Real Analysis Exchange
Vol. 29(2), 2003/2004, pp. 647–656

Burkhard Lenze, Fachhochschule Dortmund, Fachbereich Informatik,
Postfach 105018, 44047 Dortmund, Germany. email: lenze@fh-dortmund.de

ON THE POINTS OF REGULARITY OF MULTIVARIATE FUNCTIONS OF BOUNDED VARIATION

Abstract

In the one-dimensional case it is well-known that functions of bounded variation on $\mathbb{R}$ possess at most a countable number of non-regular points. In this paper we will show that multivariate functions $f : \mathbb{R}^n \to \mathbb{R}$ of bounded variation satisfying the condition $\lim_{|x|\to\infty} f(x) = 0$ are non-regular at most on a subset of $\mathbb{R}^n$ of Lebesgue measure zero. Moreover, we will point out that this result is best possible.

1 Introduction

Since Jordan's famous theorem on the representation of a one-dimensional function of bounded variation as the difference of two nondecreasing functions it is well-known that univariate functions of bounded variation possess at most a countable number of discontinuities, all of the first kind (cf. [5], p. 188, Theorem 10 or [21], p. 19, Theorem (2.8)). Especially, for $f \in BV(\mathbb{R})$ and $x \in \mathbb{R}$ the one-sided limits

$$\lim_{h\to 0+} f(x+h) \text{ and } \lim_{h\to 0+} f(x-h) \tag{1}$$

exist, and for all $x \in \mathbb{R}$ except at most a countable number of points f satisfies the one-sided continuity condition

$$f(x) = \lim_{h\to 0+} f(x+h) = \lim_{h\to 0+} f(x-h), \tag{2}$$

resp. – in view of the title of our paper – the so-called regularity condition

$$f(x) = \frac{1}{2}\left(\lim_{h\to 0+} f(x+h) + \lim_{h\to 0+} f(x-h)\right). \tag{3}$$

Key Words: bounded variation, multivariate functions, points of regularity
Mathematical Reviews subject classification: 26B05, 26B35
Received by the editors May 29, 2003
Communicated by: Udayan B. Darji

In the following, we will take a look at the related question in case of multivariate functions of bounded variation. We will give weak sufficient conditions which imply something like one-sided continuity resp. regularity in this multidimensional setting, too. Therefore, this paper may be seen in the quite classical tradition of [22] and the main motivation to write it is due to the fact that uni- and multivariate functions of bounded variation are of increasing interest in different fields of application (without claim of completeness we mention [11, 12, 13, 16, 8, 14, 19, 23, 9, 3, 4, 7]).

2 Notation

First of all, we have to decide what we mean when we speak of functions of bounded variation in more than one variable. Generally, there are a number of different, but related definitions which all may have their advantages in special situations. (For an entire presentation of these definitions and a discussion of their relations compare [6], [1], and [2], and the references given there.) In the sequel we will follow the definition connected with the names of Vitali, Lebesgue, Fréchet, and de la Vallée Poussin which proved to be useful in the theory of measure and integration, too (for original references see [17] and [20]; classical textbooks which cover these topics are primary [18], and, moreover, [10] and [15]).

We start with some preliminaries. Let $n \in \mathbb{N}$ be given arbitrarily. For $a, b, x \in \mathbb{R}^n$ with $a \leq b$ (i.e., $a_i \leq b_i,\ 1 \leq i \leq n$) we define

$$\begin{aligned} (a,b) :=& \{x \in \mathbb{R}^n \mid a_i < x_i < b_i, 1 \leq i \leq n\}, \\ [a,b] :=& \{x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i\ ,\ 1 \leq i \leq n\}, \\ Cor[a,b] :=& \{x \in \mathbb{R}^n \mid x_i = a_i \vee x_i = b_i\ ,\ 1 \leq i \leq n\} \\ \gamma(x,a) :=& |\{i \in \{1,...,n\} \mid x_i = a_i\}|. \end{aligned} \tag{4}$$

In (4) $|\cdot|$ denotes the number of distinct elements of the set under consideration. Now, for a given function $f : \mathbb{R}^n \to \mathbb{R}$ the so-called corresponding interval function $\triangle_f$ of f is defined for all bounded intervals $[a,b] \subset \mathbb{R}^n$ by

$$\triangle_f[a,b] := \sum_{x \in Cor[a,b]} (-1)^{\gamma(x,a)} f(x) \quad . \tag{5}$$

In case $n = 1$ definition (5) reduces to

$$\triangle_f[a,b] = f(b) - f(a) = f(b_1) - f(a_1)$$

and in case $n = 2$ to

$$\triangle_f[a,b] = f(b_1,b_2) - f(b_1,a_2) - f(a_1,b_2) + f(a_1,a_2).$$

Let us note that the interval function $\triangle_f$ is known to be additive, i.e., $[a,b] = [a^{(1)}, b^{(1)}] \cup [a^{(2)}, b^{(2)}]$, $(a^{(1)}, b^{(1)}) \cap (a^{(2)}, b^{(2)}) = \emptyset$, implies

$$\triangle_f[a,b] = \triangle_f[a^{(1)}, b^{(1)}] + \triangle_f[a^{(2)}, b^{(2)}].$$

Now we are prepared to give the following basic definition.

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called
– monotone increasing on $\mathbb{R}^n$, if for all $[a,b] \subset \mathbb{R}^n$ we have $\triangle_f[a,b] \geq 0$
– of (uniform) bounded variation on $\mathbb{R}^n$ (or simply $f \in BV(\mathbb{R}^n)$), if there exists a constant $K > 0$ such that the interval function $\bar{\triangle}_f$ defined for all $[a,b] \subset \mathbb{R}^n$ by

$$\begin{aligned}\bar{\triangle}_f[a,b :=& \sup\Big\{ \sum_{i=1}^{r} |\triangle_f[a^{(i)}, b^{(i)}]| : \big([a^{(i)}, b^{(i)}] \subset [a,b] \wedge \\ & (a^{(i)}, b^{(i)}) \cap (a^{(j)}, b^{(j)}) = \emptyset, i \neq j\big),\ 1 \leq i,\, j \leq r,\ r \in \mathbb{N}\Big\}\end{aligned} \tag{6}$$

satisfies the inequality $\sup\{\bar{\triangle}_f[a,b] \mid [a,b] \subset \mathbb{R}^n\} \leq K$.

It can be easily shown that in case of $f \in BV(\mathbb{R}^n)$ the interval function $\bar{\triangle}_f$ defined in (6) is additive and nonnegative. Moreover, the same is true for the interval functions $\triangle_{P_f}$ and $\triangle_{N_f}$ induced by f via

$$\begin{aligned}\triangle_{P_f}[a,b] :=& \frac{1}{2}(\bar{\triangle}_f[a,b] + \triangle_f[a,b]),\ [a,b] \subset \mathbb{R}^n, \\ \triangle_{N_f}[a,b] :=& \frac{1}{2}(\bar{\triangle}_f[a,b] - \triangle_f[a,b]),\ [a,b] \subset \mathbb{R}^n.\end{aligned} \tag{7}$$

Therefore, we have the so-called Jordan decomposition of $\triangle_f$ into the difference of two nonnegative additive interval functions $\triangle_{P_f}$ and $\triangle_{N_f}$,

$$\triangle_f[a,b] = \triangle_{P_f}[a,b] - \triangle_{N_f}[a,b],\ [a,b] \subset \mathbb{R}^n. \tag{8}$$

At this point we recall that to any nonnegative additive interval function $\triangle$ there corresponds an outer (Carathéodory) measure $m^*_\triangle$ on $\mathbb{R}^n$ defined by

$$m^*_\triangle(M) := \inf\left\{\sum_{i=1}^{r} \triangle[a^{(i)}, b^{(i)}] \mid M \subset \bigcup_{i=1}^{r} (a^{(i)}, b^{(i)})\ ,\ r \in \mathbb{N} \cup \{\infty\}\right\}$$

for $\emptyset \neq M \subset \mathbb{R}^n$ and $m^*_\triangle(\emptyset) := 0$. As usual the σ-algebra of sets $M \subset \mathbb{R}^n$ satisfying

$$m^*_\triangle(N) = m^*_\triangle(N \cap M) + m^*_\triangle(N \setminus M),\ N \subset \mathbb{R}^n, \tag{9}$$

is called the set of measurable sets with respect to $m^*_\triangle$ (or $m_\triangle$) and the set function $m_\triangle$ defined for all $M \subset \mathbb{R}^n$ satisfying (9) by

$$m_\triangle(M) := m^*_\triangle(M) \tag{10}$$

is referred to as the (Carathéodory) measure induced by $\triangle$. Now, we return to an arbitrary function $f \in BV(\mathbb{R}^n)$. By means of (7) – (10) the (i.g. signed) finite measure m_f,

$$m_f := m_{\triangle_{P_f}} - m_{\triangle_{N_f}}, \tag{11}$$

is well-defined on the σ-algebra of sets measurable with respect to $m_{P_f} := m_{\triangle_{P_f}}$ and $m_{N_f} := m_{\triangle_{N_f}}$. The measure m_f is called Lebesgue-Stieltjes measure induced by f and is, by definition, intimately connected with the interval function $\triangle_f$. For example, we have the following result where the formulation "for almost all" has to be interpreted as usual with respect to the ordinary n-dimensional Lebesgue measure.

Lemma 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be monotone increasing on $\mathbb{R}^n$ and $a, b \in \mathbb{R}^n$, $a < b$. Then we have the inequality $m_f((a,b)) \leq \triangle_f[a,b] \leq m_f([a,b])$. Moreover, for almost all $a \in \mathbb{R}^n$ and almost all $b \in \mathbb{R}^n, b > a$, we have the equality $m_f((a,b)) = \triangle_f[a,b] = m_f([a,b])$.*

Proof. Compare [18], p. 68, Theorem (6.2) and p. 62, Theorem (4.1) (ii). □

3 Points of Regularity and $BV(\mathbb{R}^n)$

First of all, a few words should be said about the order of smoothness which can be expected for functions $f \in BV(\mathbb{R}^n)$. On the one hand, it is well-known that any function $f : \mathbb{R}^n \to \mathbb{R}$ which is independent of one of the variables x_i, $i \in \{1, \ldots, n\}$ may behave arbitrary worse with respect to the other variables x_j, $j \neq i$, but always belongs to $BV(\mathbb{R}^n)$ (cf. [15], 46.6s, p. 249). This implies, for example, that in case $n = 2$ the space $BV(\mathbb{R}^2)$ contains bounded functions which are everywhere discontinues (both in x_1 and x_2) or which are non-measurable in the sense of Lebesgue (cf. [1], Theorem 15, p. 722). These results show that we may not be too optimistic in view of strong smoothness results (especially, in view of continuity or regularity). On the other hand, the space $BV(\mathbb{R}^n)$ should reflect at least some of the nice properties of $BV(\mathbb{R})$ and, therefore, under some appropriate restrictions we would expect a few positive results. The first (and, as far as we know, the only) paper which takes a detailed look at these problems is [22]. In this paper the authors completely determine the structure of the discontinuities of different types of so-called multivariate increasing functions and implicitly also for the difference of two of these functions. (For details compare [22] and, moreover, the references given there; see also [1], Paragraph 6, pp. 721ff.) In the following, although using a similar terminology as in [22], we will differ from [22] in two essential points. First, we will use a different definition of monotonicity (compare Definition 2.1) and, caused by this, we will only consider functions $f \in BV(\mathbb{R}^n)$ which satisfy a kind of regularity condition at infinity, precisely $\lim_{|x|\to\infty} f(x) = 0$. Secondly, we will prove our results with the help of a measure theoretic approach which wasn't utilized in [22].

After these preliminaries we start with some simple definitions essentially taken from [22]. From now on let $z^{(s)} := (z_1^{(s)}, \dots, z_n^{(s)}), 1 \le s \le 2^n$, denote the 2^n distinct vectors out of $\mathbb{R}^n$ satisfying $|z_k^{(s)}| = 1, 1 \le k \le n, 1 \le s \le 2^n$. For example, in case $n = 2$ these vectors are $(1,1), (-1,1), (1,-1)$, and $(-1,-1)$. Moreover, for any fixed $x \in \mathbb{R}^n$ we define the open resp. closed quadrants of x with respect to $z^{(s)}$ by

$$\begin{aligned} Q^o(x, z^{(s)}) &:= \{t \in \mathbb{R}^n \mid (t_k - x_k) z_k^{(s)} > 0, 1 \le k \le n\}, 1 \le s \le 2^n, \\ Q(x, z^{(s)}) &:= \{t \in \mathbb{R}^n \mid (t_k - x_k) z_k^{(s)} \ge 0, 1 \le k \le n\}, 1 \le s \le 2^n. \end{aligned}$$

With the help of these definitions we are prepared to prove the following theorem.

Theorem 3.1. *Let $f \in BV(\mathbb{R}^n)$ with $\lim_{|t|\to\infty} f(t) = 0$ be given. Then for all $x \in \mathbb{R}^n$ and all $s \in \{1, 2, \dots, 2^n\}$ we have*

$$\lim_{\substack{t \to x \\ t \in Q^o(x, z^{(s)})}} f(t) = (-1)^n \left(\prod_{k=1}^n z_k^{(s)} \right) m_f(Q^o(x, z^{(s)})). \tag{12}$$

PROOF. Let $x \in \mathbb{R}^n$ and $s \in \{1, \dots, 2^n\}$ be given and $m_f = m_{P_f} - m_{N_f}$ the decomposition of m_f according to (11). Now, for each $\alpha > 0$ and each $t \in Q^o(x, z^{(s)})$ we consider the cubes $W_{s,\alpha}(t) := [\min\{t, t + \alpha z^{(s)}\}, \max\{t, t + \alpha z^{(s)}\}]$ and their interior $W^o_{s,\alpha}(t) := (\min\{t, t + \alpha z^{(s)}\}, \max\{t, t + \alpha z^{(s)}\})$. By means of (11) and Lemma 2.2 we immediately get the inequality

$$\begin{aligned} m_{P_f}(W^o_{s,\alpha}(t)) - m_{N_f}(W_{s,\alpha}(t)) &\le \triangle_f[\min\{t, t + \alpha z^{(s)}\}, \max\{t, t + \alpha z^{(s)}\}] \\ &\le m_{P_f}(W_{s,\alpha}(t)) - m_{N_f}(W^o_{s,\alpha}(t)). \end{aligned} \tag{13}$$

For $\alpha \to \infty$ we have

$$\begin{aligned} \lim_{\alpha\to\infty} W_{s,\alpha}(t) &= Q(t, z^{(s)}), \\ \lim_{\alpha\to\infty} W^o_{s,\alpha}(t) &= Q^o(t, z^{(s)}). \end{aligned} \tag{14}$$

Moreover, since $\lim_{|y|\to\infty} f(y) = 0$ we also get by means of (5)

$$\begin{aligned}
&\lim_{\alpha\to\infty} \triangle_f[\min\{t, t+\alpha z^{(s)}\}, \max\{t, t+\alpha z^{(s)}\}] \\
=& \lim_{\alpha\to\infty} \sum_{y\in Cor(W_{s,\alpha}(t))} (-1)^{\gamma(y,\min\{t,t+\alpha z^{(s)}\})} f(y) \\
=&(-1)^n \left(\prod_{k=1}^{n} z_k^{(s)}\right) f(t).
\end{aligned} \tag{15}$$

Using (14) and (15) we conclude from (13) for $\alpha \to \infty$ that

$$\begin{aligned}
m_{P_f}(Q^o(t, z^{(s)})) - m_{N_f}(Q(t, z^{(s)})) \leq\ & (-1)^n \left(\prod_{k=1}^{n} z_k^{(s)}\right) f(t) \\
\leq m_{P_f}(Q(t, z^{(s)})) - m_{N_f}(Q^o(t, z^{(s)})).&
\end{aligned} \tag{16}$$

If we now make use of the set identities

$$Q^o(x, z^{(s)}) = \lim_{\substack{t\to x \\ t\in Q^o(x,z^{(s)})}} Q^o(t, z^{(s)}),$$

$$Q^o(x, z^{(s)}) = \lim_{\substack{t\to x \\ t\in Q^o(x,z^{(s)})}} Q(t, z^{(s)}),$$

we obtain from (16) for $t \to x, t \in Q^o(x, z^{(s)})$ that

$$m_{P_f}(Q^o(x, z^{(s)})) - m_{N_f}(Q^o(x, z^{(s)})) = (-1)^n \left(\prod_{k=1}^{n} z_k^{(s)}\right) \lim_{\substack{t\to x \\ t\in Q^o(x,z^{(s)})}} f(t),$$

which implies (12) by means of (11). □

Remark. The result of the above theorem may be read as follows. For $f \in BV(\mathbb{R}^n)$ with $\lim_{|x|\to\infty} f(x) = 0$ the limits

$$\lim_{\substack{t\to x \\ t\in Q^o(x,z^{(s)})}} f(t), 1 \leq s \leq 2^n, \tag{17}$$

exist for all $x \in \mathbb{R}^n$. On the other hand, we have no information about the existence (or behavior) of limits like (17) if we allow the approaching points t to lie on one of the hyperplanes $H(x, k)$,

$$H(x, k) := \{t \in \mathbb{R}^n \mid t_k = x_k\}, 1 \leq k \leq n. \tag{18}$$

Since in case $n = 1$ the limits (17) coincide with the one-sided limits (1) and the hyperplanes (18) degenerate to the point x, the result of Theorem 3.1 seems to be an appropriate extension to the multivariate case (see also [22] in this context).

In the following we will take a look at the second point of interest in connection with continuity and regularity of functions $f \in BV(\mathbb{R}^n)$, $\lim_{|x|\to\infty} f(x) = 0$, more precisely, we want to examine under which conditions the limits (17) are equal to $f(x)$ (compare (2)) or — in formal analogy of (3) — to find those points $x \in \mathbb{R}^n$ which satisfy $f(x) = 2^{-n} \sum_{s=1}^{2^n} \lim_{\substack{t\to x \\ t\in Q^o(x,z^{(s)})}} f(t)$.

Theorem 3.2. *Let $f \in BV(\mathbb{R}^n)$ with $\lim_{|t|\to\infty} f(t) = 0$ be given. Then for almost all $x \in \mathbb{R}^n$ we have*

$$f(x) = \lim_{\substack{t\to x \\ t\in Q^o(x,z^{(s)})}} f(t), 1 \le s \le 2^n. \tag{19}$$

Especially, almost all $x \in \mathbb{R}^n$ satisfy

$$f(x) \;=\; 2^{-n} \sum_{s=1}^{2^n} \lim_{\substack{t\to x \\ t\in Q^o(x,z^{(s)})}} f(t) \;\;. \tag{20}$$

Proof. The proof is similar to that of Theorem 3.1. Let $s \in \{1, \dots, 2^n\}$ be given and $m_f = m_{P_f} - m_{N_f}$ the decomposition of m_f according to (11). Now, by means of Lemma 2.2 it is easy to show that for almost all $x \in \mathbb{R}^n$ we can find a monotone increasing sequence $(\alpha_j)_{j\in\mathbb{N}}$, $0 < \alpha_1 < \cdots < \alpha_j < \alpha_{j+1} < \dots$, $\lim_{j\to\infty} \alpha_j = \infty$, with

$$W^o_{s,\alpha_j}(x) := (\min\{x, x + \alpha_j z^{(s)}\}, \max\{x, x + \alpha_j z^{(s)}\}),\ j \in \mathbb{N},$$

satisfying

$$\begin{aligned} &m_{P_f}(W^o_{s,\alpha_j}(x)) - m_{N_f}(W^o_{s,\alpha_j}(x)) \\ =&\triangle_{P_f}[\min\{x, x + \alpha_j z^{(s)}\}, \max\{x, x + \alpha_j z^{(s)}\}] \\ &- \triangle_{N_f}[\min\{x, x + \alpha_j z^{(s)}\}, \max\{x, x + \alpha_j z^{(s)}\}], j \in \mathbb{N}; \end{aligned}$$

i. e., via (11) and (8)

$$m_f(W^o_{s,\alpha_j}(x)) = \triangle_f[\min\{x, x + \alpha_j z^{(s)}\}, \max\{x, x + \alpha_j z^{(s)}\}], j \in \mathbb{N}. \tag{21}$$

By means of (14) and (15) (with $t = x$ and $\alpha = \alpha_j$) identity (21) implies for $j \to \infty$ that $m_f(Q^o(x, z^{(s)})) = (-1)^n \left(\prod_{k=1}^{n} z_k^{(s)}\right) f(x)$. Using Theorem 3.1 we finally obtain (19). □

4 Concluding Remarks

(1) In generalization of the one-dimensional case (cf. [10], p. 161, or [21], p. 224) we may call $x \in \mathbb{R}^n$ *regular* with respect to $f \in BV(\mathbb{R}^n), \lim_{|t|\to\infty} f(t) = 0$, if

$$f(x) = 2^{-n} \sum_{s=1}^{2^n} \lim_{\substack{t \to x \\ t \in Q^o(x, z^{(s)})}} f(t). \tag{22}$$

In this terminology Theorem 3.2 may be reformulated as follows. Almost all $x \in \mathbb{R}^n$ are regular with respect to $f \in BV(\mathbb{R}^n), \lim_{|t|\to\infty} f(t) = 0$.

(2) Let us note that, in contrast to the one-dimensional case, it is not possible to prove for $n > 1$ that (22) is valid for all $x \in \mathbb{R}^n$ except at most a countable number of points. For example, this may be seen by considering the function $g : \mathbb{R}^2 \to \{0, 1\}$,

$$g(x_1, x_2) := \begin{cases} 1 & \text{for } (0, -1) \le (x_1, x_2) \le (0, 1) \\ 0 & \text{elsewhere,} \end{cases}$$

which obviously belongs to $BV(\mathbb{R}^2)$, satisfies $\lim_{|t|\to\infty} g(t) = 0$, and possesses more than a countable number of non-regular points.

(3) We point to the fact that the condition $\lim\limits_{|t|\to\infty} f(t) = 0$ is essential for $f \in BV(\mathbb{R}^n)$ to have limits as considered in (17) and to make the definition of regularity in sense of (22) possible. For example, the function $h : \mathbb{R}^2 \to \{0, 1\}$,

$$h(x_1, x_2) := \begin{cases} 1 & \text{for } x_1 \in \mathbb{Q} \\ 0 & \text{elsewhere,} \end{cases}$$

satisfies $\triangle_h[a, b] = 0, [a, b] \subset \mathbb{R}^2$, and, therefore, belongs to $BV(\mathbb{R}^2)$. But, the limits considered in (17) do not exist for any point $x \in \mathbb{R}^2$.

(4) Finally, let us note that it is possible to formulate a sufficient condition for a point $x \in \mathbb{R}^n$ to satisfy (19) resp. (20) in terms of the so-called vanishing oscillation condition introduced in [18], p. 60. Precisely, the equations (19) resp. identity (20) are valid for a function $f \in BV(\mathbb{R}^n)$ with $\lim_{|t|\to\infty} f(t) = 0$ at a point $x \in \mathbb{R}^n$ if the oscillation of the interval function $\triangle_f$ vanishes for all hyperplanes $H(x, k)$, $1 \le k \le n$, defined in (18) (and this is the case at least almost everywhere; for details compare [18], Chapter III, Paragraphs 1–6).

References

[1] C. R. Adams and J. A. Clarkson, *Properties of functions $f(x, y)$ of bounded variation*, Trans. Amer. Math. Soc., **36** (1934), 711–730.

[2] C. R. Adams and J. A. Clarkson, *A correction to "Properties of functions $f(x, y)$ of bounded variation"*, Trans. Amer. Math. Soc., **46** (1939), 468.

[3] S. Baron, E. Liflyand and U. Stadtmüller, *Complementary spaces and multipliers of double Fourier series for functions of bounded variation*, J. Math. Anal. Appl., **250** (2000), 706–721.

[4] S. A. Belov and V. V. Chistyakov, *A selection principle for mappings of bounded variation*, J. Math. Anal. Appl., **249** (2000), 351–366.

[5] C. Carathéodory, *Vorlesungen über reelle Funktionen*, Chelsea Publishing Company, New York, 1968, third edition.

[6] J. A. Clarkson and C. R. Adams, *On definitions of bounded variation of functions of two variables*, Trans. Amer. Math. Soc., **35** (1933), 824–854.

[7] S. Y. Galkina, *On the Fourier-Haar coefficients of functions of several variables with bounded Vitali variation*, Math. Notes, **70** (2001), 733–743.

[8] P. Gora, *Importance of bounded variation condition for stability of an absolutely continuous invariant measure*, J. Math. Anal. Appl., **181** (1994), 422–428.

[9] V. Gupta and R. P. Pant, *Rate of convergence for the modified Szász-Mirakyan operators on functions of bounded variation*, J. Math. Anal. Appl., **233** (1999), 476–483.

[10] E. Kamke, *Das Lebesgue-Stieltjes Integral*, B. G. Teubner Verlagsgesellschaft, Leipzig, 1956.

[11] B. Lenze, *On multidimensional Lebesgue-Stieltjes convolution operators*, in: "Multivariate Approximation Theory IV" (Eds.: C. K. Chui, W. Schempp, K. Zeller), ISNM 90, Birkhäuser Verlag, Basel–Boston–Berlin, 1989, 225–232.

[12] B. Lenze, *On constructive one-sided approximation of multivariate functions of bounded variation*, Num. Funct. Anal. a. Optimiz., **11** (1990), 55–83.

[13] B. Lenze, *A hyperbolic modulus of smoothness for multivariate functions of bounded variation*, J. Appr. Theory and its Appl., **7:1** (1991), 1–15.

[14] A. S. Leonov, *An application of functions of bounded variation of several variables to the piecewise-uniform regularization of ill-posed problems*, Dokl. Math. Akad. Nauk., **54** (1996), 918–922.

[15] E. J. McShane, *Integration*, Princeton University Press, Princeton, 1974, eighth edition.

[16] G. Prasad, E. R. Love and A. Sahai, *An improved estimate of the rate of convergence of the integrated Meyer-König and Zeller operators for functions of bounded variation*, J. Math. Anal. Appl., **187** (1994), 1–16.

[17] J. Radon, *Theorie und Anwendungen der absolut additiven Mengenfunktionen*, Sitzungsberichte der kaiserlichen Akademie der Wissenschaften, Abteilung 2a, Band 122, 2-ter Halbband, Wien, 1913, 1295–1438.

[18] S. Saks, *Theory of the Integral*, Hafner Publishing Company, New York, 1937, second edition.

[19] V. N. Temlyakov and S. A. Telyakovskii, *Convergence of multiple Fourier series for functions of bounded variation*, Math. Notes, **61** (1997), 484–494.

[20] Ch. J. de la Vallée Poussin, *Intégrales de Lebegue, fonctions d'ensemble, classes de Baire*, Paris, 1916.

[21] R. L. Wheeden and A. Zygmund, *Measure and Integral*, Marcel Dekker, Inc., New York–Basel, 1977.

[22] W. H. Young and G. C. Young, *On the discontinuities of monotone functions of several variables*, Proc. London Math. Soc., **(2), 22** (1923), 124–142.

[23] X. M. Zeng, *On the rate of convergence of the generalized Szász-type operators for functions of bounded variation*, J. Math. Anal. Appl., **226** (1998), 309–325.