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ON FIRST RETURN PATH SYSTEMS

Abstract

It is known that for a first return system of paths $\{\mathrm{R_x} : \mathrm{x} \in [0,1]\}$ the right path systems R^+ (the left path system R^-) is right (is left) continuous and R satisfies $I.I.C.$ property. In this paper we consider path systems that are continuous and satisfy $I.I.C.$ and investigate the possibility of containing first return path systems. We also study the effect of turbulence on trajectories by treating them as sequences

1 Introduction.

Motivated by the Poincaré first return map of differentiable dynamics, R. J. O'Malley introduced a new type of path systems which he calls first return systems[12]. He shows that, though these are extremely thin paths, the systems possess an interesting intersection property that makes their differentiation theory as rich as those of much thicker path systems. For example, every first return path differentiable function is in DB_1^* and every first return path derivative is in DB_1. First return systems have been extensively investigated in a series of papers (for example see [7,8,9,10] and some of their references).

In this paper we consider two types of problems. The first problem relates to a question raised by professor O'Malley at the 26th Summer Symposium in Real Analysis, when the author was presenting his talk. The concept of continuous systems of paths was introduced in [1] and was generalized by Milan Matejdes [11]. We showed that for a first return system of paths $R = \{R_x : x \in [0,1]\}$, the system $R^+ = \{R_x^+ : x \in [0,1]\}$ and $R^- = \{R_x^- : x \in [0,1]\}$ are

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right and left continuous respectively (see [2]), thus in the presence of path differentiability of f, f'_R enjoys the nice properties shared by path derivatives where the path system has the internal intersection property as well as being continuous. Professor O'Malley asked if there is a continuous system of paths which is not a first return system of paths. In fact in [2] we show that such system of paths do exist. However, our example does not have any intersection condition property, thus it would be interesting to know if a path system with some nice properties (in particular, a path system with intersection condition or preferably a continuous path system with internal intersection condition) contains a first return system of paths. The second problem involves the effect of turbulence on trajectories by considering them as sequences.

2 Preliminaries.

By $\mathbb{R}$ we mean the set of real numbers, $\overline{A}$ denotes the closure of A, and A' is the set of accumulation points of A. By an interval (a, b) we mean an interval with a and b as an end points, not necessarily $a < b$.

Definition 2.1. (see [12]) A trajectory is a sequence $P_n,\ n = 0, 1, \ldots,$ with the following properties:

(i) $P_0 = 0$, $P_1 = 1$,

(ii) $P_n \neq P_m$, $n \neq m$,

(iii) $0 \leq P_n \leq 1$ for all n,

(iv) $\{P_n : n = 0, 1, \ldots\}$ is dense in [0,1].

Our notation for a trajectory will be $\{P_n\}$. For a given $k \geq 1, \Pi_k$ will represent the partition of the interval [0,1] generated by the initial segment $\{P_0, P_1, \ldots, P_k\}$. The ith interval of that partition will be denoted as $\Pi_{k,i}$. For each partition $\Pi_k =; P_0 = 0 < P_{\alpha_1} < P_{\alpha_2} < \cdots < P_{\alpha_{k-1}} < P_1 = 1$ we assign the code $(0, \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, 1)$.

Definition 2.2. (see [4]) Let x belong to $[0, 1]$. A path at x is a set $R_x \subseteq [0, 1]$ such that $x \in R_x$ and x is a point of accumulation of R_x. A system of paths R is a collection $\{R_x : x \in [0, 1]\}$, where each R_x is a path at x. For two path systems $R = \{R_x : x \in [0, 1]\}$ and $E = \{E_x : x \in [0, 1]\}$ we say R contains E and denote it by $E \subset R$ if $E_x \subset R_x$ for each $x \in [0, 1]$.

Definition 2.3. (see [1]) Let $R = \{R_x : x \in [0,1]\}$ be a system of paths, with each R_x compact. We endow R with the Hausdorff metric d_H to form a metric space. If the function $P : x \to R_x$ is a continuous function, we say R is a continuous system of paths. The left continuous and right continuous systems of paths are defined similarly.

Definition 2.4. (see [3]) A system of paths $E = \{E_x : x \in \mathbb{R}\}$ is said to be
(1) of congruent type if each $E_x = T + x = \{t + x : t \in T\}$ for each x where T is a set having 0 as a member and as a point of accumulation.
(ii) of sequential type if $E_x = T + x$ for each x where $T = \{0, h_1, h_2, h_3, \dots\}$ and $\{h_n\}$ is a fixed sequence converging to 0.
Note that sequential system of paths and congruent system of paths when T is a closed subset of the real line, are special cases of continuous system of paths.

Definition 2.5. (see [12]) Let $\{P_n\}$ be a fixed trajectory. For a given interval $(a,b) \subset [0,1]$, $r(a,b)$ will be the first element of the trajectory in (a,b). For $0 \leq y < 1$, the right first return path to y, R_y^+, is defined recursively via $y_1^+ = y$, $y_2^+ = 1$ and $y_{k+1}^+ = r(y, y_k^+)$ for $k \geq 2$. For $0 < y \leq 1$, the left first return path to y, R_y^-, is defined similarly. For $0 < y < 1$, we set $R_y = R_y^- \cup R_y^+$, and $R_0 = R_0^+$, $R_1 = R_1^-$. The path systems $R^+ = \{R_x^+ : x \in [0,1)\}$, $R^- = \{R_x^- : x \in (0,1]\}$ and $R = \{R_x : x \in (0,1)\} \cup \{R_0^+, R_1^-\}$ are called the right first return system, the left first return system and the first return system of paths generated by $\{P_n\}$, respectively.

Definition 2.6. (see [4]) A path system R is said to have the external intersection condition denoted by $E.I.C.$ (intersection condition denoted by $I.C.$, internal intersection condition denoted by $I.I.C.$) if there is a positive function $\delta(x)$ on $[0,1]$ such that $R_x \cap R_y \cap (y, 2y-x) \neq \emptyset$ and $R_x \cap R_y \cap (2x-y, x) \neq \emptyset$ ($R_x \cap R_y \cap [x,y] \neq \emptyset$, $R_x \cap R_y \cap (x,y) \neq \emptyset$, respectively), whenever $0 < y - x < \min\{\delta(x), \delta(y)\}$.

Definition 2.7. (see [4,12]) Let F:[0,1]$\to \mathbb{R}$ and let R be any path system. If the

$$\lim_{y \to x, y \in R_x \setminus \{x\}} \frac{F(y) - F(x)}{y - x} = f(x)$$

exists and is finite, then we say F is $R-$differentiable at x. If F is $R-$differentiable at every point x, then we say that F is path differentiable and f is the path derivative of F and is denoted by $F'_R = f$. If the system of paths is a first return system, then f is called the first return path derivative of F.

We consider the following known metrics on the space of sequences.

Definition 2.8. Let $\mathcal{S}$ be the set of sequences defined on $[0,1]$ and let $S = \{s_i\}_{i=1}^{\infty}$ and $T = \{t_i\}_{i=1}^{\infty}$ be two sequences. We define
$d_1(S,T) = \sup\{\frac{|s_1-t_1|}{1+|s_1-t_1|}, \frac{|s_2-t_2|}{1+|s_2-t_2|}, \dots, \frac{|s_k-t_k|}{1+|s_k-t_k|}, \dots\}$,
$d_2(S,T) = \sum_{i=1}^{\infty} \frac{1}{2^i} \mid s_i - t_i \mid$. Clearly $(\mathcal{S},d_1)$ and $(\mathcal{S},d_2)$ are metric spaces.

Definition 2.9. Let $P = \{p_n\}$ and $Q = \{q_n\}$ be two sequences. We say P and Q differ in a finite number of terms if there exist positive integers m and n so that $p_{m+i} = q_{n+i}$ for each $i \geq 0$. We also call the path systems $R = \{R_x\}$ and $E = \{E_x\}$ eventually the same from right(left), if for each $x \in [0,1)$ the paths R_x^+ and E_x^+ ($x \in (0,1]$ the paths R_x^- and E_x^-) are eventually the same, that is for each x there exists $\delta_x > 0$ so that $R_x^+ \cap [x, x+\delta_x) = E_x^+ \cap [x, x+\delta_x)$ ($R_x^- \cap (x-\delta_x, x] = E_x^- \cap (x-\delta_x, x]$). The path systems R and E are said to be eventually the same, if they are eventually the same from right and left. We call the path systems R and E eventually the same in a uniform way, if $\delta = \inf_{x\in[0,1]} \delta_x > 0$, that is, if there exists $\delta > 0$ so that for each $x \in [0,1]$, $R_x \cap (x-\delta, x+\delta) = E_x \cap (x-\delta, x+\delta)$.

Definition 2.10. (see [4]) Let δ be a positive function and let X be a set of real numbers. By a $\delta-$decomposition of X we shall mean a sequence of sets $\{X_n\}$, which is a relabelling of the countable collection

$$Y_{m,j} = \{x \in X : \delta(x) > \tfrac{1}{m}\} \cap [\tfrac{j}{m}, \tfrac{j+1}{m}],\ m = 1,2,3,\dots \text{ and } j = 0, \pm1, \pm2, \pm3, \dots.$$

The key features of such a decomposition of a set X are:
(i) $\cup_{n=1}^{\infty} X_n = X$;
(ii) if x and y belong to the same set X_n, then $\mid x - y \mid < \min\{\delta(x), \delta(y)\}$, and
(iii) if $x \in \overline{X_n}$, then there are points $y \in X_n$ with $\mid x - y \mid < \min\{\delta(x), \delta(y)\}$.

3 Results.

Theorem 3.1. *Let $R = \{R_x : x \in [0,1]\}$, $R_x = x + T$ is a path system of congruent type with any sort of intersection property, then $(T+T) - (T+T)$ contains an interval of positive length.*

PROOF. Suppose R satisfies some sort of intersection condition property, and δ is the positive function associated with such property. Let $\{A_n\}$ be the

$\delta-$decomposition of $[0,1]$, thus $[0,1] = \cup_{n=1}^{\infty} A_n$. By Baire Category Theorem, there exists n_0 so that $\overline{A_{n_0}}$ contains an interval (c,d) of positive length. Take $x_{n_0} \in A_{n_0}$ and let $x \in (c,d)$ be arbitrary, then we have $y_x \in A_{n_0}$ so that $\mid x - y_x \mid < \min\{\delta(x), \delta(y_x)\}$, thus $R_x \cap R_{y_x} \neq \emptyset$. For $z \in R_x \cap R_{y_x} = x + T \cap y_x + T$, we have $z = x + t_1 = y_x + t_2$. Thus $x = (t_2 - t_1) + y_x$ with t_1, t_2 in T. Since y_x and x_{n_0} are in A_{n_0} we have $\mid x_{n_0} - y_x \mid < \min\{\delta(x_{n_0}), \delta(y_x)\}$, thus $R_{x_{n_0}} \cap R_{y_x} \neq \emptyset$. For $z \in R_{x_{n_0}} \cap R_{y_x} = x_{n_0} + T \cap y_x + T$, we have $z = x_{n_0} + t_3 = y_x + t_4$, thus $y_x = (t_3 - t_4) + x_{n_0}$ with t_3, t_4 in T. Therefore $x = (t_2 - t_1) + y_x = (t_2 - t_1) + (t_3 - t_4) + x_{n_0} = (t_2 + t_3) - (t_1 + t_4) + x_{n_0}$, implying $x \in [(T+T) - (T+T)] + x_{n_0}$, thus $(c,d) - x_{n_0} \subseteq (T+T) - (T+T)$. Hence $(T+T) - (T+T)$ contains an interval of positive length. $\square$

Corollary 3.2. *A path system of congruent type $R = \{R_x : x \in [0,1]\}$, $R_x = x + T$ does not contain any first return system of path when $(T+T) - (T+T)$ does not contain an interval of positive length.*

The following theorem is a straight forward application of Theorem 3.1.

Theorem 3.3. *Any path system of congruent type $R = \{R_x\} = \{x + T : x \in \mathbb{R}\}$ with T countable (in particular any path system of sequential type) cannot have any intersection property, thus it cannot contain any first return system of paths.*

Proof. Let $R_x = T + x$ for each x where $T = \{h_i\}_{i=1}^{\infty}$ with $0 \in (T)'$. Let $B_i = \{h_j + h_i\}_{j=1}^{\infty}$, then for each i, B_i is a countable set and $B_i = T + h_i$. Thus $T + T = \cup_{i=1}^{\infty} B_i$ is a countable set. Similarly we can show that for a countable set T the set $T - T$ is also countable. Hence $(T+T) - (T+T)$ is a countable set, so it does not contain an interval of positive length. Hence R does not have any intersection property, in particular it does not contain any path with internal intersection property. Thus it cannot contain a first return system of paths. $\square$

Let $R = \{R_x\}$ be a first return path system. It is known that R has the internal intersection property (see [12]), R^+ and R^- are right and left continuous, respectively(see[2]). Thus in the presence of path differentiability of f, f'_R enjoys the nice properties shared by path derivatives, where the path system has the intersection property as well as being continuous. Now we want to look at the reverse problem, that is for a continuous path system $R = \{R_x\}$ which satisfies $I.I.C.$ and f'_R exists with $f'_R \in B_1$, is it possible to find a first return path system $E = \{E_x\}$ so that $E_x \subseteq R_x$ for each x and $f'_E = f'_R$. In [6] Darji and Evans gave an example of a function f and a path system

E satisfying $E.I.C.$ so that f'_E exists, but f is not first return differentiable to f'_E. The path system given there does not satisfy $I.I.C.$. The following example is given by Cordy in[5].

Example 3.4. There exists a continuous function F and a path system E which satisfies $I.C.$, yet $F'_E \notin B_1$.

Due to the existence of a continuous function with $F'_E \notin B_1$, the path system E given in Example 3.4, neither contains a first return nor contains a continuous system of paths. It is also easy to see that E satisfies $I.C.$, but it does not satisfy $I.I.C.$. In the following example, which is a modification of an example of Darji and Evans, we show that even in the presence of $F'_E \in B_1$, the $I.I.C.$ alone is not sufficient to guarantee that E contains a first return path system. Theorem 3.3 also indicates that there are continuous path systems not containing any first return path system.

Example 3.5. There exists a function F and a bilateral system of paths E having the internal intersection condition so that $F'_E \in B_1$, yet E does not contain any first return system of paths.

PROOF. Let $P \subset [0,1]$ be a cantor like set of positive measure containing a countable dense subset $\{s_i\}_{i=1}^{\infty}$ such that P has density 1 at each s_i. Let $\{(a_i,b_i)\}_{i=1}^{\infty}$ be the intervals contiguous to P. Define

$$F(x) = \begin{cases} 0 & \text{if } x \in P, \\ \sin[\frac{1}{(x-a_n)(b_n-x)}] & \text{if } x \in (a_n,b_n) \ \text{ for } n = 1,2,\cdots, \end{cases}$$

and

$$f(x) = \begin{cases} F'(x) & \text{if } x \in \cup_{n=1}^{\infty}(a_n,b_n), \\ \frac{1}{2^i} & \text{if } x = s_i \text{ for } i = 1,2,\cdots, \\ 0 & \text{if } x \in P \setminus \{s_i\}_{i=1}^{\infty}, \end{cases}$$

It is clear that $f \in B_1$.

For $s_k \in \{s_i\}_{i=1}^{\infty}$, let $\epsilon_k = \min\{\frac{1}{20}, \mid s_i - s_k \mid : 1 \leq i < k\}$. For $\epsilon_k > 0$, there exists $0 < \delta(s_k) < \epsilon_k$ so that $\mid \frac{P \cap (s_k-h, s_k+h)}{2h} \mid > \frac{19}{20}$, when $\mid h \mid < \delta(s_k)$. Let

$$E_x = \begin{cases} [0,1] & \text{if } x \in [0,1] \setminus P, \\ \{x\} \cup \{t : \frac{F(t)-F(x)}{t-x} = \frac{1}{2^i}\} & \text{if } x = s_i,\ i = 1,2,\cdots \\ \cup_{i=1}^{\infty} T_{s_i}(x) \cup P & \text{if } x \in P \setminus (\{s_i\}_{i=1}^{\infty} \cup \{a_i,b_i\}_{i=1}^{\infty}), \\ \cup_{i=1}^{\infty} T_{s_i}(x) \cup P \cup (\{F^{-1}(0) \cap (a_i,b_i)\}) & \text{if } x = a_i \text{ or } x = b_i \text{ for } i = 1,2,\cdots. \end{cases}$$

Where $T_{s_i}(x) = \begin{cases} E_{s_i} \cap (\frac{x+s_i}{2}, s_i) & \text{if } \mid x - s_i \mid < \delta(s_i), \\ \emptyset & \text{if } \mid x - s_i \mid \geq \delta(s_i). \end{cases}$

Let $1 \leq i < \infty$, to see that E_{s_i} is a path at s_i, let $I = (s_i, s_i + h)$ be an interval with $\mid h \mid < \delta(s_i)$, then P has infinitely many points in I, thus for infinitely many n, we have $(a_n, b_n) \subset I$, implying $E_{s_i} \cap I \neq \emptyset$, hence $s_i \in (E_{s_i})'$ and E_{s_i} is a bilateral path at s_i. It is also clear that for each $x \in [0,1] \setminus \{s_i\}_{i=1}^{\infty}$, E_x is a bilateral path at x.
Define $\delta : [0.1] \to R^+$ as

$$\delta(x) = \begin{cases} dist(x, P) & \text{if } x \notin P, \\ \delta(s_i) & \text{if } x = s_i \text{ for } i = 1, 2, \cdots, \\ 1 & \text{if } x \in P \setminus (\{s_i\}_{i=1}^{\infty} \cup \{a_i, b_i\}_{i=1}^{\infty}), \\ \min\{1, \frac{b_n - a_n}{3}\} & \text{if } x = a_n \text{ or } x = b_n \text{ for some } n. \end{cases}$$

Now we show that E satisfies $I.I.C.$ Let $x, y \in [0,1]$ be such that $\mid x - y \mid < \min\{\delta(x), \delta(y)\}$, then either x and y are both in $[0,1] \setminus P$ or both belong to P. In the case that both belong to P, if for some i, $x = a_i$, then $y \neq b_i$ and if $x = b_i$, then $y \neq a_i$, for the same i.
If $x, y \in [0,1] \setminus P$, then $E_x \cap E_y \cap (x, y) = (x, y) \neq \emptyset$. If $x, y \in P$, with $x \in \{s_i\}_{i=1}^{\infty}$, then $y \notin \{s_i\}_{i=1}^{\infty}$, and vise versa. Thus we consider the following cases;
(i) $x = s_m$ for some m and $y \in P \setminus (\{s_i\}_{i=1}^{\infty} \cup \{a_i, b_i\}_{i=1}^{\infty})$.
(ii) $x = s_m$ for some m and $y = a_i$ or $y = b_i$ for some i .
(iii) $x, y \in P \setminus (\{s_i\}_{i=1}^{\infty} \cup \{a_i, b_i\}_{i=1}^{\infty})$.
(iv) $x \in P \setminus (\{s_i\}_{i=1}^{\infty} \cup \{a_i, b_i\}_{i=1}^{\infty})$, but $y = a_i > x$ or $y = b_i < x$ for some i.
(v) $x = a_i$ and $y = a_j < x$ or $y = b_j < x$.
(vi) $x = b_i$ and $y = a_j > x$ or $y = b_j > x$.
In case (i), if $\mid s_m - y \mid < \min\{\delta(s_m), \delta(y)\}$, we have
$E_{s_m} \cap E_y \cap (s_m, y) \supset \{s_m\} \cup \{t : \frac{F(t) - F(s_m)}{t - s_m} = \frac{1}{2^m}\} \cap (\cup_{i=1}^{\infty} T_{s_i}(y) \cup P)$
$\cap (s_m, y) \supset T_{s_m}(y) \neq \emptyset$.
In case (ii), we have $E_{s_m} \cap E_y \cap (s_m, y) \supset \{s_m\} \cup \{t : \frac{F(t) - F(s_m)}{t - s_m} = \frac{1}{2^m}\} \cap (\cup_{i=1}^{\infty} T_{s_i}(x) \cup P \cup (\{F^{-1}(0) \cap (a_i, b_i)\}) \cap (s_m, y) \supset T_{s_m}(y) \neq \emptyset$.
In case (iii), we have $E_x \cap E_y \cap (x, y) \supset (\cup_{i=1}^{\infty} T_{s_i}(y) \cup P) \cap (\cup_{i=1}^{\infty} T_{s_i}(x) \cup P) \cap (x, y) \supset P \cap (x, y) \neq \emptyset$.
In case (iv), we have $E_x \cap E_y \cap (x, y) = (\cup_{i=1}^{\infty} T_{s_i}(x) \cup P) \cap (P \cup \{F^{-1}(0) \cap (a_i, b_i)\}) \cap (x, y) \supset P \cap (x, y) \neq \emptyset$.
In both cases (v) and (vi), we have $E_x \cap E_y \cap (x, y) \supset P \cap (x, y) \neq \emptyset$.
Thus E satisfies $I.I.C.$ property. To see $F'_E(x) = f(x)$. This is clear when $x \in [0,1] \setminus P$ or $x \in \{s_i\}_{i=1}^{\infty}$. Suppose $x \in P \setminus \{s_i\}_{i=1}^{\infty}$, then for $y \in P$ or $y \in \{F^{-1}(0) \cap (a_i, b_i)$ we have $\lim_{y \to x} \frac{F(y) - F(x)}{y - x} = \lim_{y \to x} 0 = f(x)$. If $y \in E_x \setminus P$, then $y \in E_{s_i} \cap (\frac{x + s_i}{2}, s_i)$ for some i, thus
$\mid \frac{F(y) - F(x)}{y - x} \mid = \mid \frac{F(y) - F(s_i)}{y - s_i} \cdot \frac{y - s_i}{y - x} \mid = \mid \frac{F(y) - F(s_i)}{y - s_i} \mid \cdot \mid \frac{y - s_i}{y - x} \mid \leq \frac{1}{2^i} \cdot 1 = \frac{1}{2^i}$. Hence $\lim_{y \to x, y \in E_x \setminus P} \mid \frac{F(y) - F(x)}{y - x} \mid \leq \lim_{i \to \infty} \frac{1}{2^i} = 0$ hence $F'_E(x) = f(x) = 0$.

We claim there is no first return path system $R = \{R_x : x \in [0,1]\}$ so that for each x, $R_x \subset E_x$. On the contrary suppose $Q = \{q_i\}$ is a trajectory and R is the first return path system generated by Q so that $R \subset E$, then we have $F'_R(x) = F'_E(x) = f(x)$. Let $A_n = \{t \in P : \forall x_i \in R_t \text{ with } i > n \;\&\; \mid \frac{F(x_i)-F(t)}{x_i-t} - f(t) \mid< 1\}$. From $F'_E(t) = f(t)$, we have $P = \cup_{n=1}^{\infty} A_n$, by Baire Category Theorem, there exists an open interval V such that $\overline{A_m} \subset V \cap P \neq \emptyset$, for some m. Choose $s_j \in V, j > m$, since P has density 1 at s_j and $f(s_j) = \frac{1}{2^j}$, we have $x_k \in (V \cap R_{s_j}), k > m$, $\mid s_j - x_k \mid< \frac{1}{m}$, with $\mid \frac{F(s_j)-F(x_k)}{s_j-x_k} \mid> \frac{1}{2 \cdot 2^j}$, and $\mid \frac{I \cap P}{I} \mid> 1 - \frac{1}{4 \cdot 2^j}$, where $I = (x_k, s_j)$. Since $\overline{A_m} \subset V \cap P$, pick $p \in A_m \setminus \{s_i\}_{i=1}^{\infty}$ such that $x_k < p < s_j$ and $\mid \frac{s_j - x_k}{p - x_k} \mid> 2 \cdot 2^j$. Then we have
$\mid \frac{F(x_k)-F(p)}{x_k-p} \mid=\mid \frac{F(x_k)-F(s_j)}{x_k-p} \mid=\mid \frac{F(x_k)-F(s_j)}{x_k-s_j} \mid \cdot \mid \frac{x_k-s_j}{x_k-p} \mid> \frac{1}{2 \cdot 2^j} \cdot (2 \cdot 2^j) > 1$. Since $x_k \in R_{s_j}$ and $x_k < p < s_j$, we have $x_k \in R_p$, contradicting $p \in A_m$. Thus E does not contain any first return path system. □

In the rest of the paper, by considering trajectories as sequences, we study the effect of turbulence on trajectories and the first return path systems generated by them.

Theorem 3.6. *Let $\mathcal{S}$ be the set of sequences defined on $[0,1]$ endowed with metrics d_1 or d_2, and let $\mathcal{A}$ be the set of trajectories on $[0,1]$, then $\mathcal{A}$ is a closed nowhere dense subset of $\mathcal{S}$.*

PROOF. Let $T_n \in \mathcal{A}$ and $\lim_{n\to\infty} T_n = T$. We show that T is a trajectory on $[0,1]$. To this end let $T_n = \{t_i^n\}_{i=1}^{\infty}$, $T = \{t_i\}_{i=1}^{\infty}$, $\epsilon > 0$ be an arbitrary real number, and let $x \in [0,1]$. It is easy to see that $t_0 = 0$ and $t_1 = 1$ and $\lim_{n\to\infty} t_i^n = t_i$ for each $i \geq 2$. Choose n_0 large enough so that $d(T_{n_0}, T) < \frac{\epsilon}{4}$, then pick $t_{i(n_0)}^{n_0} \in T_{n_0}$ such that $\mid x - t_{i_{n_0}}^{n_0} \mid< \frac{\epsilon}{4}$. Then we have $\mid t_{i(n_0)} - x \mid\leq\mid t_{i(n_0)} - t_{i(n_0)}^{n_0} \mid + \mid t_{i(n_0)}^{n_0} - x \mid\leq \frac{\epsilon}{4} + \frac{\epsilon}{2} < \epsilon$. Hence T is a trajectory and thus $\mathcal{A}$ is closed. To show $\mathcal{A}$ is nowhere dense let $S = \{s_i\}_{i=1}^{\infty} \in \mathcal{A}$. Pick $z \in (0,1)$, $0 < \eta < \frac{\epsilon}{4}$ and a sequence $\{z_i\}_{i=1}^{\infty} \subseteq (z-\eta, z+\eta) = I_p$ so that it is not dense in I_p. Then take the sequence $T = \{t_n\}$ as $t_n = s_n$ for all n with $s_n \in [0,1] \setminus I_p$ and $t_n = z_n$ for all n with $s_n \in I_p$. It is easy to see that $d_1(S,T) < \epsilon$, $d_2(S,T) < \epsilon$, however $T \cap I_p \subseteq \{z_i\}_{i=1}^{\infty}$, implying $\overline{T \cap I_p} \subseteq \overline{\{z_i\}_{i=1}^{\infty}} \neq I_p$, so T can not be a trajectory on $[0,1]$. □

Lemma 3.7. *Let $\{S_m\}_{m=1}^{\infty}$ be a sequence of trajectories, S be a trajectory, Π_k and Π_k^m be the partitions of $[0,1]$ obtained from the first k terms of S and S_m, respectively. If $\lim_{m\to\infty} d_1(S_m, S) = 0$, then for each $k \geq 2$ there exist* *a natural number n_k so that for all $m \geq n_k$, the partitions Π_k^m and Π_k have the same code.*

Proof. Let δ be the length of the smallest subinterval of Π_k. Then choose n_k so large that for all $m \geq n_k$, $d_1(S_m, S) < \delta/4$. It is clear that for $m \geq n_k$, Π_K^m has the same code as Π_k. $\square$

Theorem 3.8. *For each $m \geq 1$, let S_m be a trajectory and let R_m be the first return system of path generated by S_m. If S is a trajectory with $\lim_{m\to\infty} d_1(S_m, S) = 0$, then $\lim_{m\to\infty} d_1(R_m, R) = 0$ and $\lim_{m\to\infty} d_H(R_m, R) = 0$ where R is the first return path system generated by S.*

Proof. We first note that for every $x \in [0,1], R_x$, the path leading to x, is a sequence as well as a closed subset of [0,1]. Let $S = \{p_n\}$, $0 \leq x \leq 1$, and let ε be an arbitrary positive number. We consider two cases.

(i) $x \in [0,1] \setminus \{p_n\}_{n=1}^{\infty}$,
(ii) $x = p_k$ for some $k = 1, 2, \ldots$.

Suppose δ_k is the length of the longest subinterval of partition Π_k. Obviously, $\lim_{k\to\infty} \delta_k = 0$. Choose k_0 large enough so that $\delta_{k_0} < \frac{\varepsilon}{2}$. In case (i), Let $\Pi_{k_0,i} = [c,d]$ be the subinterval of Π_{k_0} containing x, and set $\delta = \frac{1}{4}\min\{\mid d - x \mid, \mid c - x \mid\}$. Now if $\mid y - x \mid < \delta$, then y and x are both contained in the interval (c,d). From lemma 3.7, we know that there exist a positive integer n_ϵ so that for all $m \geq n_\epsilon$, $\Pi_{k_0}^m$ and Π_{k_0} have the same code, so the corresponding terms of the paths $(R_m)_x$ and R_x that do not fall in the interval (c,d) have a distance less than $\varepsilon/2$. On the other hand the rest of the terms of all the sequences $(R_m)_x$, R_x lie in the interval (c,d) that also has length less than $\varepsilon/2$, thus we have $d_1((R_m^+)_x, R_x^+) < \varepsilon$, $d_1((R_m^-)_x, R_x^-) < \varepsilon$, $d_H((R_m^+)_x, R_x^+) < \varepsilon$, $d_H((R_m^-)_x, R_x^-) < \varepsilon$.

In case (ii), let $x = p_{k_1}$. Since $R_x^- = \{x_k^-\}$ and $R_x^+ = \{x_k^+\}$ are monotone subsequences of $\{p_n\}$ converging to x, there exists a positive integer N_1 such that $\mid x_k^- - x \mid < \frac{\varepsilon}{2}$ and $\mid x_k^+ - x \mid < \frac{\varepsilon}{2}$ for all $k \geq N_1$. Let k_2 be a positive integer so that Π_{k_2} contains the points $x_1^-, x_1^+, \ldots, x_{N_1}^-, x_{N_1}^+, x$ as end points and $\delta_{k_2} < \frac{\varepsilon}{2}$. Suppose δ'_{k_2} is the length of the smallest subinterval of Π_{k_2}, and $\delta = \min\{\delta_{k_2}, \delta'_{k_2}\}$. From lemma 3.7 we know that there exist a positive integer n_ϵ so that for all $m \geq n_\epsilon$, $\Pi_{k_2}^m$ and Π_{k_2} have the same code, so the corresponding terms of the paths $(R_m)_x$ and R_x that do not fall in the intervals adjacent to p_{k_1} have a distance less than $\varepsilon/2$. On the other hand the rest of the terms of all the sequences $(R_m)_x$, R_x lie in the two intervals adjacent to p_{k_1}, each with length less than $\frac{\varepsilon}{2}$, thus we have $d_1((R_m^+)_x, R_x^+) < \varepsilon$, $d_1((R_m^-)_x, R_x^-) < \varepsilon$, $d_H((R_m^+)_x, R_x^+) < \varepsilon$, $d_H((R_m^-)_x, R_x^-) < \varepsilon$. Thus the result follows in both cases. □

As an application of the turbulence on trajectories one may consider two trajectories that differ in a finite number of terms, expecting the corresponding first return systems to be eventually the same. As the following theorem shows the result is as expected and follows immediately from the hypothesis and the definition of first return.

Theorem 3.9. *Let $P = \{p_n\}$ and $Q = \{q_n\}$ be two trajectories that differ in a finite number of terms, and let R^+ and E^+ be the right (R^- and E^- be the left) first return systems generated by P and Q, respectively. Then R^+ and E^+ (R^- and E^-) are eventually the same.*

PROOF. Since P and Q differ in a finite number of terms, there exist positive integers m and n so that $p_{m+i} = q_{n+i}$ for $i \geq 1$. Let $\Pi_k =; 0 = r_0 < r_1 < r_2, < \cdots < r_l = 1$ be a partition of $[0,1]$ containing the first m terms of P and the first n terms of Q. Let $x \in [0,1)$ be an arbitrary point, then $x \in [r_i, r_{i+1})$ for some i. Let η be the first element of P that lies in (x, r_{i+1}), we show $\eta \in R_x^+ \cap E_x^+$. To see this let $t_0 = x, t_1 = 1$ and $t_2 = r(x, t_1)$. If $t_2 \geq r_{i+1}$, then t_2 is a point of the partition to the right of x and $t_2 - x < t_1 - x$. Since there are finitely many members of the partition to the right of x. By performing in this way there is i_0 so that $t_{i_0} \geq r_{i+1}$, but $t_{i_0+1} = r(x, t_{i_0}) < r_{i+1}$. Since t_{i_0+1} is the first element of P in the interval (x, t_{i_0}), and there is no prior element of P in the interval $[r_{i+1}, t_{i_0})$, we have $t_{i_0+1} = \eta$. In a similar way we can show that $\eta \in E_x^+$. From this point on, the paths leading to x from right, R_x^+, and E_x^+ will have exact same element, since $P \cap (x, \eta) = Q \cap (x, \eta)$. Thus the right hand first systems are eventually the same. Similarly we can show that the first return left path systems generated by P and Q are eventually the same. Thus the first return path systems generated by the trajectories P and Q are eventually the same. □

The converse to Theorem 3.9 is not true in general, however with some extra condition on the generated first return path systems, the converse is true.

Theorem 3.10. *Suppose P and Q are two trajectories and $R = \{R_x : x \in [0,1\}$ and $E = \{E_x : x \in [0,1]\}$ are the first return path systems generated by P and Q, respectively. If R and E are eventually the same in a uniform way, then the trajectories P and Q differ in a finite number of terms.*

Proof. First we show that for the trajectory P we have $P = \cup_{x\in[0,1]}R_x^+ \setminus \{x\} = \cup_{x\in[0,1]}R_x^- \setminus \{x\}$. To show this, it is enough to show that for each $p_i \in P$ we have $p_i \in R_x^+ \setminus \{x\}$ for some x. Similarly one can show $p_i \in R_x^- \setminus \{x\}$. We note that if r is an element of the trajectory and $r \in R_{x_0}^+$ for some $x_0 < r$, then for each $x < y < r$ we have $r \in R_y^+$. Let $x_0 < p_i$ be a point in the trajectory so that $p_i \notin R_{x_0}^+$, then there exist r_m and r_{m+1} in $R_{x_0}^+$ so that $x_0 < r_{m+1} < p_i < r_m$. This means that in the ordering of P, r_{m+1} appears after r_m and before p_i. Choose $x_1 \in (r_{m+1}, p_i)$ so that $p_i - x_1 < \frac{1}{2}(p_i - x_0)$. Obviously $r_m \in R_{x_1}^+$. If $p_i \notin R_{x_1}^+$, then there exist some $r_{l_1} \in P$ such that $r_{m+1} < r_{l_1} < p_i$ and r_{l_1} appears after r_{m+1} and before p_i in the trajectory P. Now we choose $x_2 \in (r_{l_1}, p_i)$ so that $p_i - x_2 < \frac{1}{2}(p_i - x_1)$. Again $r_m \in R_{x_2}^+$ and if $p_i \notin R_{x_2}^+$ there exists $r_{l_2} \in P$ such that $x_2 < r_{l_2} < p_i$ and r_{l_2} appears after r_{m+1} and before p_i. Pick $x_3 \in (r_{l_2}, p_i)$ and continue the same process. Due to the fact that there are only finitely many members of P that appear before p_i, in the process of choosing the sequence $x_i, i = 0, 1, 2, \ldots$ that increases to p_i, we reach a point x_k with $p_i \in R_{x_k}^+$. Thus we have $P = \cup_{x\in[0,1]}R_x^+ \setminus \{x\}$. Similarly we have $Q = \cup_{x\in[0,1]}E_x^+ \setminus \{x\}$.
Let δ be the positive number resulting from the path systems R and E being eventually the same, in a uniform way, and let Π_k be a partition of [0,1] by the first k terms of P with largest subinterval less than δ. Then for each subinterval $[c_i, d_i) = \Pi_{k,i}$, we have $R_x^+ \cap [d_i, 1] = R_y^+ \cap [d_i, 1]$ and $R_x^- \cap [0, c_i] = R_y^- \cap [0, c_i]$ for each $x, y \in (c_i, d_i)$. On the other hand, for each i, $d_i - c_i < \delta$, thus, for each $x \in (c_i, d_i)$ we have $R_x \cap (c_i, d_i) = E_x \cap (c_i, d_i)$, hence $(\cup_{x\in[c_i,d_i)}R_x^+ \setminus \{x\}) \setminus [c_i, d_i)$ is a finite set and therefore $(\cup_{x\in[0,1]}R_x^+ \setminus \{x\} \setminus \cup_i (c_i, d_i)) = \cup_i (\cup_{x\in[c_i,d_i)}R_x^+ \setminus \{x\}) \setminus [c_i, d_i)$ that is a finite union of finite sets and thus a finite set. This implies that $P \setminus Q \subseteq (\cup_{x\in[0,1]}R_x^+ \setminus \{x\}) \setminus (\cup_{x\in[0,1]}E_x^+ \setminus \{x\}) \subseteq (\cup_i \cup_{x\in[c_i,d_i)}R_x^+ \setminus \{x\}) \setminus [c_i, d_i)$. Implying that $P \setminus Q$ is a finite set. Similarly we may show that $Q \setminus P$ is a finite set. Thus the trajectories P and Q differ in a finite number of terms. □

Example 3.11. There exist a trajectory P with two rearrangements P_1 and P_2 so that the first return path systems generated by P_1 and P_2 are not eventually the same in a uniform way.

Proof. Let $P = \{0, 1\} \cup \{\frac{k}{2^m} : m = 0, 1, 2, 3, \ldots \& k = 0, 1, 3, 5, \ldots, 2^m - 1\}$. It is easy to see that P is a trajectory. We rearrange P in two different ways and call them P_1 and P_2. Take
$P_1 = \{0, 1\} \cup \{\frac{k}{2^{2m}}, 1 \le k < 2^{2m},\ k \text{ odd integer },\ \frac{k}{2^{2m-1}}, 1 \le k < 2^{2m-1},\ k \text{ odd integer }\}_{m\ge1}$
$= \{0, 1, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{16}, \frac{3}{16}, \ldots, \frac{15}{16}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{64}, \frac{3}{64}, \ldots, \frac{63}{64}, \frac{1}{32}, \ldots, \frac{1}{256}, \ldots, \frac{255}{256}, \frac{1}{128}, \ldots\}$
and $P_2 = \{0, 1, \frac{1}{2}\} \cup \{\frac{k}{2^{2m+1}}, 1 \le k < 2^{2m+1},\ k \text{ odd integer },\ \frac{k}{2^{2m}}, 1 \le k <$

$2^{2m},\ k$ odd integer $\}_{m\geq 1}$
$=\{0,1,\frac{1}{2},\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8},\frac{1}{4},\frac{3}{4},\frac{1}{32},\dots,\frac{31}{32},\frac{1}{16},\dots,\frac{15}{16},\frac{1}{128},\dots,\frac{127}{128},\frac{1}{64},\dots,\frac{63}{64},\frac{1}{512},\dots\}$.
Let R and Q be the first return systems generated by P_1 and P_2, respectively. We show that at each $x\in P$ the right first return paths R_x^+ and Q_x^+ are eventually different. In fact we have:
$R^+_{\frac{1}{2}}=\{\frac{1}{2},1,\frac{3}{4},\frac{9}{16},\frac{33}{64},\dots\}=\{\frac{1}{2},1\}\cup\{\frac{2^{(2k-1)}+1}{2^{2k}}\}_{k\geq 1}$,
$Q^+_{\frac{1}{2}}=\{\frac{1}{2},1,\frac{5}{8},\frac{17}{32},\frac{65}{128},\frac{257}{512},\dots\}=\{\frac{1}{2},1\}\cup\{\frac{2^{2k}+1}{2^{(2k+1)}}\}_{k\geq 1}$,
$R^+_{\frac{1}{4}}=\{1,\frac{1}{4},\frac{3}{4},\frac{1}{2},\frac{5}{16},\frac{17}{64},\dots\}=\{\frac{1}{4},\frac{3}{4},1\}\cup\{\frac{2^{2k}+1}{2^{2(k+1)}}\}_{k\geq 0}$,
$Q^+_{\frac{1}{4}}=\{\frac{1}{4},1,\frac{1}{2},\frac{3}{8},\frac{9}{32},\frac{33}{128},\dots\}=\{\frac{1}{4},1,\frac{1}{2}\}\cup\{\frac{2^{2k-1}+1}{2^{2(k+1)}}\}_{k\geq 1}$,

$$R^+_{\frac{3}{4}}=\{\tfrac{3}{4},1,\tfrac{13}{16},\tfrac{49}{64},\tfrac{209}{256},\dots\},\ Q^+_{\frac{3}{4}}=\{\tfrac{3}{4},1,\tfrac{7}{8},\tfrac{25}{32},\tfrac{97}{128},\tfrac{385}{512},\dots\},$$

$$R^+_{\frac{1}{8}}=\{\tfrac{1}{8},1,\tfrac{1}{4},\tfrac{3}{16},\tfrac{9}{64},\tfrac{33}{256},\dots\},\ Q^+_{\frac{1}{8}}=\{\tfrac{1}{8},1,\tfrac{1}{2},\tfrac{3}{8},\tfrac{1}{4},\tfrac{5}{32},\tfrac{17}{128},\tfrac{65}{512},\dots\},$$

$$R^+_{\frac{3}{8}}=\{\tfrac{3}{8},1,\tfrac{3}{4},\tfrac{1}{2},\tfrac{7}{16},\tfrac{25}{64},\dots\},\ Q^+_{\frac{3}{8}}=\{\tfrac{3}{8},1,\tfrac{1}{2},\tfrac{5}{8},\tfrac{13}{32},\tfrac{49}{128},\tfrac{133}{512},\dots\},$$

$$R^+_{\frac{5}{8}}=\{\tfrac{5}{8},1,\tfrac{3}{4},\tfrac{11}{16},\tfrac{41}{64},\tfrac{161}{256},\dots\},\ Q^+_{\frac{5}{8}}=\{\tfrac{5}{8},1,\tfrac{7}{8},\tfrac{3}{4},\tfrac{21}{32},\tfrac{81}{128},\tfrac{321}{512},\dots\},$$

$$R^+_{\frac{7}{8}}=\{\tfrac{7}{8},1,\tfrac{15}{16},\tfrac{57}{64},\tfrac{225}{256},\dots\},\ Q^+_{\frac{7}{8}}=\{\tfrac{7}{8},1,\tfrac{29}{32},\tfrac{113}{128},\tfrac{449}{512},\dots\},$$

We see that for all $x\in P$, the right first return paths R_x^+ and Q_x^+ are different after a few first terms. This is also true for R_x^- and Q_x^-. Now let $x\in[0,1]\setminus P$, then for each $\delta>0$ choose k large enough so that the length of the largest subinterval in Π_k is less than δ, where Π_k is the partition obtained from the first k terms of P. Suppose $x\in\Pi_{k,i}$ so there exist c and d members of Π_k so that $c<x<d$. Choose $r\in P\cap(c,d)$, then by Lemma 1 of [10], we have $R_x^+\cap[d,1]=R_r^+\cap[d,1]$, $R_x^-\cap[0,c]=R_r^-\cap[0,c]$, $Q_x^+\cap[d,1]=Q_r^+\cap[d,1]$ and $Q_x^-\cap[0,c]=Q_r^-\cap[0,c]$. Thus from the fact that P is dense in [0,1], and for each $r\in P$, R_r and Q_r are different except for a few terms, it follows that we can not have $\delta>0$ so that for all $x\in[0,1]$, $R_x\cap(x-\delta,x+\delta)=Q_x\cap(x-\delta,x+\delta)$. Hence R and Q are not eventually the same in a uniform way. □

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