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FIRST-RETURN LIMITING NOTIONS IN REAL ANALYSIS

Abstract

During the past decade several authors have explored some ramifications of borrowing the "first-return" notion from the study of the dynamics of chaotic maps on an interval and applying the concept to the core areas of real analysis: approximation, continuity, differentiation, and integration. This survey attempts to summarize the results obtained and to point toward possible future directions for exploration.

1 Introduction

It has been roughly a decade since the appearance of the first paper [36] exploring the applications the "first-return" notion, borrowed from the study of the dynamics of chaotic maps on an interval, to one of the core areas of classical real analysis, namely to the theory of differentiation. In the following ten years several authors have fruitfully pursued applications of this concept, not only to differentiation, but also approximation, continuity, and integration. The purpose of this survey is to present a summary of the results obtained. The presentation does not reflect the chronological order of the discoveries, but rather attempts to organize results in a logical framework that will be beneficial to a reader looking to gain insight into this area of investigation.

The pronoun "we" will be used frequently and liberally in this survey. This is because many of the works referenced here are projects of at least one of the

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two authors of this paper, often in collaboration with a subset of a collection of our colleagues, Udayan Darji, Paul Humke, and Chris Freiling. Thus, instead of listing which particular subset of authors proved a particular theorem or considered a particular problem, we will immodestly take advantage of our longtime friendship with these folks and simply use the pronoun "we." We trust that our citations and bibliography sufficiently identify the appropriate antecedent.

1.1 Underlying Definitions and Notation

To give an accessible background in which it becomes clear how the ideas of chaotic trajectories concatenate nicely with historical developments in classical real analysis will require a few definitions and notational conventions. Throughout this survey we shall use I to denote the interval $[0,1]$ and most of the functions we consider will be real-valued with domain I. We shall use Z to denote the collection of all nondegenerate closed subintervals of I. Any real-valued function defined on Z will be called an interval function. Various notions of convergence of intervals can be formulated. Here, we find it useful to say that the sequence $\{I_n\}_{n=0}^{\infty}$ of distinct elements in Z converges to a point x_0 if the sequence of lengths $\{|I_n|\}_{n=0}^{\infty}$ converges to 0 and $\bigcap_{n=0}^{\infty} I_n = \{x_0\}$. An interval function s will be labeled convergent to a real-valued function g provided that for all x_0 we have $\{s(I_n)\}_{n=0}^{\infty}$ converges to $g(x_0)$ for all $\{I_n\}_{n=0}^{\infty}$ convergent to x_0. Any interval function s which satisfies $a < s([a,b]) < b$ on Z is convergent and will be called a selection on I; furthermore, $s(H)$ will be called the preferred point in H [34]. The mean value theorem applied to any differentiable function F on I provides an easy example of a selection being used to construct a nontrivial interval function convergent to a derivative F'.

Also underlying all of our subsequent definitions is the notion of what we shall call a *trajectory*. A *trajectory* is any sequence $\overline{x} = \{x_n\}_{n=0}^{\infty}$ of distinct points in I, whose range is is dense in I. Any countable dense subset D of I is called a support set and, of course, any enumeration of D becomes a trajectory. Let $\overline{x} = \{x_n\}$ be a fixed trajectory. Here is how we will choose a preferred point, based on that trajectory: for a given interval, or finite union of intervals, $H \subseteq [0,1]$, $r(\overline{x}, H)$ (or simply $r(H)$ if $\overline{x}$ is understood) will be the first element of the trajectory $\{x_n\}$ in H; this is called the first-return selection of $\overline{x}$ to H. It is clear that the first-return selection is a special example of a selection. It is worth noting that first-return selections have the following property that we label hereditary. For any interval, $[a,b] \in Z$ let $p = r([a,b])$; then for any interval $[c,d] \in Z : a < c < p < d < b\}$ we have $r([c,d]) = p$. The reader should observe that the hereditary property on a selection insures that the range of the selection function will be countable (Consider the points selected in intervals with rational endpoints.) and clearly dense. In other words, the range of an hereditary selection function can serve as a support set D.

1.2 Background

In the later part of the nineteenth century, there was a belief in the deterministic, clockwork precision of the universe. From this belief arose an interest in establishing the stability of the planetary motions of our solar system. Oscar II, King of Sweden and Norway, initiated a mathematical competition in 1887 to celebrate his sixtieth birthday in 1889. One of the problems, proposed by Karl Weierstrass, dealt with this stability: "Given a system of arbitrarily many mass points that attract each other according to Newton's laws, assuming that no two points ever collide, give the coordinates of the individual points for all time as the sum of a uniformly convergent series whose terms are made up of known functions."

Henri Poincaré attacked this problem for our solar system but found this too complicated to solve. He switched to a three body system and employed a technique, which later became to be called inserting a Poincaré section. This technique involved inserting a section, or plane, transverse to the motion of one of the masses. The intersections of the mass's trajectory with this plane defines a discrete time subtrajectory that he could study. Indeed, these intersections with the section form a sequence of "first-returns" providing one motivation for the study we have subsequently undertaken and describe in this survey. From his analysis Poincaré first submitted a solution to the 3 body problem. In 1888 he wrote to Mittag-Leffler:

> in this particular case I have found a rigorous demonstration of stability and a method of placing limits on the elements of the third body.

He was awarded the prize but in the publication review of the work, Phragmen found an error. The original manuscript had to be discarded and Poincaré came to understand that a concept which we now call sensitive dependence on initial conditions was involved. Stated in terms that apply to our study here, it is the lack of complete information about initial conditions and the pointwise, rather than uniform, convergence of functions defined by series that rendered impossible the long term prediction of the third body motion. In particular, Poincaré noted that the lack of complete information and nonuniform convergence allows the the motion to be recurrent but not periodic and leads to the realization that the first-returns can be dense. He rewrote the paper outlining these new observations and was still awarded the prize. (Readers wishing to learn more about this competition or more about Poincaré may wish to consult [3] or visit the University of St. Andrews MacTutor History of Mathematics website: `www-history.mcs.st-andrews.ac.uk/history/`.) These events may also indicate why in his introduction to Science and Method [39] Poincaré quotes Tolstoy:

> We cannot know all the facts, since they are practically infinite in number. We must make a selection, and that being so, can this selection be governed by the mere caprice of our curiosity. Is it not better to be guided by utility, by our practical, and more especially our moral necessities.

Applying this advice to the study of local behavior of any real function defined on an interval would lead us naturally to consider a restriction of our observation to less than the whole domain. Furthermore, since we wish to draw conclusions on the local behavior of the function, the minimal condition would be to make a selection, or choice, of one preferred point in each nondegenerate subinterval of our domain. This desire still leaves open the question of how to make this choice of preferred point meaningfully in the study of real functions. In this we can rely on the work of other contemporaries of Poincaré.

Throughout the late 1800's and early 20th century it can be said that the implications of the fact that a pointwise limit of continuous functions can yield a function that is "less than continuous" was the study of a multitude of researchers. These pointwise limits of continuous functions are called Baire class one functions. Researchers such as Darboux (who was a member of Poincaré's doctoral thesis committee), Denjoy, Khintchine, Lebesgue, Lusin, and Borel sought ways to understand and characterize the behavior of derivatives, approximately continuous functions and other subclasses of Baire one functions. An important result was obtained by A. Gleyzal [29]. He proved that there is an equivalence between the class of limits of convergent interval functions and Baire one functions. As mentioned above, an application of this theorem is to derivatives. For any interval, the difference quotient of the primitive defines the desired interval function. Moreover, the Mean Value Theorem provides an immediate method to select a preferred point in the interval where the value of the derivative is this difference quotient. This idea was noted and extended by C. J. Neugebauer [33] to Baire one, Darboux functions. He said that a function $f : I \to \mathbb{R}$ has property C_1 if there is a selection s such that, for every $x_0 \in I$, we have that $\{I_n\}_{n=0}^{\infty}$ of distinct elements in Z converging to a point x_0 implies that $\{f(s(I_n))\}$ converges to $f(x_0)$. He then showed that functions having the C_1 property are precisely the Baire one, Darboux functions. This realization that a Baire one, Darboux function can be determined at each point by knowing its values at only the selected points, along with the other ideas noted in this section, paved the way for us to launch the investigation described in the sequel.

2 The Sequential Essence of Baire Class One Functions

A continuous function $f : I \to I$ is, of course, uniquely determined by its values on any countable dense set in I. The core question to be examined in this section is whether a Baire class one function is also completely determined by its values on a countable dense set, and if so, how to determine that countable dense set and how to compute its value at any point in I. In brief, the answer is that a Baire class one function is uniquely determined by its values on some countable dense set and the algorithm for computing values everywhere is extremely simple. Of course, unlike the situation for continuous functions, not just any countable dense set will work for a Baire class one function. However, if a function is Baire one, there are certain countable dense sets which carry enough information to permit computation of the value of the function at any point, provided that the countable dense set is carefully enumerated into a useful sequence. Thus, a Baire class one function appears to have what we heuristically have called a "sequential essence" in the title of this section. A secondary, but fascinating, issue is how well the algorithm for computing the function can mimic the behavior of the function. This issue, which we refer to as the "fineness" of the algorithm will be made more precise as we continue. Along the way we shall see that new characterizations are obtained for the class of Baire one functions, as well as for several of its most familiar subclasses.

2.1 First-Return Recoverability

If one knows the value of a continuous function at each point in a trajectory, then there is a very natural algorithm for computing the value of the function at any point in I. This algorithm we call first-return recovery and define it as follows. Here $B_\rho(x) \equiv \{y \in [0,1] : |x - y| < \rho\}$, for $x \in [0,1]$, and $\rho > 0$.

Definition 2.1.1. Let $x \in I$ and let $\{x_n\}$ be a fixed trajectory. The *first return route to* x, $\mathcal{R}_x = \{w_k\}_{k=1}^{\infty}$, is defined recursively via

$$w_1 = x_0,$$

$$w_{k+1} = \begin{cases} r\left(B_{|x-w_k|}(x)\right) & \text{if } x \neq w_k \\ w_k & \text{if } x = w_k. \end{cases}$$

We say that f is *first return recoverable with respect to* $\{x_n\}$ at x provided that

$$\lim_{k\to\infty} f(w_k) = f(x),$$

and if this happens for each $x \in I$, we say that f is *first return recoverable with respect to* $\{x_n\}$. Finally, we say that f is *first-return recoverable* if it is first-return recoverable with respect to some trajectory.

As alluded to above, it is easy to see that a function $f : I \to \mathbb{R}$ is continuous if and only if it is first-return recoverable with respect to *every* trajectory. Upon changing *every* to *some*, we were able to establish the following result, which adds to the list of attractive characterizations of the class of Baire one functions.

Theorem 2.1.1. *[14] A function $f : I \to \mathbb{R}$ belongs to Baire class one if and only if f is first return recoverable.*

Before continuing to the next section, we pause to note two improvements on this result. First, it holds in a more general setting, if we interpret Baire class one to mean Borel class one in the metric space setting:

Theorem 2.1.2. *[6] If X is a compact metric space and Y is a separable metric space, then a function $f : X \to Y$ belongs to Borel class one if and only if f is first return recoverable.*

Next, a theorem due to Chris Freiling and Robert Vallin shows that there is a certain flexibility in selecting the trajectory to recover a Baire one function; indeed, given a countable collection of Baire one functions, there is a trajectory that will serve to recover them all.

Theorem 2.1.3. *[28] Given a countable collection of real-valued Baire class one functions on I, there is a trajectory with respect to which all of the functions in the collection are first-return recoverable.*

Lastly, we note that it would, of course, be nice to know if Baire class two is amenable to a characterization based on some weakening of first-return recovery. We have yet to find such a characterization. Instead, we next turn to strengthenings of first-return recovery.

2.2 First-Return Approachability

Note that if a Baire one function is such that the point $(z, f(z))$ is an isolated point in the graph of f, then z will have to be in any trajectory used to recover f. In this case the *first return route to* z, $\mathcal{R}_z$, is eventually constantly z. A natural question is whether or not a function with no isolated points on its graph can be recovered in such a way that the chosen sequence approaching each point is not eventually constant. Here is a definition, making this more precise, and a theorem, showing a positive answer to this question.

Definition 2.2.1. For each $x \in [0,1]$ the *first return approach to x based on* $\{x_n\}$, $\mathcal{A}_x = \{u_k\}$, is defined recursively via

$$u_1 = r((0,1) \setminus \{x\}), \text{ and } u_{k+1} = r(B_{|x-u_k|}(x) \setminus \{x\}).$$

We say that f is *first return approachable at x with respect to the trajectory* $\{x_n\}$ provided

$$\lim_{\substack{u \to x \\ u \in \mathcal{A}_x}} f(u) = f(x).$$

We say that f is *first return approachable with respect to* $\{x_n\}$ provided it is first return approachable with respect to $\{x_n\}$ at each $x \in [0,1]$. Likewise, f is said to be *first return approachable* provided there exists a trajectory with respect to which f is first return approachable.

We have the following characterization.

Theorem 2.2.1. *[10] The function $f : I \to \mathbb{R}$ is Baire one with no isolated points on its graph if and only if f is first return approachable.*

2.3 First-Return Continuity

Continuing the theme started in the previous subsection, we next consider the Baire one, Darboux functions; that is, those Baire one functions having no points on the graph which are isolated from even one side. Here again, the results are pleasingly positive.

Definition 2.3.1. Let $\{x_n\}$ be a trajectory. For $0 < x \leq 1$, the *left first return path to x based on* $\{x_n\}$, $\mathcal{P}_x^l = \{t_k\}$, is defined recursively via

$$t_1 = r(0,x), \text{ and } t_{k+1} = r(t_k, x).$$

For $0 \leq x < 1$, the *right first return path to x based on* $\{x_n\}$, $\mathcal{P}_x^r = \{s_k\}$, is defined analogously. We say that f is *first return continuous from the left [right] at x with respect to the trajectory* $\{x_n\}$ provided

$$\lim_{\substack{t \to x \\ t \in \mathcal{P}_x^l}} f(t) = f(x) \quad \left[\lim_{\substack{s \to x \\ s \in \mathcal{P}_x^r}} f(s) = f(x)\right].$$

We say that for any $x \in (0,1)$, f is *first return continuous at x with respect to the trajectory* $\{x_n\}$ provided it is both left and right first return continuous at x with respect to the trajectory $\{x_n\}$. (For $x = 0$ or $x = 1$ we only require the appropriate one-sided first-return continuity.)

We say that f is *first return continuous with respect to* $\{x_n\}$ provided it is first return continuous with respect to $\{x_n\}$ at each $x \in [0,1]$. Likewise, f is said to be *first return continuous* provided there exists a trajectory with respect to which f is first return continuous.

The Baire one, Darboux functions turn out to be precisely the first-return continuous functions.

Theorem 2.3.1. *[11] A function $f : I \to \mathbb{R}$ is Baire one, Darboux if and only if it is first-return continuous.*

Let us pause here to note that while many of the results in this survey quite readily to real-valued functions defined on $\mathbb{R}^n$, the natural extension of Theorem 2.3.1 has eluded us. For example, what is the best notion of first-return continuity for a real-valued function defined on the square? Will it characterize the connected Baire one functions?

2.4 Fineness

With the notions of the three previous subsections in mind, one might now ask if given a Baire one function, is it possible to find a trajectory, $\{x_n\}$, with respect to which f will not only be first-return recoverable but also first-return continuous at each point x for which $(x, f(x))$ is isolated on neither side in the graph and first-return approachable at each point x for which $(x, f(x))$ is isolated on exactly one side in the graph? Again, the answer is yes; a trajectory can be chosen to "finely" recover the function in this manner. We need to pin down some terminology to make our statement precise.

Definition 2.4.1. Given an $f : I \to \mathbb{R}$, the type 1 points are those $x \in (0,1)$ for which $(x, f(x))$ is isolated on neither the left nor the right. (The point $x = 0$ [$x = 1$] will be a type 1 point if $(0, f(0))$ [$(1, f(1))$] is not isolated on the right [left].) Type 2 points are those $x \in (0,1)$ for which $(x, f(x))$ is isolated on exactly one side. (The points 0 and 1 are never considered type 2.) Type 3 points are those $x \in I$ for which $(x, f(x))$ is isolated. We use the notation $T_i(f)$ to denote the set of type i points for f, $i = 1, 2, 3$.

Definition 2.4.2. We say that f is *finely recoverable with respect to the trajectory* $\{x_n\}$ provided that, with respect to $\{x_n\}$, f is first-return recoverable *and* first-return continuous at each point in $T_1(f)$ as well as first-return approachable at each point in $T_2(f)$. We say that f is *finely recoverable* if there exists a trajectory $\{x_n\}$ such that f is finely recoverable with respect to $\{x_n\}$.

With this notation we have the following refinement of Theorem 2.1.1.

Theorem 2.4.1. *[24] A function $f : I \to \mathbb{R}$ belongs to Baire class one if and only if f is finely recoverable.*

The following lemma was used to establish this result and, as we shall see in Section 6.1, has some independent interest.

Lemma 2.4.1. *[24] If $f : I \to \mathbb{R}$ is Baire one, then there exists a trajectory $\{x_n\}$ such that for each $x \in I$, except the x_n's, f is first-return continuous at x with respect to the $\{x_n\}$.*

Theorems 2.1.1 and 2.4.1 indicate that there is no difference between being first-return recoverable and being finely recoverable. Note that this is different from saying that that there is no difference between being first-return recoverable with respect to a particular trajectory $\{x_n\}$ and being finely recoverable with respect to that same trajectory $\{x_n\}$. Indeed, the latter is not true. An example was constructed in [10] to illustrate this:

Example 2.4.1. [10] There is a Baire one function $f : I \to I$ such that $(x, f(x))$ is isolated on neither the left nor the right for all but countable many $x \in I$ and a trajectory $\{x_n\}$ such that

(a) f is first-return approachable with respect to $\{x_n\}$.

(b) If $\{v_n\}$ is any rearrangement of $\{x_n\}$ (including the original $\{x_n\}$ itself), then the set B of points at which f fails to be first-return continuous with respect to $\{v_n\}$ is perfectly dense in I; i.e., every open subinterval of I contains a perfect subset of B.

On the other hand, the set of points at which a function is first-return approachable with respect to a given trajectory, but not first return continuous with respect to that same trajectory must be small in both measure and category; indeed, it must be σ-porous, as shown by the following:

Theorem 2.4.2. *[10] Let $f :\to \mathbb{R}$ and let $\{x_n\}$ be a trajectory. The set of points at which f is first-return approachable with respect to $\{x_n\}$ but not first-return continuous with respect to $\{x_n\}$ is a σ-porous set.*

As a foretaste of the upcoming section on consistency and universality, let us give some thought as to how much freedom one has in identifying a trajectory to either recover or finely recover a given Baire one function. We can view the identification of any trajectory as a two-step process: First, we identify a countable set D which is dense in I, i.e., a support set D. Then we enumerate D into a trajectory, $\{x_n\}$.

> If recovery is the goal, it turns out that there is a great deal of freedom in selecting D; ordering is the tricky part. If the goal is fine recovery, more care must be applied in selecting D, itself.

Note that if a function f is going to be first-return recoverable with respect to an enumeration of a given support set D, then clearly the graph of $f|D$ must be dense in the entire graph. For a Baire one function, that condition is actually sufficient; indeed, the recovery can be made somewhat "finely," as the following theorem illustrates.

Theorem 2.4.3. *[10] Let $f : I \to \mathbb{R}$ be a Baire one function and let D be a support set with the property that the graph of $f|D$ is dense in the graph of f. Then there is an enumeration of $\{x_n\}$ of D such that*

(a) f is first-return recoverable with respect to $\{x_n\}$.

(b) If $(x, f(x))$ is not an isolated point on the graph of f, then f is first-return approachable at x with respect to $\{x_n\}$.

If we seek fine recovery for f, the situation is more complex and centers on the set $T_2(f)$. For example, one positive result is the following:

Theorem 2.4.4. *[9] If $f : I \to \mathbb{R}$ is a Baire one function and D is a support set such that $T_2(f) \subseteq D$ and the graph of $f|D$ is dense in the graph of f, then there is an enumeration of D with respect to which f is finely recoverable.*

Let us give a name to those functions for which the recovery in Theorem 2.4.3 can be fine.

Definition 2.4.3. We say that a function $f : I \to \mathbb{R}$ is *always finely recoverable* provided that whenever a support set D has the property that the graph of $f|D$ is dense in the graph of f, then there is an enumeration $\{x_n\}$ of D such that f is finely recoverable with respect to $\{x_n\}$.

It turns out that if $T_2(f)$ is a scattered set, then f is always finely recoverable, and, therefore, that any Baire one, Darboux function is always finely recoverable. However, the scatteredness of $T_2(f)$ is a little too strong for a characterization. We need to get a little more technical.

Definition 2.4.4. For a given $f : I \to \mathbb{R}$ and for each natural number n we set

$$E_n \equiv \{x : \max\{\liminf_{y \to x^-} |f(y) - f(x)|, \liminf_{y \to x^+} |f(y) - f(x)|\} < 1/2^n\},$$

where the two-sided condition is reduced to the appropriate one-sided condition at the endpoints 0 and 1, and

$$F_n \equiv \backslash E_n.$$

The characterization for always finely recoverable functions is

Theorem 2.4.5. *[9] A function $f : I \to \mathbb{R}$ is always finely recoverable if and only if $F_n \cap T_2(f)$ is a scattered set for each natural number n.*

2.5 Consistency and Universality

As noted in Subsection 2.1, a function $f : I \to \mathbb{R}$ is continuous if and only if it is first-return recoverable with respect to *every* trajectory, and upon changing *every* to *some*, we obtain a characterization of the class of Baire one functions. What about intermediate conditions? To motivate what we have in mind, it might be helpful to recast these two results, using the notion of support sets:

> A function $f : I \to \mathbb{R}$ is continuous if and only if for every support set D and for every enumeration $\{x_n\}$ of D, f is first-return recoverable with respect to $\{x_n\}$. On the other hand, a function $f : I \to \mathbb{R}$ is of Baire class one if and only if for some support set D and for some enumeration $\{x_n\}$ of D, f is first-return recoverable with respect to $\{x_n\}$.

Thus, two intermediate properties suggest themselves. First, we might consider the class of functions having the property that there exists a support set D with the property that the function is first-return recoverable with respect to every ordering of D. Next, we might look for the class of functions having the property that for every support set D there is an enumeration $\{x_n\}$ of D, f is first-return recoverable with respect to $\{x_n\}$. Let's focus on the former first.

Definition 2.5.1. Let D be a support set. We shall say that D *consistently recovers* a function f provided that f is first-return recoverable with respect to every ordering of D. A function is said to be consistently recoverable if there exists a support set D which consistently recovers f.

Consistently recoverable functions are not far removed from being continuous as the following characterization shows.

Theorem 2.5.1. *[23] A function $f : I \to \mathbb{R}$ is consistently recoverable if and only if it is continuous at all but countably many points.*

Let us now turn to functions satisfying the second intermediate property mentioned above. Notice that such a function cannot possibly have an isolated point on its graph, and thus, by Theorem 2.2.1 the class of such functions is going to be a subclass of the first-return approachable functions. Therefore, we define terms as follows:

Definition 2.5.2. A function $f : I \to \mathbb{R}$ is called *universally first-return approachable* if for every support set D there is an enumeration $\{x_n\}$ of D, f is first-return approachable with respect to $\{x_n\}$.

It turns out that in order for a Baire one function to be universally first-return approachable, not only must the graph have no isolated points, but the graph of the function restricted to its set of points of continuity, $C(f)$, must be dense in the whole graph. That is, the function needs to be *quasicontinuous* in the sense of Kempisty[30]:

Definition 2.5.3. A function $f : I \to \mathbb{R}$ is *quasicontinuous at* x if every neighborhood of $(x, f(x))$ contains a point of the graph of $f|C(f)$. We let $Q(f)$ denote the set of points of quasicontinuity of f and $NQ(f) = [0,1] \setminus Q(f)$. If $Q(f) = I$, we say that f is quasicontinuous.

Theorem 2.5.2. *[10] A function $f : I \to \mathbb{R}$ is universally first-return approachable if and only if it is Baire one and quasicontinuous.*

If we define universal first-return continuity in the following natural manner, then we obtain an analogous theorem.

Definition 2.5.4. A function $f : I \to \mathbb{R}$ is called *universally first return continuous* if for every support set D there is an enumeration $\{x_n\}$ of D, f is first-return continuous with respect to $\{x_n\}$.

Theorem 2.5.3. *[12] A function $f : I \to \mathbb{R}$ is universally first-return continuous if and only if it is Baire one, Darboux, and quasicontinuous.*

We close the discussion of universal type notions with the following enhancement of the previous theorem.

Theorem 2.5.4. *[12] For a function $f : I \to \mathbb{R}$ the following are equivalent:*

(a) f is Baire one, Darboux, and quasicontinuous.

(b) f is universally first-return continuous.

(c) There exists a trajectory $\{x_n\} \subset (0,1) \cap C(f)$ such that f is first-return continuous with respect to $\{x_n\}$, where $C(f)$ denotes the set of continuity points of f.

(d) *There exists an increasing homeomorphism h of I onto itself such that $f \circ h$ is continuous in the a.e. topology.*

Finally, we could return to the first type of intermediate property and define consistently first-return approachable and consistently first-return continuous functions in the obvious manner. However, a slight modification of the proof given for Theorem 2.5.1 in [23] shows that such functions must be everywhere continuous!

2.6 A Unifying Theorem

The proofs for each of the theorems 2.1.1, 2.2.1, 2.3.1, 2.4.1, 2.4.3, 2.4.4, 2.4.5, 2.5.2, and 2.5.3 given in the articles cited in the theorem statements have been rather involved and quite distinct from one another. For example, the proof of Theorem 2.3.1 in [11] utilizes the powerful Maximoff-Preiss Theorem ([32], [40]), while those for 2.1.1, 2.2.1, and 2.4.1 entail intricate decomposition arguments, each somewhat different from the others. The goal of [9] was to prove one theorem which would have all of these theorems (or at least the more challenging direction in the case of the characterizations) as fairly immediate corollaries. The following theorem is such a result. Despite its daunting appearance, it is easily applied to yield the results listed above.

Theorem 2.6.1. *[9] Suppose $f : I \to \mathbb{R}$ is a Baire one function, $D = \{d_n\}_{n=1}^{\infty}$ is a support set such that the graph of $f|D$ is dense in the graph of f, $\{A_n\}$ is a decreasing sequence of F_σ sets, and $\{B_n\}$ is an increasing sequence of F_σ sets such that for each n, $A_n \subseteq E_n \cup D$, $d_n \in A_n \cup B_n$, $A_n \cap B_n = \emptyset$, and $A_{n+1} \cup B_{n+1} \subseteq A_n \cup B_n \subseteq I$. Then there is an enumeration of D with respect to which f is first-return continuous at every point of $T_1(f) \cap (\bigcap_{n=1}^{\infty} A_n)$, and first-return approachable at every point of $(\bigcup_{n=1}^{\infty} B_n) \setminus T_3(f)$.*

To see this theorem in action, consider how it is used to prove the "if" direction of Theorem 2.4.4: For each n let $B_n = F_n \cap T_2(f)$ and $A_n = I \setminus B_n$. Since each $F_n \cap T_2(f)$ is scattered, each is both an F_σ and a G_δ. Hence each A_n is an F_σ and each B_n is an F_σ, Now apply the theorem. To see how to apply the theorem to obtain all the above listed results, the interested reader may consult [9].

3 Differentiation

It is somewhat ironic that we have opted not to discuss first-return differentiation first in this survey since it was our long-term interest in differentiation theory that led to our original fascination with first-return analysis and since the first results in the theory were related to differentiation. We have chosen to base the order of presentation on an intuitive flow of concepts rather than on the chronology of their development. Another reason for postponing the differentiation results has been to allow us to set the stage for what we found to be one of the more remarkable results in this area, a result of Freiling, which we shall take up shortly. First, the definition.

3.1 The First-Return Derivatives

The first application of the first-return selection method was with respect to differentiation in [36], where the first-return derivative of a function $f : I \to \mathbb{R}$ was defined as follows.

Definition 3.1.1. Let $\{x_n\}$ be a given trajectory. Then, using the notation of Section 2.3, we define the first-return path to $x \in I$ by

$$\mathcal{P}_x = \mathcal{P}_x^l \cup \mathcal{P}_x^r,$$

where we take $\mathcal{P}_0^l = \mathcal{P}_1^r = \emptyset$. Finally, if each $x \in I$

$$\lim_{\substack{t \to x \\ t \in \mathcal{P}_x}} \frac{f(t) - f(x)}{t - x} = g(x),$$

then we say that f is *first-return differentiable to g with respect to the trajectory* $\{x_n\}$. If f and g are functions on I and there exists a trajectory $\{x_n\}$ for which f is first return differentiable to g on $[0, 1]$, then we say that f is first-return differentiable to g and call g a first-return derivative of f.

Among other things, the following results were established in [36].

Theorem 3.1.1. *If $f : I \to \mathbb{R}$ has the finite function g as a first-return derivative on I, then f is a Baire*-one, Darboux function and g is a Baire-one, Darboux function.*

Thus, first-return derivatives share properties with ordinary and most generalized derivatives. However, as noted in [36], a function f *can* have more than one first-return derivative on I. Nonetheless, the first-return derivative is essentially unique and independent of trajectory as the following result from [36] indicates.

Theorem 3.1.2. *Let $\{x_n\}$ and $\{t_n\}$ be two trajectories and suppose that a function $f : I \to \mathbb{R}$ has g as a first-return derivative with respect to $\{x_n\}$ and h as a first-return derivative with respect to $\{t_n\}$. Then $\{x \in I : g(x) \neq h(x)\}$ is countable.*

3.2 Relationships to Other Generalized Derivatives

Theorem 3.1.1 opened the avenue of inquiry into whether certain other useful generalized derivations could be achieved via this method of differentiation. In particular, we became curious as to whether finite approximate and Peano derivatives are first-return derivatives. It seemed a daunting task to find a trajectory that would yield the approximate derivative, for example. However, as sometimes happens in mathematics, it was easier to prove a more general result, namely the following [11].

Theorem 3.2.1. *Let $f : I \to \mathbb{R}$ be compositely differentiable to $g : I \to \mathbb{R}$ and suppose that for each $x \in I$, $g(x)$ is a bilerateral derived number of f at x. Then f is first-return differentiable to g on I.*

O'Malley and Weil coined the term composite differentiation in [37] and the reader not familiar with this concept is referred to that paper for the definition and properties. O'Malley showed that an approximate derivative is both a composite derivative [35] and a selective derivative [34]. The latter implies that the bilateral condition of the hypotheses for Theorem 3.2.1 is satisfied. Likewise, H. Fejzić [25] has shown that each k^{th} Peano derivative of a function f is both a composite derivative and a selective derivative of the $(k-1)^{\text{th}}$ Peano derivative of f. Subsequently, Fejzić [26] superseded both of these results by showing that each approximate k^{th} Peano derivative of a function f is both a composite derivative and a selective derivative of the $(k-1)^{\text{th}}$ approximate Peano derivative of f. Likewise, Ewa Łazarow and Aleksander Maliszewski [31] have shown that the $\mathcal{I}$-approximate derivatives are composite derivatives. With these results, we immediately obtain the following corollary to Theorem 3.2.1.

Corollary 3.2.1. *Every approximate derivative, every $\mathcal{I}$-approximate derivative, every Peano derivative of every order, and, indeed, every approximate Peano derivative of every order is a first-return derivative.*

A deeper problem is to determine precisely which selective derivatives are first-return derivatives. Some partial results along these lines as well as relationships between first-return and other types of generalized derivatives can be found in [7] and [15].

3.3 Consistency and Universality

Just as we defined and explored *consistently* and *universally* first-return recoverable, first-return approximable, and first-return continuous functions, it seems natural to define consistent and universal versions of first-return differentiability. That is, we could make the following definitions.

Definition 3.3.1. If f and g have the property that there is a support set D such that for every ordering $\{x_n\}$ of D, f is first-return differentiable to g with respect to $\{x_n\}$, then we say that f is *consistently first-return differentiable to* g and call g a consistent first-return derivative of f.

Definition 3.3.2. If f and g have the property that for every support set D there is an ordering $\{x_n\}$ of D such that f is first-return differentiable to g with respect to $\{x_n\}$, then we say that f is *universally first return differentiable to* g and call g a universal first-return derivative of f.

Alas, it is not at all hard to modify the technique of proof used for Theorem 2.5.1 in [23] to show that if f is consistently first-return differentiable to g, then $g = f'$, the ordinary derivative of f.

The universal notion is considerably more exciting. Based on our experience with approximability and continuity, we assumed that this notion would be richer than first-return differentiability. In particular, when we proved Theorem 3.2.1 and, consequently, Corollary 3.2.1, the conclusion was actually that f has the appropriate g as a *universal* first-return derivative. For some time the relationship between first return differentiability and universal first-return differentiability was unknown. The following elegant result of Chris Freiling [27] settled the problem.

Theorem 3.3.1. *If f has g as a first-return derivative, then f is universally first-return differentiable to g.*

4 Is the Term *Trajectory* Warranted?

Before proceeding to the section on integration, we pause here to explain why we have chosen to refer to any sequence $\{x_n\}$ having a dense range in I as a *trajectory*. The term originally surfaced in our discussions with one another because the types of sequences that we had in mind, were, indeed, *trajectories*, i.e., the sequence of iterates $\{g^n(z)\}$ of a transitive continuous function $g : I \to I$ at some point z having a dense orbit or trajectory. However, in the context of this survey only two things are necessary to create a trajectory: a support set D and an enumeration of D into a sequence $\{x_n\}$ of distinct points. For any such D it is easy to create enumerations $\{x_n\}$ of D having the property that there is no continuous function g such that $g^n(x_0) = x_n$ for all n. Therefore, it is plausible to anticipate that those first-return path systems generated by trajectories of transitive continuous functions would possess nicer properties than general first return path systems. However, this is not the case, as the following theorem illustrates. Recall that in the terminology of dynamics two mappings f and g are said to be topologically conjugate if there is a homeomorphism h from I onto I such that $f = h \circ g \circ h^{-1}$.

Theorem 4.0.2. *[11] Let D be a support set and let $\{x_n\}_{n=0}^{\infty}$ be an enumeration of D. Let $g : I \to I$ be a transitive continuous map. Then there is a function $f : I \to I$, topologically equivalent to g, such that*

(a) the range of the trajectory of x_0 under f is D; i.e., the range of the sequence $\{x_0, f(x_0), f^2(x_0), f^3(x_0), \dots\}$ is D.

(b) the first return path system determined by $\{x_n\}_{n=0}^{\infty}$ is identical to that determined by $\{x_0, f(x_0), f^2(x_0), f^3(x_0), \dots\}$.

5 Integration

Given that certain functions carry sufficient information on a trajectory to recover them or compute meaningful derivatives, it is only natural to investigate whether some sort of first-return procedure might be employed to compute the integral of an integrable function, or, even more boldly to explore the possibility of a theory of integration based on a first-return procedure of some sort. Here, the surface has barely been scratched. Time (and effort) will tell if this is a fruitful area for exploration.

5.1 A First-Return Method to Capture Lebesgue Integrals

The first positive result that we obtained in this area is the following, which shows that the integral of a Lebesgue integrable function can, indeed, be computed from its values on a trajectory using a very Riemann-like first-return process.

Definition 5.1.1. Let $f : I \to \mathbb{R}$ be Lebesgue integrable. We say a trajectory $\{t_n\}$ on I *first-return Riemann yields the Lebesgue integral of f* (or simply that $\{t_n\}$ yields the Lebesgue integral of f) if for each interval $J \subseteq I$ the following condition holds: For each $\epsilon > 0$, there exists a $\delta > 0$ such that for every partition $\mathcal{P} = (x_0 < x_1 < \cdots < x_n)$ of J of mesh $\|\mathcal{P}\| < \delta$ we have

$$\left| \left(\sum_{i=1}^{n} f(r([x_{i-1}, x_i]))(x_i - x_{i-1}) \right) - \int_J f \right| < \epsilon.$$

Theorem 5.1.1. *[8] Suppose $f : I \to \mathbb{R}$ is a Lebesgue integrable function. Then there is a support set D and an enumeration $\{x_n\}$ of D which first-return Riemann yields the integral of f.*

We remark that in [8] this is actually proved for functions defined on the unit "square" in $\mathbb{R}^n$. It also only treats the integral over the entire square, but a careful reading of the proof shows that it applies to sub-"intervals" in the sense of our definition above, as well.

Next, there is an interesting link between this process of capturing the Lebesgue integral and the process of recovering a function.

Theorem 5.1.2. *[18] Let $f : I \to \mathbb{R}$ be bounded and measurable. A trajectory $\{x_n\}$ recovers f almost everywhere, if and only if $\{x_n\}$ yields the Lebesgue integral of f.*

From these two theorems, it is not difficult to deduce the following new characterization of Lebesgue measurable functions:

Theorem 5.1.3. *[18] A function $f : I \to \mathbb{R}$ is Lebesgue measurable if and only if it is a.e. recoverable.*

The following analogous result also holds and has a significantly easier proof.

Theorem 5.1.4. *[18] A function $f : I \to \mathbb{R}$ has the Baire property if and only if it is recoverable except at a first category set of points.*

5.2 Consistency and Universality

Just as we commented in Section 2.1 that a continuous function can easily be characterized as one that is first-return recoverable with respect to every enumeration of every support set, so too can Riemann integrable functions be characterized as those such that every enumeration of every support set yields the integral. In other words, there is a number A such that for every support set D and every enumeration $\{x_n\}$ of D we have that for each $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathcal{P}$ is a partition of I having norm less than δ, then $\left|\sum_{J\in\mathcal{P}} f(r(J))|J| - A\right| < \epsilon$. Motivated by previous applications of the adverb "consistently," we make the following definition.

Definition 5.2.1. Let $f : I \to \mathbb{R}$ be Lebesgue integrable. We say a support set D *consistently yields the integral of f* if every enumeration of D yields the integral of f. In this situation we say that D is a *consistent support set for f*.

Not surprisingly, it turns out that a function with a consistent support set must be quite close to being Riemann integrable.

Theorem 5.2.1. *[19] Let $f : I \to \mathbb{R}$ be Lebesgue integrable. The following are equivalent.*

1. *f is almost everywhere equal to a Riemann integrable function.*

2. *f has a consistent support set D.*

3. *There is a set W of full measure in I such that if T is any support set lying in W, then T is a consistent support set for f.*

Continuing our "consistent/universal" subtheme, it might make sense to say that a Lebesgue integrable $f : I \to \mathbb{R}$ satisfies the universal first-return integration criterion if every support set D has an enumeration which yields the integral of f. This seems rather confining, however, since even the characteristic function of the set of rationals would fail to meet this criterion. *Perhaps* a more fruitful line of inquiry might be pursued if we consider the following modification.

Definition 5.2.2. Let $f : I \to \mathbb{R}$ be Lebesgue integrable. We say that f satisfies the *a.e.-universal first-return integration criterion* if there is a full measure set $E \subset I$ such that every support set D lying in E has an enumeration which yields the integral of f.

We do not know much about this class. However, note that from Theorem 5.2.1 it follows that every Lebesgue integrable function having a consistent support set satisfies the a.e.-universal first-return integration criterion. On the other hand, the function

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & x \in (0,1] \\ 0, & x = 0 \end{cases},$$

satisfies the universal first-return integration criterion, but does not have a consistent support set. For an example of a function that does not satisfy the a.e.-universal first-return integration criterion, consider the characteristic function of a Cantor set of positive measure.

The reader is invited to classify either the class of functions satisfying the a.e.-universal first-return integration criterion or the class of functions satisfying the universal first-return integration criterion.

5.3 More Questions

With so few results established to date, there are obviously many open problems available in this integration area. For example, can the procedure of Theorem 5.1.1 be applied to compute Henstock-Kurzweil integrals, as well? Is there a characterization of Lebesgue integrability that involves this first-return procedure? Perhaps using the first-return process in conjunction with the Riemann process is not the best way to proceed. Is there a better way to involve a first-return process, one that could yield a characterization of Lebesgue integrable functions? Or, less ambitiously, is there a different first-return process, which is more productive for computing integrals?

6 An Application: Polygonal Approximation of Functions

While it is clear that any Baire class one function f is the pointwise limit of polygonal (continuous, piecewise linear) functions, it is not immediately obvious that one can choose the vertices of the polygons to be on the graph of f itself. Agronsky, Ceder, and Pearson [1] raised this question and showed that, indeed, the vertices can be chosen to lie on the graph. They achieved this result by applying a theorem on first-return recoverability.

6.1 Additional Characterizations for Baire One and Baire One, Darboux functions

Consider the following definitions.

Definition 6.1.1. Let $f : I \to \mathbb{R}$.

a) We say that a function $h : [0, 1] \to \mathbb{R}$ is a *polygonal function for* f if there is a partition $\tau = \{0 = a_0 < a_1 < a_2 < \cdots < a_m = 1\}$ such that h agrees with f at each partition point and is linear on the intervening closed intervals. We call $a_0, a_1, \ldots, a_m$ the *nodes of* h and $(a_0, h(a_0)), (a_1, h(a_1)), \ldots, (a_m, h(a_m))$ the *vertices of* h. The maximum distance between consecutive nodes is called the *mesh of* h and the maximum distance between consecutive vertices is called the *graph-mesh of* h. These are denoted $mesh(h)$ and graph-mesh(h), respectively.

b) We say that a sequence $\{h_n\}$ of polygonal functions for f *polygonally approximates* f on I if $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in I$ and $lim_{n\to\infty} mesh(h_n) = 0$. In this case we say that f *is polygonally approximable*

c) If graph-mesh(h_n) replaces $mesh(h_n)$ in b) then we obtain the notion of a *strongly polygonally approximable function*.

The classes of polygonally approximable and strongly polygonally approximable functions were characterized in [1] as follows:

Theorem 6.1.1. *[1] A function $f : I \to \mathbb{R}$ is*

(a) Baire one iff it is polygonally approximable.

(b) Baire one, Darboux iff it is strongly polygonally approximable.

Agronsky, Ceder, and Pearson applied Lemma 2.4.1 and Theorem 2.3.1 to obtain (a) and (b), respectively in a straightforward manner.

6.2 Consistency and Universality

Let us first examine a subclass of each of these classes based on the following "universal" versions of these concepts.

Definition 6.2.1. Let $f : I \to \mathbb{R}$. We say that

a) f is *universally polygonally approximable* (UPA) if for every dense subset D in I there is a sequence $\{h_n\}$ of polygonal functions for f, having nodes in $D \cup \{0, 1\}$ which polygonally approximates f on I.

b) f is *strongly universally polygonally approximable* (SUPA) if for every dense subset D in I there is a sequence $\{h_n\}$ of polygonal functions for f, having nodes in $D \cup \{0, 1\}$ which strongly polygonally approximates f on I.

Recall that in Section 2.5 we saw that quasicontinuity plays a major role in understanding universally first-return approachable functions and universally first-return continuous functions. Not surprisingly, the points of quasicontinuity of the function important to the investigation of both UPA and SUPA.

The next result is not deep, but provides alternate ways of thinking about UPA and SUPA.

Theorem 6.2.1. *[21] Let $f : [0, 1] \to \mathbb{R}$. The following are equivalent:*

i) f is UPA [SUPA].

ii) There is a sequence $\{h_n\}$ of polygonal functions for f, having all nodes other than 0 and 1 in $C(f)$, which [strongly] polygonally approximates f.

iii) There is a sequence $\{h_n\}$ of polygonal functions for f, having all nodes other than 0 and 1 in $Q(f)$, which [strongly] polygonally approximates f.

The classification of SUPA functions is well in hand; indeed, it turns out to be precisely the class of universally first-return continuous functions. More precisely, we have shown the following:

Theorem 6.2.2. *[21] The following are equivalent for a function $f : I \to \mathbb{R}$:*

(a) f is Baire one, Darboux, and quasi-continuous.

(b) f is Darboux and UPA.

(c) f is SUPA.

(d) f is universally first return continuous.

Upon seeing this theorem, one might quickly conjecture that the class of UPA functions will be the class of Baire one, quasicontinuous functions, i.e., the universally first-return approachable functions. However, it is easy to see that while the Baire one, quasicontinuous functions are UPA, the class of UPA functions contains functions that are not quasicontinuous. However, we can show the following.

Theorem 6.2.3. *[21] Suppose $f : I \to \mathbb{R}$ is universally polygonally approximable. Then for each $\epsilon > 0$, the set of points of nonquasicontinuity is $\sigma - (1 - \epsilon)$-symmetrically porous.*

A result tantalizingly close to a converse for this is

Theorem 6.2.4. *[16] Suppose $E = \cup_{n=1}^{\infty} F_n$ where each F_n is 1-symmetrically porous and closed in I. Then there is a UPA function $f : I \to I$ for which $E = NQ(f)$, i.e., E is precisely the set of points at which f fails to be quasi-continuous.*

We have been less-than-totally-successful in our attempts to aesthetically classify the UPA functions. We invite the interested reader to join the quest. The following references *might* prove useful. [21], [16], [22], [17].

Finally, what shall we do with the "consistent" concept for polygonal approximation? Here is one way to proceed:

Definition 6.2.2. Let D be a support set and let us say that a function f is *consistently polygonally approximable* if there exists a support set D such that for every enumeration $\{x_k\}$ of D, we have that the polygonal functions h_n for f determined by the partitions $\mathcal{P}_n$, consisting of the points $\{0, 1, x_1, x_2, \ldots, a_n\}$, polygonally approximate f.

With this as our definition, we obtain the following.

Theorem 6.2.5. *[23] The following are equivalent for a function $f : I \to \mathbb{R}$:*

(I) f is consistently first-return recoverable.

(II) f is consistently polygonally approximable.

(III) f is continuous except at countably many points.

7 Recent Work

In this final section we wish to call attention to recent works which build upon topics described in earlier sections. We shall first examine some recent projects by other authors and then mention some recent activity by Evans and Humke.

7.1 Recent Work by Others

Aliasghar Alikhani-Koopaei [2] has extended the first-return derivative concept from Section 3 to extreme first-return derivatives and Ilona Ćwiek, Ryszard J. Pawlak, and Bożena Świątek have refined some ideas from Section 2, introducing the notion of a Baire one point of a function and have used this notion to introduce and examine some spaces both larger and smaller than Baire class one.

7.1.1 Extreme First-Return Derivatives

In a natural manner Alikhani [2] has extended the notion of first-return derivatives to (possibly infinite) extreme first return derivatives and has examined their properties. To do this, given a trajectory $\{x_n\}$, he first appends each x to its first-return paths to form

$$R_x^- = \mathcal{P}_x^l \cup \{x\},\ R_x^+ = \mathcal{P}_x^r \cup \{x\},\ R_x = \mathcal{P}_x^l \cup \mathcal{P}_x^r.$$

Then $R = \cup\{R_x : x \in [0,1]\}$ $[R^- = \cup\{R_x^- : x \in [0,1]\}$, $R^+ = \cup\{R_x^+ : x \in [0,1]\}]$ becomes a [left, right] path system in the sense of [4] with each R_x^-, R_x^+, R_x^- compact. He then defines the six extreme first-return derivates with respect to these path systems as one can with any path system, and shows that the function P from $[0,1]$ to R (equipped with the Hausdorff metric), given by $P(x) = R_x$ is continuous. Likewise, the map $x \to R_x^-$ $[x \to R_x^+]$ is left [right] continuous. He uses this result to establish the following:

Theorem 7.1.1. *Given a Borel [Lebesgue] measurable function f and a trajectory $\{x_n\}$, the six extreme first-return derivatives of f with respect to $\{x_n\}$ are Borel [Lebesgue] measurable. Specifically, if f belongs to Borel class B_α then these six derivatives belong to Borel class $B_{\alpha+2}$.*

Specializing to continuous functions, he then proves the following two theorems:

Theorem 7.1.2. *If $f : [0,1] \to \mathbb{R}$ is continuous and if one of the extreme first-return extreme derivatives with respect to some trajectory $\{x_n\}$ is finite and continuous at x_o, then f is differentiable at x_o.*

Theorem 7.1.3. *If $f : [0,1] \to \mathbb{R}$ is continuous and if one of the extreme first-return extreme derivatives with respect to some trajectory $\{x_n\}$ is nonnegative on $[0,1]$, then f is nondecreasing.*

7.1.2 Baire One Points of Functions

Ćwiek, Pawlak and Świątek [5] say that a point $x \in I$ is a Baire one point of a function $f : I \to \mathbb{R}$ with respect to a trajectory $\{x_n\}$ if f is first-return recoverable with respect to $\{x_n\}$ at x. The symbol $\mathcal{B}_1(f, \{x_n\})$ denotes the set of all Baire one points of f with respect to $\{x_n\}$. Using this terminology, one could restate Theorem 2.1.1 as "*f belongs to Baire class one if and only if for some trajectory $\{x_n\}$, $\mathcal{B}_1(f, \{x_n\}) = I$.*" In [5] the authors are interested in investigating functions f for which $\mathcal{B}_1(f, \{x_n\})$ is not all of I, but "most" of it. They observe that if the exceptional set is so small that it has countable closure, then one still obtains precisely the class of Baire one functions; that is, they improve upon Theorem 2.1.1 by showing that *f belongs to Baire class one if and only if for some trajectory $\{x_n\}$, $I \setminus \mathcal{B}_1(f, \{x_n\})$ has countable closure.* (See the next subsection of this survey for a further improvement of this result.)

More generally, let $\mathcal{F}$ denote the ideal of subsets of I, each having empty interior, and define a function $f : I \to \mathbb{R}$ to be strongly $\mathcal{F}$ - almost everywhere first-return recoverable with respect to a trajectory $\{x_n\}$ if the closure of $I \setminus \mathcal{B}_1(f, \{x_n\}) \in \mathcal{F}$. Let $\mathcal{B}_1^{\mathcal{F}}(\{x_n\})$ denote the class of all functions which are strongly $\mathcal{F}$ - strongly first return recoverable with respect to $\{x_n\}$. In addition to introducing this class, the authors define a rather technical notion of weak first-return continuity with respect to a trajectory and show that for a Baire one function weak first-return continuity is equivalent to first-return continuity. Then in [38] Pawlak and Świątek let $\mathcal{B}_1^{\mathcal{F}}$ denote the class of all functions f for which there exists at least one trajectory $\{x_n\}$ such that $f \in \mathcal{B}_1^{\mathcal{F}}(\{x_n\})$. The class $\mathcal{B}_1^{\mathcal{F}}$ is clearly quite a bit larger than the class of Baire one functions, as may be seen by noting that it clearly contains all those functions whose points of discontinuity are a nowhere dense set. The following result gives an indication of how large this class is.

Theorem 7.1.4. *[38] If $\mathcal{L}$ denotes the collection of members of $\mathcal{B}_1^{\mathcal{F}}$ which are Lebesgue measurable, then $\mathcal{L}$ is superporous in $\mathcal{B}_1^{\mathcal{F}}$.*

On the other hand, they show that $\mathcal{B}_1^{\mathcal{F}}$ is a vector space, closed under multiplication, and shares the following property with the class of Baire one functions:

Theorem 7.1.5. *If $f \in \mathcal{B}_1^{\mathcal{F}}$, then the discontinuities of f form a meagre set.*

7.2 More or Less "Recoverable Functions"

Having seen in Theorems 5.1.3 and 5.1.4 that a couple of standard classes of functions are characterized by the notions of being recoverable except on a set of measure zero or first category, Evans and Humke [20] have recently explored the situation when even smaller exceptional sets are employed. For example, it is shown there that

Theorem 7.2.1. *[20] A function $f : I \to \mathbb{R}$ belongs to honorary Baire class two if and only if it is recoverable except at a countable set of points.*

Compare this with the result from [5] mentioned in the previous subsection which states that if the exceptional set has countable closure then we are right back to the class of Baire one functions. More generally, we have

Theorem 7.2.2. *[20] A function $f : I \to \mathbb{R}$ belongs to Baire class one if and only if it is recoverable except at a scattered set of points.*

It would be interesting to know if other types of exceptional sets (σ-porous sets, for example) for recoverability produce characterizations of other familiar classes of functions.

Along another line, after considering the notion of recoverability except on a set, small in some sense, it seems natural to explore functions which are consistently or universally recoverable except on a small set. Numerous results concerning these notions are obtained in [20]. We conclude this survey by noting a couple of representative examples of these findings.

Theorem 7.2.3. *[20] A function $f : I \to \mathbb{R}$ universally recoverable almost everywhere if and only if it is quasicontinuous almost everywhere.*

Theorem 7.2.4. *[20] A function $f : I \to \mathbb{R}$ consistently recoverable except at countably many points if and only if it there is a co-countable set T such that $f|_T$ is continuous.*

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