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A NONHEREDITARY BOREL-COVER γ-SET

Abstract

In this paper we prove that if there is a Borel-cover γ-set of cardinality the continuum, then there is one which is not hereditary.

In this paper we answer some of the questions raised by Bartoszyński and Tsaban [1] concerning hereditary properties of sets defined by certain Borel covering properties.

Definition. An ω-cover of a set X is a family of sets such that every finite subset of X is included in an element of the cover but X itself is not in the family.

Definition. A γ-cover of a set X is an infinite family of sets such that every element of X is in all but finitely many elements of the family.

Definition. A set X is called a Borel-cover γ-set iff every countable ω-cover of X by Borel sets contains a γ-cover.

These concepts were introduced by Gerlits and Nagy [5] for open covers.

Being a Borel-cover γ-set is equivalent to saying that for any ω-sequence of countable Borel ω-covers of X we can choose one element from each and get a γ-cover of X – this is denoted $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$. The equivalence was proved by Gerlitz and Nagy [5] for open covers but the proof works also for Borel covers, as was noted in Scheepers and Tsaban [8]:

Let $\mathcal{B}_n$ be Borel ω-covers of X. Since $\{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ is an ω-cover if $\mathcal{U}$ and $\mathcal{V}$ are, we may assume that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$. Let x_n for $n < \omega$ be

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distinct elements of X and let $\mathcal{B} = \{A \setminus \{x_n\} : n < \omega, A \in \mathcal{B}_n\}$. It is easy to check that $\mathcal{B}$ is an ω-cover of X. Now let $\mathcal{C}$ be a γ-subcover of $\mathcal{B}$. Note that for any fixed n at most finitely many of the elements of $\mathcal{C}$ can be of the form $A \setminus \{x_n\}$. By refining $\mathcal{C}$, we may assume at most one thing is taken from each $\mathcal{B}_n$ and since they are refining, we can fatten up $\mathcal{C}$ to take exactly one element of each $\mathcal{B}_n$.

Definition. A family of subsets of X. $\mathcal{U}$ is a τ cover of X iff every element of X is in infinitely many elements of $\mathcal{U}$ and for every $x, y \in X$ at least one of the sets $\{U \in \mathcal{U} : x \in U,\ y \notin U\}$ or $\{U \in \mathcal{U} : x \notin U,\ y \in U\}$ is finite.

Clearly any γ-cover is a τ-cover. These covers were introduced in Tsaban [9].

Theorem 1. *Suppose there is a Borel-cover γ-set of size the continuum. Then there is a Borel-cover γ-set X and subset Y of X which is not a Borel-cover γ-set. In fact, there is an open ω-cover of Y with no τ-subcover.*

PROOF. For $X \subset P(\omega)$ let $\widetilde{X} = \{\omega \setminus a : a \in X\}$ be the dual of X, i.e., the set of complements of elements of X. Let $P \subseteq [\omega]^\omega$ be a perfect set of independent subsets of ω. This means that for every disjoint pair F_1, F_2 of finite subsets of P the set $\bigcap F_1 \cap \bigcap \widetilde{F_2}$ is infinite. Such a set was first constructed by Fichtenholtz, Kantorovich, and Hausdorff, see Kunen [6]. To construct one, let $Q = \{(n, s) : n \in \omega, s \subseteq P(n)\}$. Define $A_x = \{(n, s) : x \cap n \in s\}$ for each $x \subseteq \omega$ and $P = \{A_x : x \subseteq \omega\} \subseteq P(Q) = 2^Q$. □

Let $Z \subseteq P$ be a Borel-cover γ-set of cardinality the continuum.

Claim 2. *$Z \cup \widetilde{Z}$ is a Borel-cover γ-set.*

PROOF. Let $\{B_n : n < \omega\}$ be a Borel ω-cover of $Z \cup \widetilde{Z}$. Then it is easy to see that $\{(B_n \cap \widetilde{B_n}) : n < \omega\}$ is an ω-cover of Z. This is because if $(F \cup \widetilde{F}) \subseteq B_n$, then $F \subseteq (B_n \cap \widetilde{B_n})$.

Since Z is a Borel-cover γ-set, there exists an $a \in [\omega]^\omega$ such that $\{(B_n \cap \widetilde{B_n}) : n \in a\}$ is a γ-cover of Z. But then it is also a γ-cover of $\widetilde{Z}$. This proves the Claim. □

Let $X = Z \cup \widetilde{Z}$ and to pick $Y \subseteq X$ as required we will choose for each $a \in Z$ to put either $a \in Y$ or $(\omega \setminus a) \in Y$ (but not both). Since Z was a subset of P and P was independent, we will have that the intersection of any finite subset of Y is infinite. In particular, $\mathcal{U} = \{U_n : n \in \omega\}$ where $U_n = \{a \subseteq \omega : n \in a\}$ is an ω-cover of Y. But $\{U_n : n \in b\}$ is a γ-cover of Y iff $b \subseteq^* a$ for every $a \in Y$. But this is easy to defeat. Using that Z has cardinality of the continuum let $Z = \{a_\alpha : \alpha < \mathfrak{c}\}$ and let $[\omega]^\omega = \{b_\alpha : \alpha < \mathfrak{c}\}$. For each α if $b_\alpha \subseteq^* a_\alpha$ put $(\omega \setminus a_\alpha)$ into Y and otherwise put a_α into Y.

Constructing Y so that $\mathcal{U}$ has no τ-subcovers can be done by using two elements a_0, a_1 of Z for each $b \in [\omega]^\omega$. First note that the set $\{U_n : n \in b\}$ is a τ-cover of Y iff b meets every element of Y in an infinite set and for every two elements a_0, a_1 of Y either $(a_0 \cap b) \subseteq^* a_1$ or $(a_1 \cap b) \subseteq^* a_0$.

Notation: $a^{(0)} = a$ and $a^{(1)} = \omega \setminus a$.

Claim 3. *There exists i, j in $\{0, 1\}$ such that*

(a) $b \cap a_i^{(j)}$ is finite or

(b) both $b \cap a_0^{(i)} \cap a_1^{(1-j)}$ and $b \cap a_0^{(1-i)} \cap a_1^{(j)}$ are infinite.

Proof. Assume case (a) fails for all i, j in $\{0, 1\}$. The four sets $a_0^{(i)} \cap a_1^{(j)}$ partition ω into infinite sets since a_0 and a_1 are independent. If all four meet b in an infinite set, then we are done. So assume that $b \cap a_0^{(i)} \cap a_1^{(j)}$ is finite for some i, j. But since $b \cap a_0^{(i)}$ is infinite it must be that $b \cap a_0^{(i)} \cap a_1^{(1-j)}$ is infinite. A similar argument shows $b \cap a_0^{(1-i)} \cap a_1^{(j)}$ is infinite. This proves the Claim. □

To kill off the possibility of b giving a τ-subcover we put $a_i^{(j)}$ into Y in case (a) or put both $a_0^{(i)}, a_1^{(j)}$ into Y in case (b). This proves the Theorem.

Remark 4. Tsaban points out the following corollary of our result. In Problem 7.9 of Bukovský, Recław, and Repický [3] it is asked whether every γ-set of reals, which is also a σ-set, is a hereditary γ-set. It is shown in Scheepers and Tsaban [8] that every Borel-cover γ-set (more generally $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$-set) is a σ-set. Hence the answer to the problem is no.

The following result is due to Brendle [2]. Our proof is a slight modification of a result of Todorčević – see Theorem 4.1 of Galvin and Miller [4], and is perhaps simpler.

Theorem 5. *(Brendle) Assume CH. Then there exists a Borel-cover γ-set of size ω_1.*

Proof. The idea is to construct an Aronszajn tree $T \subseteq \omega^{<\omega_1}$ labeled by perfect sets. We construct perfect subtrees $p_s \subseteq 2^{<\omega}$ for $s \in T$ and $X_s \subseteq [p_s]$ countable dense sets such that;

1. if $s \subseteq t$, then $p_s \supseteq p_t$,
2. if s and t are incomparable, then $[p_s] \cap [p_t] = \emptyset$,
3. if $\alpha < \beta < \omega_1$ and $n < \omega$, then for every $s \in T_\alpha$ there exists $t \in T_\beta$ with $s \subseteq t$ and $p_s \cap 2^n = p_t \cap 2^n$, and

4. for every sequence $(B_n : n < \omega)$ of Borel subsets of 2^ω there exists $\alpha < \omega_1$ such that either for some finite $F \subseteq \cup\{X_s : s \in T_{\leq\alpha+1}\}$ no B_n covers F or there exists an $a \in [\omega]^\omega$ such that $\{B_n : n \in a\}$ is a γ-cover of
$$\cup\{[p_s] : s \in T_{\leq\alpha+1}\} \;\cup\; \bigcup\{X_s : s \in T_{\leq\alpha}\}.$$

After the construction is completed we will let $X = \cup\{X_s : s \in T\}$. The last item guarantees that X will be a Borel-cover γ-set. The first three items are simply to guarantee that our construction can continue at limit levels. To do the last item we use the following Lemma.

Define for any perfect tree p and $s \in p$, $p\langle s\rangle = \{t \in p : t \subseteq s \text{ or } s \subseteq t\}$.

Lemma 6. *Suppose $\langle p_n : n < \omega\rangle$ are perfect trees and $(\mathcal{B}_n : n < \omega)$ is a sequence of countable Borel ω-covers of $\cup_{n<\omega}[p_n]$. Then there exists perfect pairwise disjoint subtrees $q_n \subseteq p_n$ and $\{B_n \in \mathcal{B}_n : n < \omega\}$ which is a γ-cover of $\cup_{n<\omega}[q_n]$.*

Proof. We can begin by refining the p_n's so that $[p_n]$ are pairwise disjoint. So, we may as well assume this to begin with. Also since Borel sets have the (relative) property of Baire with respect to each perfect set, by passing to perfect subsets we may assume that each of our Borel covers is an open cover.

Note that for finite sequences $(k_i : i < n)$ and $(t_i \in p_{k_i} : i < n)$, there exists a $U_n \in \mathcal{B}_n$ and $r_i \supseteq t_i$ with $r_i \in p_{k_i}$ such that $\cup_{i<n}[p_{k_i}\langle r_i\rangle] \subseteq U_n$ Using this observation it is easy to construct a fusion sequence which produces the q_n and the required γ-cover. This proves the Lemma. □

Let $T_\alpha = T \cap \omega^\alpha$. In our construction of the tree we start by assuming that $\{\mathcal{B}_\alpha : \alpha < \omega_1\}$ is a list containing all countable families of Borel subsets of 2^ω. At limit ordinals α, we use the usual fusion arguments to produce p_s for $s \in T_\alpha$. We take care of condition (4) as follows.

Suppose by induction we have already constructed $(p_s : s \in T_\alpha)$ and $(X_s : s \in T_{<\alpha})$. To obtain condition (4), let $\{x_n : n < \omega\} = \cup\{X_s : s \in T_{<\alpha}\}$, and define $\mathcal{B}^n_\alpha = \{B \in \mathcal{B}_\alpha : \{x_i : i < n\} \subseteq B\}$

If some $\mathcal{B}^n_\alpha$ is not an ω-cover of $\cup\{[p_s] : s \in T_\alpha\}$, then there exists a finite subset of $\{x_n : n < \omega\} \cup \bigcup\{[p_s] : s \in T_\alpha\}$ which is not covered by any $B \in \mathcal{B}_\alpha$. In this case, we choose $X_s \subseteq p_s$ so that this finite set is included in $\cup\{X_s : s \in T_{\leq\alpha}\}$. We then choose p_{sn} so that $p_{sn} \cap 2^n = p_s \cap 2^n$, $p_{sn} \subseteq p_s$, and $[p_{sn}]$ for $n < \omega$ are pairwise disjoint. We don't need to worry about $\mathcal{B}_\alpha$ because it cannot be an ω-cover of X.

So we may assume each $\mathcal{B}^n_\alpha$ is an ω-cover of $\cup\{[p_s] : s \in T_\alpha\}$. Apply the Lemma to the sequence $(p_s\langle t\rangle : s \in T_\alpha \text{ and } t \in p_s)$. Then for each $s \in T_\alpha$ and $n < \omega$ let $p_{sn} = \cup\{q_{s,t} : t \in 2^n \cap p_s\}$. In this case we can take each $X_{sn} \subseteq [p_{sn}]$ to be an arbitrary countable dense subset. This proves Theorem 5. □

Theorem 7 is probably known but we include its proof here for completeness.

Theorem 7. *Suppose that M is any countable standard model of ZFC. Then there exists a ccc poset $\mathbb{P}$ in M of size continuum such if G is any $\mathbb{P}$-filter generic over M, then $X = M \cap 2^\omega$ in M is a Borel-cover γ-set in $M[G]$. Note that forcing with $\mathbb{P}$ does not change the size of the continuum in $M[G]$.*

PROOF. This is really a corollary of result noted by Gerlitz and Nagy [5] that assuming MA (or even just MA(σ-centered)) that every set X of size less than continuum is a γ-set.

Let $\{B_n : n < \omega\}$ be an ω-cover of X. For each $x \in X$ let $a_x = \{n : x \in B_n\}$. The family $\{a_x : x \in X\}$ has the finite intersection property. So there is a well-known ccc poset of size $|X|$ (see Kunen [7]) which adds an infinite $a \in [\omega]^\omega$ such that $a \subseteq^* a_x$ for each $x \in X$. Then $\{B_n : n \in a\}$ is a γ-cover of X. To obtain the model $M[G]$ simply iterate continuum many times, with the usual dovetailing argument to take care of all sequences of Borel sets in $M[G]$. $\square$

Question 8. Does MA imply there exists a Borel-cover γ-set of size the continuum?

The theorems in this section show that it is consistent that the classes $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$, $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\mathrm{T})$, and $\mathsf{S}_{\mathrm{fin}}(\mathcal{B}_\Omega, \mathcal{B}_\mathrm{T})$ are not hereditary.

I don't know about the other classes in Bartoszyński and Tsaban [1], for example:

Question 9. Is the class $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ hereditary?

For the definitions of these classes see [1].

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