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QUANTIZATION DIMENSION VIA QUANTIZATION NUMBERS

Abstract

We give a characterization of the quantization dimension of Borel probability measures on $\mathbb{R}^d$ in terms of ε-quantization numbers. Using this concept, we show that the upper rate distortion dimension is not greater than the upper quantization dimension of order one. We also prove that the upper quantization dimension of a product measure is not greater than the sum of that of its marginals. Finally, we introduce the notion of the ε-essential radius for a given measure to construct an upper bound for its quantization dimension.

1 Introduction

Quantization problems originate in engineering technologies such as signal processing or data compression. In return, mathematical results concerning quantization have a large variety of applications to other sciences (see [5]). Mathematically, the quantization problem is to approximate a given probability measure μ by a finitely supported probability measure ν with respect to the L_r–Wasserstein (or Kantorovich) metric given by

$$\rho_r(\mu,\nu) := \inf_Q \left(\int \|x-y\|^r \, dQ(x,y) \right)^{1/r},$$

where the infimum is taken over all Borel probability measures Q on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ,ν, and $1 \le r < \infty$. One of the main goals is to determine the *n-th quantization error*

$$V_{n,r}(\mu) := \inf_{\nu\in\mathcal{P}_n} (\rho_r(\mu,\nu))^r,$$

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where $\mathcal{P}_n$ denotes the set of probability measures with at most $n \in \mathbb{N}$ supporting points. Note that this number is also determined by the following formula

$$V_{n,r}(\mu) = \inf\left\{\int \min_{a\in\alpha} \|x-a\|^r \ d\mu : \ \alpha \subset \mathbb{R}^d, \mathrm{card}(\alpha) \leq n\right\}, \qquad (1.1)$$

which is more suitable for our purposes (cf. [2, Lemma 3.1]). The efficiency of this approximation can be expressed by the convergence rate of $e_{n,r}(\mu) := (V_{n,r}(\mu))^{1/r}$ tending to 0 as n increases. This leads to the notion of quantization dimension first introduced by ZADOR (cf. [9]). For a Borel probability measure μ on $\mathbb{R}^d$ fulfilling the *moment condition* $\int \|x\|^r \ d\mu < \infty$ the *upper* and *lower quantization dimension of* μ *of order* $r \geq 1$ are defined by

$$\overline{D}_r(\mu) := \limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\mu)}, \quad \underline{D}_r(\mu) := \liminf_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\mu)}.$$

The *upper* and *lower quantization dimension of order infinity* are defined in the same fashion by replacing $e_{n,r}(\mu)$ with the n*-th covering radius* $e_{n,\infty}(\mu)$ given by

$$e_{n,\infty}(\mu) := \inf\left\{\sup_{x\in\mathrm{supp}(\mu)} \min_{a\in\alpha} \|x-a\| : \ \alpha \subset \mathbb{R}^d, \mathrm{card}(\alpha) \leq n\right\},$$

where $\mathrm{supp}(\mu)$ denotes the topological support of the measure μ. Several authors, especially GRAF and LUSCHGY, have treated the quantization dimension systematically (see e.g. [1, 2, 3, 4, 7, 8]).

This paper is organized as follows. In Theorem 2.1 we give as the main result a description of the quantization dimension of finite order in terms of quantization numbers defined in (2.1) below. As a first application of this theorem we solve a question on the upper rate distortion dimension which is left open in [2, p. 163]. As a second application we prove an inequality for the upper quantization dimension of product measures. Finally, we introduce the ε-essential radius of order r of a probability measure to give an upper bound for its quantization dimension by the ε-essential covering rate. An example is included to show that our concept can be used to give a good upper bound for the quantization dimension when the r-th moment is finite but all the $(r+\delta)$-moments, $\delta > 0$, are infinite.

2 Quantization Numbers

For $r \in [1,\infty]$ let us call

$$n_{r,\varepsilon}(\mu) := \inf\{n \geq 1 : \ e_{n,r}(\mu) \leq \varepsilon\} \qquad (2.1)$$

the ε*-quantization number of* μ *of order* r. Note that this quantity has previously been used in the proof of [2, Theorem 11.10]. From [2, Theorem 11.7] we already know that

$$\overline{D}_\infty(\mu) = \limsup_{n\to\infty} \frac{\log n_{\infty,\varepsilon}(\mu)}{-\log\varepsilon}, \quad \underline{D}_\infty(\mu) = \liminf_{n\to\infty} \frac{\log n_{\infty,\varepsilon}(\mu)}{-\log\varepsilon}.$$

It is natural to ask whether analogous equalities also hold for the quantization dimension of finite order. We state the answer in the following theorem.

Theorem 2.1. *Let* $1 \le r < \infty$*, and let* μ *be a Borel probability measure on* $\mathbb{R}^d$ *with* $\int \|x\|^r \, d\mu < \infty$. *Then for* $n_{r,\varepsilon}(\mu)$ *defined as above we have*

$$\overline{D}_r(\mu) = \limsup_{\varepsilon\to 0} \frac{\log n_{r,\varepsilon}(\mu)}{-\log\varepsilon} \text{ and } \underline{D}_r(\mu) = \liminf_{\varepsilon\to 0} \frac{\log n_{r,\varepsilon}(\mu)}{-\log\varepsilon}.$$

The proof of this theorem relies on an elementary observation, stated in the following lemma.

Lemma 2.2. *Let* $(\beta_n)_{n\ge 1}$ *be a non-increasing sequence of non-negative real numbers with* $\lim_{n\to\infty}\beta_n = 0$ *and define* $B(\varepsilon) := \inf\{n \in \mathbb{N} : \beta_n \le \varepsilon\}$. *Suppose either of the two conditions holds.*

1. *There exists* $N \ge 1$ *such that* $\beta_n = 0$ *for all* $n \ge N$.

2. *The sequence* $(\beta_n)_{n\ge 1}$ *is strictly decreasing.*

We then have

$$\limsup_{n\to\infty} \frac{\log n}{-\log\beta_n} = \limsup_{\varepsilon\to 0} \frac{\log B(\varepsilon)}{-\log\varepsilon}, \quad \liminf_{n\to\infty} \frac{\log n}{-\log\beta_n} = \liminf_{\varepsilon\to 0} \frac{\log B(\varepsilon)}{-\log\varepsilon}.$$

PROOF. First suppose that Condition 1 holds. Without loss of generality we assume that N is the smallest integer fulfilling Condition 1. If $N = 1$, then the lemma trivially holds. Otherwise we have $\beta_{N-1} > 0$. Since for any $0 < \varepsilon < \beta_{N-1}$, we have $B(\varepsilon) = N$ and $\beta_N = 0$, the equalities in the lemma hold.

Now we assume that $(\beta_n)_{n\ge 1}$ is strictly decreasing. Then for all $n \in \mathbb{N}$ we have $\beta_n > 0$ and by the definition of $B(\varepsilon)$, we know that

$$\text{(I)} \quad \beta_{B(\varepsilon)} \le \varepsilon, \quad \text{(II)} \quad \beta_{B(\varepsilon)-1} > \varepsilon, \quad \text{and} \quad \text{(III)} \quad B(\beta(n)) = n.$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty}\frac{\log n}{-\log\beta_n} &\overset{\text{by (III)}}{=} \limsup_{\varepsilon\to 0}\frac{\log B(\varepsilon)}{-\log\beta_{B(\varepsilon)}} \overset{\text{by (I)}}{\leq} \limsup_{\varepsilon\to 0}\frac{\log B(\varepsilon)}{-\log\varepsilon}\\ &\overset{\text{by (II)}}{\leq} \limsup_{\varepsilon\to 0}\frac{\log B(\varepsilon)}{-\log\beta_{B(\varepsilon)-1}} = \limsup_{\varepsilon\to 0}\frac{\log(B(\varepsilon)-1)}{-\log\beta_{B(\varepsilon)-1}}\\ &\overset{\text{by (III)}}{=} \limsup_{n\to\infty}\frac{\log n}{-\log\beta_n},\end{aligned}$$

proving the first equality stated in the lemma. Since the argument above also holds if we interchange "$\limsup$" with "$\liminf$", the second equality in the lemma follows. □

PROOF OF THEOREM 2.1. For any Borel probability measure μ fulfilling the moment condition $\int \|x\|^r \, d\mu < \infty$, which is additionally supported on a set with infinite cardinality, we have that $(e_{n,r}(\mu))_{n\in\mathbb{N}}$ is a strictly decreasing sequence converging to zero. This is a direct consequence of [2, Theorem 4.1, Theorem 4.12, Lemma 6.1]). Hence, Condition 2 of Lemma 2.2 is satisfied for this sequence.

If on the other hand $\operatorname{card}\operatorname{supp}(\mu) < \infty$, then clearly Condition 1 of Lemma 2.2 is satisfied. Combining both observations the theorem follows. □

Remark. We remark that the crucial property of strict monotonicity is in general not shared by the sequence of covering radii $e_{n,\infty}(\mu)$. This follows from a simple counter example - the classical Cantor set - where $e_{n,\infty}(\mu) = e_{n-1,\infty}(\mu)$ for infinitely many $n \in \mathbb{N}$.

3 Applications

In this section, we will use the observation of Theorem 2.1 to prove four propositions, stated within the following three subsections. In there we make use of the notion of n-optimal sets. If the infimum in the definition (1.1) of $V_{n,r}(\mu)$ is attained for some set α, then we call α an *n-optimal set of order r*. The collection of all n-optimal sets of order r is denoted by $C_{n,r}(\mu)$. Note that under the moment condition $\int \|x\|^r \, d\mu < \infty$ the set $C_{n,r}(\mu)$ is never empty and that we have $\lim_{n\to\infty} V_{n,r}(\mu) = 0$.

Rate Distortion Dimension

Let us recall a question left open in [2] concerning an upper bound for the upper rate distortion dimension. We start by giving its definition.

Again, let μ be a Borel probability on $\mathbb{R}^d$ with $\int \|x\|^r \, d\mu < \infty$ and Q a Borel probability on $\mathbb{R}^d \times \mathbb{R}^d$. By Q_1, Q_2 we denote the first and second marginal of Q, respectively. If $Q_1 = \mu$, then the *average mutual information* $I(\mu, Q)$ of Q is given by

$$I(\mu, Q) := \int h(x, y) \log h(x, y)\, d(\mu \otimes Q_2)(x, y)$$

whenever Q is absolutely continuous with respect to $\mu \otimes Q_2$ and h is the corresponding Radon-Nikodym derivative, otherwise $I(\mu, Q) := \infty$. Now, the *upper* and *lower rate distortion dimension of order* r of μ are defined to be

$$\overline{\dim}_R(\mu) := \limsup_{\varepsilon \to 0} \frac{R_{\mu,r}(\varepsilon^r)}{-\log \varepsilon}, \quad \underline{\dim}_R(\mu) := \liminf_{\varepsilon \to 0} \frac{R_{\mu,r}(\varepsilon^r)}{-\log \varepsilon},$$

where $R_{\mu,r}(t)$ is the *rate distortion function of order* r defined by

$$R_{\mu,r}(t) := \inf \left\{ I(\mu, Q) :\ Q_1 = \mu, \int ||x - y||^r \, dQ(x, y) \leq t \right\}.$$

KAWABATA and DEMBO proved in [6] that the upper and lower rate distortion dimension do not depend on r and are equal to the upper and lower Rényi information dimension respectively. In [2, Theorem 11.10] it is proved that

$$\underline{\dim}_R(\mu) \leq \underline{D}_1(\mu) \leq \underline{D}_r(\mu).$$

The following proposition covers the corresponding inequalities for the upper rate distortion dimension questioned in [2] and will prove to be a straightforward consequence of Theorem 2.1.

Proposition 3.1. *Let $1 \leq r < \infty$ and let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int \|x\|^r \, d\mu < \infty$. Then we have*

$$\overline{\dim}_R(\mu) \leq \overline{D}_1(\mu) \leq \overline{D}_r(\mu).$$

PROOF. We use the fact from [2, p. 163] that $e_{n,r}(\mu) \leq \varepsilon$ implies $R_{\mu,r}(\varepsilon^r) \leq \log n$. By observing the definition of $n_{r,\varepsilon}(\mu) = \inf\{n \geq 1 :\ e_{n,r}(\mu) \leq \varepsilon\}$ we clearly have $R_{\mu,r}(\varepsilon^r) \leq \log n_{r,\varepsilon}(\mu)$. It follows that

$$\overline{\dim}_R(\mu) \leq \limsup_{\varepsilon \to 0} \frac{\log n_{r,\varepsilon}(\mu)}{-\log \varepsilon}.$$

Thus, the inequalities follow from Theorem 2.1. □

Quantization Dimension of Product Measures

As another application we will give bounds for the upper quantization dimension of product measures. In [2, Lemma 4.15], the authors have already studied the relationship between the n-th quantization error of a random variable on $\mathbb{R}^d$ and that of its one-dimensional marginals, but the quantization dimension of product measures is not considered there.

Let μ_1, μ_2 be Borel probability measures respectively on $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$. Let $\mu := \mu_1 \otimes \mu_2$ be the product measure of μ_1, μ_2 on $\mathbb{R}^{d_1+d_2}$. Especially, we have $\mu(A \times B) = \mu_1(A)\mu_2(B)$ for all measurable sets A and B. Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two arbitrary norms respectively on $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$. For any $w = (x, y) \in \mathbb{R}^{d_1+d_2}$ we define

$$\|w\| := \|x\|_1 + \|y\|_2 .$$

Then $\|\cdot\|$ is a norm on $\mathbb{R}^{d_1+d_2}$. Since, on finite-dimensional spaces, quantization dimensions do not depend on the norms used, we will henceforth adopt the norms introduced above.

Proposition 3.2. *Let $1 \le r < \infty$, and let μ_i be a Borel probability measure on $\mathbb{R}^{d_i}$ satisfying the moment condition $\int \|x\|^r \, d\mu_i < \infty, \ i = 1, 2$. Then*

$$\max\left\{\overline{D}_r(\mu_1), \overline{D}_r(\mu_2)\right\} \le \overline{D}_r(\mu_1 \otimes \mu_2) \le \overline{D}_r(\mu_1) + \overline{D}_r(\mu_2).$$

PROOF. Let $\alpha \in C_{n,r}(\mu_1 \times \mu_2)$ be an n-optimal set of order r and let α_1, α_2 respectively denote the projections of α onto $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$. Then clearly $\alpha \subset \alpha_1 \times \alpha_2$ and $\mathrm{card}(\alpha_i) \le n$, $i = 1, 2$. Using this and the fact that $(A + B)^r \ge A^r + B^r$ for any $A, B \ge 0$ and $r \ge 1$, we have

$$\begin{aligned} V_{n,r}(\mu_1 \otimes \mu_2) &= \int \min_{a \in \alpha} \|w - a\|^r \, d(\mu_1 \otimes \mu_2)(w) \\ &\ge \int \min_{a \in \alpha_1 \times \alpha_2} \|w - a\|^r \, d(\mu_1 \otimes \mu_2)(w) \\ &\ge \int_{\mathbb{R}^{d_1}} \min_{b \in \alpha_1} \|x - b\|_1^r \, d\mu_1(x) + \int_{\mathbb{R}^{d_2}} \min_{c \in \alpha_2} \|y - c\|_2^r \, d\mu_2(y) \\ &\ge V_{n,r}(\mu_1) + V_{n,r}(\mu_2) \ge \max\left\{V_{n,r}(\mu_1), V_{n,r}(\mu_2)\right\}. \end{aligned}$$

Hence, the first inequality follows. To show the second inequality, let $\beta_i \in C_{n_i,r}(\mu_i)$ for $n_1, n_2 \in \mathbb{N}$. We then have $\mathrm{card}(\beta_1 \times \beta_2) = n_1 n_2$ such that by Fubini's theorem we get

$$\begin{aligned} V_{n_1 n_2,\, r}(\mu_1 \otimes \mu_2) \leq & \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \min_{a\in\beta_1\times\beta_2} \|w-a\|^r \; d(\mu_1\otimes\mu_2)(w) \\ \leq & 2^r \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} (\min_{b\in\beta_1} \|x-b\|_1^r + \min_{c\in\beta_2} \|y-c\|_2^r)\, d\mu_1(x)\, d\mu_2(y) \\ = & 2^r \int_{\mathbb{R}^{d_1}} \min_{b\in\beta_1} \|x-b\|_1^r \; d\mu_1 + 2^r \int_{\mathbb{R}^{d_2}} \min_{c\in\beta_2} \|y-c\|_2^r \; d\mu_2 \\ = & 2^r V_{n_1,r}(\mu_1) + 2^r V_{n_2,r}(\mu_2). \end{aligned}$$

It follows that

$$n_{r,\varepsilon 2^{(r+1)/r}}(\mu_1 \otimes \mu_2)) \leq n_{r,\varepsilon}(\mu_1) n_{r,\varepsilon}(\mu_2).$$

Using Theorem 2.1 we conclude

$$\begin{aligned} \overline{D}_r(\mu_1\otimes\mu_2) = & \limsup_{\varepsilon\to 0} \frac{\log n_{r,\varepsilon}(\mu_1\otimes\mu_2)}{-\log\varepsilon} \\ = & \limsup_{\varepsilon\to 0} \frac{\log n_{r,\varepsilon 2^{(r+1)/r}}(\mu_1\otimes\mu_2)}{-\log\varepsilon - ((r+1)/r)\log 2} \\ \leq & \limsup_{\varepsilon\to 0} \frac{\log n_{r,\varepsilon}(\mu_1) + \log n_{r,\varepsilon}(\mu_2)}{-\log\varepsilon} \\ \leq & \overline{D}_r(\mu_1) + \overline{D}_r(\mu_2). \end{aligned}$$

□

An Upper Bound for the Quantization Dimension

Finally, we give an upper bound for the quantization dimension in terms of the ε-essential covering rate of order r which involves the ε-essential radius defined in (3.1) below.

In general, the upper quantization dimension of a Borel probability measure on $\mathbb{R}^d$ is not bounded by d if its support is not compact. This is illustrated by [2, Example 6.4], where the lower quantization dimension equals infinity since the n-th quantization error of order r is comparable with $\log n$. On the other hand, for Borel probability measures μ with $\int \|x\|^{r+\delta} d\mu < \infty$ for some positive δ we have $\overline{D}_r(\mu) \leq d$ (cf. [2, Theorem 6.2]). In particular, if the absolutely continuous part with respect to λ^d does not vanish, we know that $D_r(\mu) = d$. By some straightforward modifications of [2, Example 6.4] it is easy to show that for arbitrary large $s \in [0,\infty]$ there exists a Borel probability measure μ with $\overline{D}_r(\mu) = s$. Therefore, it is significant to examine when the upper quantization dimension is finite or even bounded by d.

We define the *ε-essential radius of μ of order r* by

$$R_{r,\varepsilon}(\mu) := \inf\left\{R: \int_{B(0,R)^C} \|x\|^r \, d\mu < \varepsilon^r\right\}, \tag{3.1}$$

where $B(0,R)$ denotes the closed ball centered at 0 with radius R, and $B(0,R)^C$ denotes its complement. Let $m_{r,\varepsilon}(\mu)$ denote the smallest number of balls with radii ε covering $\operatorname{supp}(\mu) \cap B(0, R_{r,\varepsilon}(\mu))$. The *upper* and *lower ε-essential covering rate* are then respectively defined by

$$\overline{\Delta}_r(\mu) := \limsup_{\varepsilon\to 0} \frac{\log m_{r,\varepsilon}(\mu)}{-\log\varepsilon}, \quad \underline{\Delta}_r(\mu) := \liminf_{\varepsilon\to 0} \frac{\log m_{r,\varepsilon}(\mu)}{-\log\varepsilon}.$$

Proposition 3.3. *Let $1 \le r < \infty$, and let μ be a Borel probability measure on $\mathbb{R}^d$ fulfilling the moment condition $\int \|x\|^r \, d\mu < \infty$. Then we have*

$$\underline{D}_r(\mu) \le \underline{\Delta}_r(\mu), \quad \overline{D}_r(\mu) \le \overline{\Delta}_r(\mu).$$

PROOF. For any $\varepsilon > 0$, by the moment condition, there exists $R > 0$ such that

$$\int_{B(0,R)^C} \|x\|^r \, d\mu < \varepsilon^r.$$

By the continuity of measures and the definition of $R_{r,\varepsilon}(\mu)$, we know that

$$\int_{B(0,R_{r,\varepsilon}(\mu))^C} \|x\|^r \, d\mu \le \varepsilon^r.$$

Let $m_{r,\varepsilon}(\mu) \in \mathbb{N}$ be defined as above. Then there exists a collection of $m_{r,\varepsilon}(\mu)$ balls with radius ε covering $\operatorname{supp}(\mu) \cap B\left(0, R_{r,\varepsilon}\left(\mu\right)\right)$. Let us denote the set of their centers by $\{x_i : \ 1 \le i \le m_{r,\varepsilon}(\mu)\}$ we have

$$\min_{1\le i\le m_{r,\varepsilon}(\mu)} \|x - x_i\| \le \varepsilon,$$

while for any point outside $B(0, R_{r,\varepsilon}(\mu))$ we have

$$\min_{0\le i\le m_{r,\varepsilon}(\mu)} \|x - x_i\| \le \|x - 0\| = ||x||.$$

It follows that

$$\begin{aligned} V_{m_{r,\varepsilon}(\mu)+1,r}(\mu) &\le \int \min_{0\le i\le m_{r,\varepsilon}(\mu)} \|x - x_i\|^r \, d\mu \\ &\le \int_{B(0,R_{r,\varepsilon}(\mu))} \varepsilon^r d\mu + \int_{B(0,R_{r,\varepsilon}(\mu))^C} \|x\|^r \, d\mu \\ &\le \varepsilon^r + \varepsilon^r \le 2\varepsilon^r. \end{aligned} \tag{3.2}$$

By the definition of $n_{r,\varepsilon}(\mu)$, we immediately have $n_{r,2^{1/r}\varepsilon}(\mu) \leq m_{r,\varepsilon}(\mu)+1$ and by Theorem 2.1 we conclude

$$\begin{aligned}\overline{D}_r(\mu) &= \limsup_{\varepsilon\to 0} \frac{\log n_{r,\varepsilon}(\mu)}{-\log\varepsilon} \\ &= \limsup_{\varepsilon\to 0} \frac{\log n_{r,2^{1/r}\varepsilon}(\mu)}{-\log\varepsilon - (1/r)\log 2} \\ &\leq \limsup_{\varepsilon\to 0} \frac{\log m_{r,\varepsilon}(\mu)}{-\log\varepsilon} = \overline{\Delta}_r(\mu).\end{aligned}$$

The inequality for the lower quantization dimension follows immediately by just replacing “$\limsup$” by “$\liminf$”. □

Remark. The inequality (3.2) in the above proof also shows the known fact that under the moment condition $\lim_{n\to\infty} V_{n,r}(\mu) = 0$.

Next, we illustrate Proposition 3.3 by an example.

Example. Let $\mathcal{C}$ be the middle-third Cantor set on $\mathbb{R}$ and ν the classical Cantor measure. Let μ_i, $i \in \mathbb{N}$ be the Cantor measure on the Cantor set $(\mathcal{C}+2^i)$, where $\mathcal{C}+2^i := \{x+2^i : x \in \mathcal{C}\}$, i.e, $\mu_i = \nu \circ S_i^{-1}$, where $S_i : x \mapsto x+2^i$. Let $\mu := \sum_{i=1}^{\infty} s_i\mu_i$, where $s_i := c\cdot(2^i i^{11})^{-r}$ and

$$c := \left(\sum_{i=1}^{\infty}(2^i i^{11})^{-r}\right)^{-1}.$$

Then we have $\int \|x\|^r \, d\mu < \infty$ and $\int \|x\|^{r+\delta} \, d\mu = \infty$ for all $\delta > 0$, but using Proposition 3.3 we get $\overline{D}_r(\mu) < s + 1/10$, where $s = \dim_H \mathcal{C}$. This can be seen as follows.

$$\begin{aligned}\int \|x\|^r \, d\mu &= \sum_{i=1}^{\infty} s_i \int x^r \, d\mu_i = \sum_{i=1}^{\infty} s_i \int_{[0,1]} (x+2^i)^r \, d\nu \\ &\leq 2^r c \sum_{i=1}^{\infty} \frac{1}{i^{11r}} < \infty \\ \int \|x\|^{r+\delta} \, d\mu &= \sum_{i=1}^{\infty} s_i \int x^{r+\delta} \, d\mu_i = \sum_{i=1}^{\infty} s_i \int_{[0,1]} (x+2^i)^{r+\delta} \, d\nu \\ &\geq c \sum_{i=1}^{\infty} \frac{2^{i\delta}}{i^{11r}} = \infty\end{aligned}$$

For any $\varepsilon > 0$, take $k(\varepsilon) := [\varepsilon^{-r/(11r-1)}] + 2$, where $[x]$ denotes the integer part of x. Then it follows that

$$\int_{B(0,2^{k(\varepsilon)}+1)^C} x^r \, d\mu < A^r \varepsilon^r,$$

where $A^r = 2^r c$, implying that $R_{r,A\varepsilon} \le 2^{k(\varepsilon)} + 1$.

On the other hand, for ε small enough, there exists some integer $K \ge 1$ such that

$$3^{-K} \le A\varepsilon < 3^{-K+1},$$

and hence each Cantor set $\mathcal{C} + 2^i$ can be covered by 2^K balls of radii $A\varepsilon$. Combining this observations we get

$$m_{r,A\varepsilon}(\mu) \le k(\varepsilon) \cdot 2^K \le k(\varepsilon) 2^{\frac{-\log(A\varepsilon)}{\log 3}+1}.$$

It follows that $\overline{D}_r(\mu) \le \overline{\Delta}_r(\mu) \le s + 1/10 < 1$.

Proposition 3.3 provides us with an upper bound for the quantization dimension by means of some covering number which is not difficult to calculate in many interesting cases. However, by a careful examination of the proof, we find that we can further refine the upper bound in terms of the quantization number. Let $\mu_{r,\varepsilon}$ be the conditional probability measure $\mu_{r,\varepsilon} = \frac{\mu(\,\cdot\, \cap B(0,R_{r,\varepsilon}(\mu)))}{\mu(B(0,R_{r,\varepsilon}(\mu)))}$ and write

$$\overline{\ell}(\mu) := \limsup_{\varepsilon \to 0} \frac{\log n_{r,\varepsilon}(\mu_{r,\varepsilon})}{-\log \varepsilon}, \ \ \underline{\ell}(\mu) := \liminf_{\varepsilon \to 0} \frac{\log n_{r,\varepsilon}(\mu_{r,\varepsilon})}{-\log \varepsilon},$$

where $n_{r,\varepsilon}(\mu_{r,\varepsilon})$ is the ε-quantization number of order r of $\mu_{r,\varepsilon}$. Clearly, we have $n_{r,\varepsilon}(\mu_{r,\varepsilon}) \le m_{r,\varepsilon}(\mu)$.

Proposition 3.4. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int \|x\|^r \, d\mu < \infty$. Then*

$$\overline{D}_r(\mu) \le \overline{\ell}(\mu), \ \ \underline{D}_r(\mu) \le \underline{\ell}(\mu).$$

PROOF. For any $\varepsilon > 0$ and each $n \in \mathbb{N}$ note that

$$\int_{B(0,R_{r,\varepsilon}(\mu))} \min_{1\le i\le n} \|x - a_i\|^r \, d\mu \le \int_{B(0,R_{r,\varepsilon}(\mu))} \min_{1\le i\le n} \|x - a_i\|^r \, d\mu_{r,\varepsilon}.$$

Thus, we have $n_{r,2^{(1/r)}\varepsilon}(\mu) \le n_{r,\varepsilon}(\mu_{r,\varepsilon}) + 1$ and the corollary follows from Theorem 2.1. $\square$

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