbb{N}} R_n \neq \emptyset$ and:

$$\bigcap_{n\in\mathbb{N}} R_n \subset \bigcap_{n\in\mathbb{N}} U_n = u \text{ and } \bigcap_{n\in\mathbb{N}} R_n \subset \bigcap_{n\in\mathbb{N}} V_n = v^{-1},$$

we have $u \cap v^{-1} \neq \emptyset$. $\square$

Example 1 (Ciesielski and Rosen [2]). *There exist Darboux G_δ functions $f, g\colon \mathbb{I} \to \mathbb{I}$ such that $f \circ g$ has no fixed point.*

Ciesielski and Rosen showed even more: they constructed Darboux G_δ function f which has no fixed point (the function constructed in [2] is also Baire class 2, see also [7]). Therefore it is enough to take the identity function $g\colon \mathbb{I} \to \mathbb{I}$ to obtain $f \circ g = g \circ f$ with no fixed point.

Example 2 (Kellum [8]). *There exist connectivity functions $f, g\colon \mathbb{I} \to \mathbb{I}$ such that $f \circ g$ has no fixed point.*

With the assumption of the Continuum Hypothesis Natkaniec showed in [9] that if the cardinality of $f^{-1}(y) \cap I$ is continuum for every $y \in \mathbb{I}$ and non-degenerated interval $I \subset \mathbb{I}$, then f is a composition of two connectivity functions. Using standard set theoretical methods it is not very hard to construct functions with the above property.

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