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ON NON-EQUILIBRATED ALMOST MONOTONIC FUNCTIONS OF THE ZYGMUND-BARY-STECHKIN CLASS

Abstract

We study quasi-monotonic functions of the Zygmund-Bary-Stechkin class Φ with the main emphasis on properties of the index numbers of functions in this class. A special attention is paid to functions whose lower and upper index numbers do not coincide with each other (non-equilibrated functions). It is proved that the bounds for functions in Φ known in terms of these indices, are exact in a certain sense. We also single out some special family of none-equilibrated functions in Φ which oscillate in a certain way between two power functions. Given two numbers $0 < \alpha \leq \beta < 1$, we explicitly construct examples of functions in Φ for which α and β serve as the index numbers.

The investigation of properties of non-equilibrated functions in Φ was evoked by applications of these properties in problems of the normal solvability of some singular integral operators in the spaces with prescribed modulus of continuity.

1 Introduction.

The famous A. Zygmund's estimate (see [22], [23]) of modulus of continuity of the conjugate function $\widetilde{f}$ (singular integral) via that of the function f has the form

$$\omega(\widetilde{f}, h) \leq c \int_0^h \frac{\omega(f,x)}{x}\, dx + ch \int_h^\ell \frac{\omega(f,x)}{x^2}\, dx$$

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where $c > 0$ does not depend on f, so that the singular integral preserves the generalized Hölder behavior

$$|f(t) - f(\tau)| \leq c\omega(|t - \tau|)$$

of a function f, if the continuous, increasing function $\omega(x)$ positive for $x > 0$ and vanishing at the origin: $\omega(0) = 0$, satisfies the conditions

$$\int_0^h \frac{\omega(x)}{x}\, dx \leq c\omega(h) \tag{Z_0}$$

and

$$\int_h^\ell \frac{\omega(x)}{x^2}\, dx \leq c\frac{\omega(h)}{h}. \tag{Z_ℓ}$$

Monotonic functions satisfying conditions (Z_0) and (Z_ℓ) were extensively studied together with some related conditions known as Lozinskii condition, Stechkin condition and others in the paper by N. Bary and S. Stechkin [1]. Thereafter the class of functions satisfying conditions (Z_0) and (Z_ℓ) was usually referred to as the Bary-Stechkin class (often denoted as Φ).

Such a class or its modification or generalizations proved to be of importance in the investigation of mapping properties of various operators in spaces of continuous functions with prescribed behavior of the modulus of continuity (the generalized Hölder space H^ω), such as the singular integral operator $Sf(x) = \frac{1}{\pi}\int\limits_a^b \frac{f(t)\,dt}{t-x}$, see for instance the papers [20], [21], or fractional integration operators [19], [7] (see also [18], Section 13.6).

In [14], in connection with the study of singular integral operators in the generalized Hölder spaces, the so called index numbers (or indices) m_ω and M_ω of an almost increasing function ω were introduced, which play the same role as, for example, the Boyd indices play for the Orlicz spaces. These indices are similar to indices known for submultiplicative functions, see [8], p. 75; [9], or [2], p. 149. In terms of the indices m_ω and M_ω in [14], [15], [17] there were obtained criterions of normal solvability of certain classes of integral equations.

Functions ω with equal indices $m_\omega = M_\omega$ were called *equilibrated*.

New applications required new properties of functions in the Zygmund-Bary-Stechkin class Φ. Some of them were developed in [14], [15], [17]. In this paper we give some new properties of functions in the class Φ, mainly related to the index numbers, with an emphasis on better understanding of the nature of non-equilibrated functions $\omega \in \Phi$.

In particular, we show that the known fact [14] that every function ω satisfying conditions (Z_0) and (Z_ℓ) oscillates between $x^{M_\omega+\varepsilon}$ and $x^{m_\omega-\varepsilon}$ is precise, in a certain sense (Theorem 4.3), but the main aspect of this paper is rather related to the question of explicit construction of non-equilibrated functions in the class Φ. Namely, given numbers $0 < \alpha < \beta < 1$, how can one explicitly construct increasing, continuous functions ω oscillating between x^α and x^β and having the indices $m_\omega = \alpha$ and $M_\omega = \beta$? Such kind of constructions are studied in Section 5.

The present study of the class Φ was motivated by applications of properties of functions from the class Φ in the theory of singular integral operators, see [15], [16], [17], and fractional and potential operators, see [6]. In particular, in Fredholmness results for singular operators, massive spectrum appears when the characteristic ω of the space is non-equilibrated. The existence of non-equilibrated characteristics satisfying the Zygmund condition was not an obvious fact (and usually disputable). We construct a family of such characteristics. This construction shows that there exist generalized Hölder spaces with "nice" characteristics (that is, belonging to Φ), whose indices m_ω and M_ω may be two à priori given numbers in the interval $(0, 1)$; in applications to the spectrum properties of singular operators this means the existence of the corresponding " lunes" defining the massiveness of the essential spectrum of the operator.

In Sections 2-3 we provide necessary definitions and known facts for functions from Φ which we need in the sequel. New results are given in Sections 4-5, the main statements being Theorem 4.3, which states that the bounds for characteristics ω given in terms of power functions are exact in a certain sense, and Theorem 5.10 in which the above mentioned family of non-equilibrated characteristics is constructed.

2 Definitions.

2.1 Zygmund-Bary-Stechkin Class Φ.

First we recall that a non-negative function φ on $[0, \ell]$ is said to be almost increasing (or almost decreasing) if there exists a constant $C \geq 1$ such that $\varphi(x) \leq C\varphi(y)$ for all $x \leq y$ (or $x \geq y$, respectively). This notion is due to S.Bernstein [3]. Let

$$W = \{\varphi \in C([0, \ell]) : \varphi(0) = 0,\ \varphi(x) > 0 \text{for } x > 0,\ \varphi(x) \text{ is almost increasing}\}. \tag{2.1}$$

Definition 2.1. A function $\omega \in W$ is said to be in the Zygmund-Bary-Stechkin class Φ if it satisfies conditions (Z_0) and (Z_ℓ).

Remark 2.2. In the notation Φ for the class introduced in Definition 2.1, we follow the book [5], p. 54, while in the original paper [1] Φ denoted the class which we denote as W.

2.2 Index Numbers of Functions $\omega \in \Phi$.

Definition 2.3. Let $\omega \in W$. The numbers

$$m_\omega = \sup_{x>1} \frac{\ln\left[\underline{\lim}_{h\to 0} \frac{\omega(xh)}{\omega(h)}\right]}{\ln\ x} \quad M_\omega = \inf_{x>1} \frac{\ln\left[\overline{\lim}_{h\to 0} \frac{\omega(xh)}{\omega(h)}\right]}{\ln\ x}$$

introduced in such a form in [12], [14], will be referred to as the *lower and upper index numbers* of a function $\omega(x) \in \Phi$. Compare these indices with the Matuszewska-Orlicz indices, see [10], p. 20, introduced for increasing, unbounded functions f defined on $(0, \infty)$, in the context of the Orlicz type spaces. We deal with characteristics ω of the generalized Hölder spaces; they are of the type of the Boyd indices, see [8], p. 75; [9], or [2], p. 149 about the Boyd indices.

We call a characteristic $\omega(x)$ *equilibrated*, if $M_\omega = m_\omega$.
It is easily seen that for $\omega_\lambda(x) = \frac{\omega(x)}{x^\lambda}$ one has

$$m_{\omega_\lambda} = m_\omega - \lambda \text{ and } M_{\omega_\lambda} = M_\omega - \lambda. \tag{2.2}$$

Remark 2.4. Note that the lower index m_ω may be expressed in terms of the upper limit

$$\overline{\Omega}(x) = \overline{\lim_{h\to 0}} \frac{\omega(xh)}{\omega(h)}.$$

Namely,

$$m_\omega = \sup_{0<x<1} \frac{\ln \overline{\Omega}(x)}{\ln\ x}. \tag{2.3}$$

(This fact was drawn to our attention by A. Karlovich.) Since the function $\overline{\Omega}(x)$ is submultiplicative; i.e., $\overline{\Omega}(xy) \le \overline{\Omega}(x)\overline{\Omega}(y)$, one may use properties of such functions (see [8], p. 75, or [4], p. 13, or [11], p. 84), which yield the following representation of the index numbers m_ω and M_ω.

$$m_\omega = \sup_{0<x<1} \frac{\ln \overline{\Omega}(x)}{\ln\ x} = \lim_{x\to 0} \frac{\ln \overline{\Omega}(x)}{\ln\ x}, \tag{2.4}$$

$$M_\omega = \sup_{x>1} \frac{\ln \overline{\Omega}(x)}{\ln\ x} = \lim_{x\to\infty} \frac{\ln \overline{\Omega}(x)}{\ln\ x}. \tag{2.5}$$

Thus always $0 \le m_\omega \le M_\omega \le \infty$ for $\omega \in W$ (compare this with (3.1)), the inequality $m_\omega \le M_\omega$ following from (2.4)-(2.5) by properties of submultiplicative functions ([8], p. 75).

3 Preliminaries about the Class Φ.

It is known that the class Φ is also characterized in some other terms. We introduce the conditions

$$\sum_{k=n+1}^{\infty} \frac{1}{k}\varphi\left(\frac{1}{k}\right) \le c\varphi\left(\frac{1}{n}\right), \tag{B}$$

$$\sum_{k=1}^{n} \varphi\left(\frac{1}{k}\right) \le cn\varphi\left(\frac{1}{n}\right), \tag{B_ℓ}$$

$$\text{there exists a } \xi > 1 \text{ such that } \underline{\lim}_{h\to 0} \frac{\varphi(\xi h)}{\varphi(h)} > 1, \tag{L}$$

$$\text{there exists a } \xi > 1 \text{ such that } \overline{\lim}_{h\to 0} \frac{\varphi(\xi h)}{\varphi(h)} < \xi, \qquad (L_\ell),$$

known as the Bary and Lozinskii conditions, see [1].

Lemma 3.1. *Let $\varphi(x) \in W$. Conditions (B), (L) and (Z) are equivalent. Similarly, conditions (B_ℓ),(L_ℓ) and (Z_ℓ) are also equivalent.*

This lemma is known (see [1]) when the class W is defined by assuming that $\varphi(t)$ is increasing; not almost increasing. However, the lemma remains true in this more general case (see the proof in [12], [13]).

The following statement proved in [14], p. 125 (see also [12]), characterizes the class Φ in terms of the indices m_ω and M_ω.

Theorem 3.2. *A function $\omega(x) \in W$ is in the class Φ if and only if*

$$0 < m_\omega \le M_\omega < 1, \tag{3.1}$$

and for $\omega \in \Phi$ and any $\varepsilon > 0$ there exist constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that

$$c_1 x^{M_\omega+\varepsilon} \le \omega(x) \le c_2 x^{m_\omega-\varepsilon}, \text{ for } 0 \le x \le \ell. \tag{3.2}$$

Besides this, condition (Z) is equivalent to $m_\omega > 0$ while condition (Z_ℓ) is equivalent to $M_\omega < 1$.

A statement similar to (3.1) was known in another situation - for similar indices of increasing unbounded functions ω defined on $(0,\infty)$, in the context of the Orlicz type spaces, see [11], p. 90. We deal here with (Z) and (Z_ℓ)-conditions of characteristics ω of the generalized Hölder spaces.

In this paper we show in particular, that bounds (3.2) for $\omega \in \Phi$ are exact in a certain sense, see Theorem 4.3.

The following statement is known ([1], Lemmas 2 and 3).

Lemma 3.3. *For a non-decreasing function $\omega \in W$ the following equivalences are valid:*

$$(Z) \iff \text{there exists a } \delta_1 > 0 \text{ such that } \frac{\omega(x)}{x^{\delta_1}} \text{ is almost increasing,} \tag{3.3}$$

$$(Z_\ell) \iff \text{there exists a } \delta_2 \in (0,1) \text{ such that } \frac{\omega(x)}{x^{\delta_2}} \text{ is almost decreasing.} \tag{3.4}$$

The following statement was also proved in [14], p. 125, see also [12].

Theorem 3.4. *Let $\omega \in W$. If $\omega(x)$ satisfies condition (Z_0), then $x^{-\delta_1}\omega(x)$ is almost increasing for any $\delta_1 < m_\omega$, while fulfillment of (Z_ℓ) implies that $x^{-\delta_2}\omega(x)$ is almost decreasing for any $\delta_2 > M_\omega$.*

4 Some Properties of the Zygmund-Bary-Stechkin Class Φ.

We show that the bounds for $\omega \in \Phi$ given in (3.2) are *exact* in the following sense. For any $\varepsilon > 0$, there exist sequences $x_n \to 0$ and $y_n \to 0$ and positive constants c_3 and c_4 not depending on ε such that

$$\omega(x_n) \geq c_3 x_n^{m_\omega+\varepsilon}, \tag{4.1}$$

$$\omega(y_n) \leq c_4 y_n^{M_\omega-\varepsilon}. \tag{4.2}$$

The following lemma shows that the bounds $\delta_1 < m_\omega$ and $\delta_2 > M_\omega$ in Theorem 3.4, cannot be improved.

Lemma 4.1. *Let a nondecreasing function ω belong to Φ. If $\frac{\omega(x)}{x^\alpha}$ is almost increasing and $\frac{\omega(x)}{x^\beta}$ is almost decreasing for some $0 < \alpha \leq \beta < 1$, then $m_\omega \geq \alpha$ and $M_\omega \leq \beta$.*

Proof. Suppose to the contrary that $m_\omega < \alpha$. Then the function $\omega_1(x) = \frac{\omega(x)}{x^{m_\omega}}$ is also almost increasing and $\omega_1(0) = 0$ since $m_\omega < \alpha$. Therefore, $\omega_1 \in W$. But also the function

$$\frac{\omega_1(x)}{x^{\delta_1}} = \frac{\omega(x)}{x^\alpha}, \ \delta_1 = \alpha - m_\omega$$

is almost increasing. Then, by Lemma 3.3, the function $\omega_1(x)$ satisfies the (Z_0)-condition. Therefore, by Theorem 3.2, its lower index m_{ω_1} is positive $m_{\omega_1} > 0$ which is impossible since $m_{\omega_1} = m_\omega - m_\omega = 0$ by (2.2).

The statement $M_\omega \le \beta$ is proved similarly. $\square$

We need also the following statement.

Lemma 4.2. *For any $\omega \in \Phi$ the existence of a sequence x_n with property (4.1) is equivalent to*

$$m_\omega = \sup\left\{\delta \in (0,1) : \frac{\omega(x)}{x^\delta} \text{ is almost increasing}\right\}, \tag{4.3}$$

and the existence of a sequence y_n with property (4.2) is equivalent to

$$M_\omega = \inf\left\{\delta \in (0,1) : \frac{\omega(x)}{x^\delta} \text{ is almost decreasing}\right\}. \tag{4.4}$$

Proof. The set $\left\{\delta \in (0,1) : \frac{\omega(x)}{x^\delta} \text{ is almost increasing}\right\}$ is a closed or semi-closed interval $[0, a]$ or $[0, a)$ for some $0 < a < 1$. Similarly, the set $\left\{\delta \in (0,1) : \frac{\omega(x)}{x^\delta} \text{ is almost decreasing}\right\}$ is either $[b, 1]$ or $(b, 1]$ with $a \le b < 1$. By Theorem 3.4, $m_\omega \le a$ and $M_\omega \ge b$. We have to prove that

$$(4.1) \iff m_\omega = a \text{ and } (4.2) \iff M_\omega = b.$$

1. The Proof of the Implication (4.1) $\implies m_\omega = a$. Suppose that $m_\omega < a$. Then for any $\varepsilon \in \left(0, \frac{\alpha - m_\omega}{2}\right)$ the function $\frac{f(x)}{x^{m_\omega + 2\varepsilon}}$ is almost increasing and consequently bounded, which is impossible since for a sequence x_n from (4.1) we have $\frac{f(x_n)}{x_n^{m_\omega+2\varepsilon}} \ge c_3 \frac{x^{m_\omega+\varepsilon}}{x_n^{m_\omega+2\varepsilon}} = \frac{c_3}{x_n^\varepsilon} \to \infty$. Therefore, $m_\omega = a$.
2. The Proof of the Implication $m_\omega = a \implies$ (4.1). Since $m_\omega = a$, the function $\omega_\varepsilon = \frac{\omega(x)}{x^{m_\omega+\varepsilon}}$ is not almost increasing for any $\varepsilon > 0$. Any non almost increasing function in W is unbounded; that is,

$$\sup_{0<x\le y\le \ell} \frac{\omega_\varepsilon(x)}{\omega_\varepsilon(y)} = \infty \implies \sup_{0<x\le \ell} \frac{\omega_\varepsilon(x)}{\omega_\varepsilon(\ell)} = \infty.$$

Since the function $\omega_\varepsilon(x)$ is continuous for $x > 0$, it may be unbounded only when $x \to 0$. Therefore, there exists a sequence $x_n \to 0$ such that $\omega_\varepsilon(x_n) \geq 1$, which is (4.1).

The equivalence (4.2) $\iff$ $M_\omega = b$ is proved similarly. □

Theorem 4.3. *For any $\omega \in W$ satisfying condition (Z_0) the upper bound $\omega(x) \leq c_2 x^{m_\omega - \varepsilon}$ given in (3.2) is exact (in the sense defined in (4.1)). Similarly, for an $\omega \in W$ satisfying condition (Z_ℓ) the lower bound $\omega(x) \geq c_2 x^{M_\omega + \varepsilon}$ is exact (in the sense defined in (4.2).*

PROOF. By Lemma 4.2, it suffices to show that equalities (4.3)–(4.4) are valid. Suppose to the contrary that the equality in (4.3) doesn't hold. Then the function $\omega_{2\varepsilon}(x) = \frac{\omega(x)}{x^{m_\omega + 2\varepsilon}}$ is almost increasing for any small $\varepsilon > 0$. Then $\omega_{2\varepsilon}(x)$ is a bounded function and therefore the function $\omega_\varepsilon(x) = x^\varepsilon \omega_{2\varepsilon}(x)$ is in W and moreover, $\omega_\varepsilon(x)$ satisfies condition (Z_0) by Lemma 3.3, since the function $\frac{\omega_\varepsilon(x)}{x^\varepsilon} = \omega_{2\varepsilon}(x)$ is almost increasing. Then $m_{\omega_\varepsilon} > 0$ by Theorem 3.2. But on the other hand by (2.2) we obtain $m_{\omega_\varepsilon} = m_\omega - (m_\omega + \varepsilon) = -\varepsilon < 0$. Consequently, (4.3) must hold.

Property (4.4) is proved similarly. □

5 Construction of a Family of Non-Equilibrated Functions $\omega \in \Phi$.

It is easy to give examples of equilibrated characteristics ω. Besides the trivial examples $\omega(x) = x^\lambda, \omega(x) = x^\lambda \left(\ln \frac{1}{x}\right)^\alpha,\ x^\lambda \left(\ln\ln \frac{1}{x}\right)^\alpha$, etc, $0 < \lambda < 1$, for which $m_\omega = M_\omega = \lambda$, we may also mention that the condition

$$\lim_{h\to 0} \frac{\omega(th)}{\omega(h)} = t^\gamma,\ \gamma = const,$$

is sufficient for $\omega(x)$ to be equilibrated. For example, the function $x^{\lambda + \frac{c}{\ln^\alpha \frac{1}{x}}}$, $\alpha \geq 1$, satisfies this condition. One may also take $\omega(x) = x^{\gamma(x)}$ with $\gamma(x)$ satisfying the Dini condition $|\gamma(x+h) - \gamma(x)| = o\left(\frac{1}{|\ln\ |h||}\right)$.

Examples of non-equilibrated characteristics are much less trivial. According to (3.2) any function in the class Φ is dominated from above and from below by power functions. Clearly, this is not a characterization of the class Φ. It is easy to give sufficient conditions for a function $\omega \in W$ to be non-equilibrated; for instance,

$$c_1 x^\beta \leq \omega(x) \leq c_2 x^\alpha,\ \text{for } 0 < \alpha < \beta < 1, \tag{5.1}$$

$$\omega(b_n) = c_1 b_n^\beta,\ \omega(a_n) = c_2 a_n^\alpha \text{ for some } a_n \to 0,\ b_n \to 0. \tag{5.2}$$

Under conditions (5.1)–(5.2) we have $m_\omega \le \alpha$ and $M_\omega \ge \beta$, so that $\omega(x)$ is certainly non-equilibrated. However, conditions (5.1)–(5.2) do not guarantee yet that $\omega \in \Phi$, because it may happen that $m_\omega = 0$ and/or $M_\omega \ge 1$. The membership in Φ of a function $\omega(x)$ oscillating between $c_1 x^\beta$ and $c_2 x^\alpha$ implies certain restrictions on the character of the oscillation of ω.

There arises the problem of finding easily verified sufficient conditions for a function $\omega \in W$ to be non-equilibrated and belong to Φ. Or, given $0 < \alpha < \beta < 1$, how can one explicitly construct an oscillation of $\omega(x)$ between $c_1 x^\beta$ and $c_2 x^\alpha$ so that ω is certainly in Φ? Or in a more restricted way; given $0 < \alpha < \beta < 1$, what should be the construction of a function $\omega \in \Phi$ oscillating between $c_1 x^\beta$ and $c_2 x^\alpha$ such that $m_\omega = \alpha$ and $M_\omega = \beta$?

We construct a family of non-equilibrated such functions in the next subsections.

5.1 On a Class of Functions Oscillating between x^β and x^α.

For simplicity we assume that $[0, \ell] = [0, 1]$. Let

$$\mathcal{P} = \{\dots, a_n, a_{n-1}, \dots, a_1, a_0\}$$

be an arbitrary partition of $[0, 1]$ by a sequence of points such that

$$\dots < a_n < a_{n-1} < \dots < a_1 < a_0 = 1 \text{ and } \lim_{n\to\infty} a_n = 0. \tag{5.3}$$

We introduce a function $\omega(x) = \omega_{\mathcal{P}}(x)$ depending on the partition $\mathcal{P}$, which is equal to x^β on subintervals $I_{2n+1} = [a_{2n+2}, a_{2n+1}]$ and is equal to x^α on subintervals $I_{2n} = [a_{2n+1}, a_{2n}]$, up to a multiplicative constant. Namely, we put

$$\omega(x) = \begin{cases} c_{2n+1} x^\beta, & \text{if } x \in I_{2n+1} \\ c_{2n} x^\alpha, & \text{if } x \in I_{2n} \end{cases} n = 0, 1, 2, \dots \tag{5.4}$$

where we take $c_0 = 1$ and afterwards the coefficients are chosen, step by step, in such a way that the resulting function $\omega(x)$ is continuous.

Definition 5.1. Any continuous function on $[0, 1]$ having the form (5.4) will be referred to for brevity as an (α, β)-**function**.

Given a partition $\mathcal{P} = \{\dots, a_n, a_{n-1}, \dots, a_1, a_0\}$, for $n = 1, 2, \dots$ the coefficients c_{2n} and c_{2n+1} of an (α, β)-function are calculated by the formulas

$$c_{2n} = \left(\frac{a_0 a_2 a_4 \cdots a_{2n}}{a_1 a_3 \cdots a_{2n-1}}\right)^{\beta-\alpha},\ c_{2n+1} = \left(\frac{a_0 a_2 a_4 \cdots a_{2n}}{a_1 a_3 \cdots a_{2n+1}}\right)^{\beta-\alpha}, \tag{5.5}$$

so that the coefficients c_m and the partition points a_m are related to each other by the formulas

$$c_{2n} = a_{2n}^{\beta-\alpha} c_{2n-1}, \; c_{2n} = a_{2n+1}^{\beta-\alpha} c_{2n+1}. \tag{5.6}$$

Lemma 5.2. *Let $0 < \alpha < \beta < 1$. The even coefficients c_{2n} of an (α, β)-function are decreasing; i.e., $c_{2n+2} < c_{2n}$, while the odd ones are increasing; i.e., $c_{2n+1} > c_{2n-1}$.*

PROOF. It suffices to note that

$$\frac{c_{2n}}{c_{2n-2}} = \left(\frac{a_{2n}}{a_{2n-1}}\right)^{\beta-\alpha} < 1, \; \frac{c_{2n+1}}{c_{2n-1}} = \left(\frac{a_{2n}}{a_{2n+1}}\right)^{\beta-\alpha} > 1, \tag{5.7}$$

which follows from (5.5) and (5.3). □

Examples.

1) $a_n = \frac{1}{(n+1)^\lambda} \Longrightarrow c_{2n} = \left[\sqrt{\pi}\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}\right]^{\lambda(\beta-\alpha)}, c_{2n+1} = \left[\sqrt{\pi}\frac{\Gamma(n+2)}{\Gamma\left(n+\frac{3}{2}\right)}\right]^{\lambda(\beta-\alpha)}$,
so that $c_{2n} \sim \left(\frac{\pi}{n}\right)^{\frac{\lambda(\beta-\alpha)}{2}}$ and $c_{2n+1} \sim (\pi n)^{\frac{\lambda(\beta-\alpha)}{2}}$ as $n \to \infty$;
2) $a_n = 2^{-\lambda n} \Longrightarrow c_{2n} = 2^{-\lambda(\beta-\alpha)n}, c_{2n+1} = 2^{\lambda(\beta-\alpha)(n+1)}$.

Lemma 5.3. *Let $0 < \alpha \leq \beta < 1$. Any (α, β)-function ω belongs to Φ and has the properties*

$$a_1^{\alpha-\beta} x^\beta \leq \omega(x) \leq x^\alpha, \; 0 \leq x \leq 1, \tag{5.8}$$

and

$$m_\omega \geq \alpha, \; M_\omega \leq \beta. \tag{5.9}$$

PROOF. First we show that $\omega \in W$. Since the function $\omega(x)$ is increasing on any one of the subintervals I_{2n} and I_{2n+1} and is continuous under the choice (5.5), it is increasing on the whole interval $[0, 1]$. Obviously, $\omega(x) > 0$ for $x > 0$. We have to show that

$$\lim_{x \to 0} \omega(x) = 0. \tag{5.10}$$

By Lemma 5.2 we have $c_{2n} \leq c_0 = 1$. Consequently, $\omega(x) \leq x^\alpha$ for $x \in I_{2n}$ according to (5.4). For $x \in I_{2n+1}$, by (5.5) we have $\omega(x) = c_{2n+1} x^{\beta-\alpha} \cdot x^\alpha \leq c_{2n+1} a_{2n+1}^{\beta-\alpha} x^\alpha = c_{2n} x^\alpha$. Therefore,

$$\omega(x) \leq \begin{cases} c_{2n} x^\alpha, & x \in I_{2n+1} \\ x^\alpha, & x \in I_{2n} \end{cases} \leq x^\alpha$$

from which (5.10) follows.

To show that $\omega \in \Phi$, we observe that the function

$$\frac{\omega(x)}{x^\alpha} = \begin{cases} c_{2n+1}x^{\beta-\alpha} & x \in I_{2n+1} \\ c_{2n} & x \in I_{2n} \end{cases} \quad n = 0, 1, 2, \dots \tag{5.11}$$

is non-decreasing and the function

$$\frac{\omega(x)}{x^\beta} = \begin{cases} c_{2n+1} & x \in I_{2n+1} \\ c_{2n}x^{\alpha-\beta}, & x \in I_{2n} \end{cases} \quad n = 0, 1, 2, \dots \tag{5.12}$$

is non-increasing. Then $\omega \in \Phi$ by Lemma 3.3.

From (5.12) we also derive that $\omega(x) \geq a_1^{\alpha-\beta} x^\beta$. □

5.2 Finding a Partition by a Given Sequence of Coefficients.

Given any decreasing sequence $c_{2n} > 0$ and an increasing sequence $c_{2n+1} > 0$, one can calculate the partition $\{\dots, a_n, a_{n-1} \dots, a_1, a_0\}$, for which the corresponding (α, β)-function has these prescribed coefficients c_{2n} and c_{2n+1}. Namely, the following lemma holds.

Lemma 5.4. *Given $0 < \alpha < \beta < 1$ and sequences*

$$1 = c_0 > c_2 > c_4 > \cdots > c_{2n} > c_{2n+2} > \cdots$$

and

$$1 < c_1 < c_3 < c_5 < \cdots < c_{2n-1} < c_{2n+1} < \cdots$$

with the property

$$\lim_{n\to\infty} \frac{c_{2n}}{c_{2n-1}} = 0, \tag{5.13}$$

there exists an (α, β)-function (that is, there exists a partition $\{\cdots < a_n < a_{n-1} < \cdots a_1 < a_0 = 1\}$ of the interval $[0, 1]$ with $a_n \to 0$), for which the numbers c_{2n} and c_{2n+1} serve as the corresponding coefficients in definition (5.4) of (α, β)-function. The points a_n are calculated from the given coefficients via the formulas

$$a_{2n} = \left(\frac{c_{2n}}{c_{2n-1}}\right)^{\frac{1}{\beta-\alpha}}, \ a_{2n+1} = \left(\frac{c_{2n}}{c_{2n+1}}\right)^{\frac{1}{\beta-\alpha}}. \tag{5.14}$$

Proof. The fact itself that the numbers c_{2n} and c_{2n+1} serve as coefficients when the numbers a_n are given by (5.14), is verified directly by substituting

(5.14) into (5.5). The only point we have to check then is to guarantee that $a_{n+1} < a_n$ for any n and $\lim\limits_{n\to\infty} a_n = 0$. The first follows from the relations

$$\frac{a_{2n+1}}{a_{2n}} = \left(\frac{c_{2n-1}}{c_{2n+1}}\right)^{\frac{1}{\beta-\alpha}} < 1, \ \frac{a_{2n}}{a_{2n-1}} = \left(\frac{c_{2n}}{c_{2n-2}}\right)^{\frac{1}{\beta-\alpha}} < 1 \tag{5.15}$$

in view of the monotonicity of c_n, while the second is consequence of (5.13) according to (5.14). $\square$

Examples.
1) $c_{2n} = \frac{1}{(n+1)^a}$, $c_{2n+1} = (n+1)^b \Longleftrightarrow a_{2n} = \frac{1}{[(n+1)^a n^b]^{\frac{1}{\beta-\alpha}}}$, $a_{2n+1} = \frac{1}{(n+1)^{\frac{a+b}{\beta-\alpha}}}$;
2) $c_{2n} = 2^{-an}$, $c_{2n+1} = 2^{bn} \Longleftrightarrow a_{2n} = 2^{\frac{b}{\beta-\alpha}} 2^{-\frac{a+b}{\beta-\alpha}n}$, $a_{2n+1} = 2^{-\frac{a+b}{\beta-\alpha}n}$;
3) $c_{2n} = A^{-aT^n}$, $c_{2n+1} = A^{bT^n} \Longleftrightarrow a_{2n} = A^{-\frac{a+\frac{b}{T}}{\beta-\alpha}T^n}$, $a_{2n+1} = A^{-\frac{a+b}{\beta-\alpha}T^n}$;
where $a > 0, b > 0, A > 1, T > 1$.

Examples of the type 3) will be essentially used below.

5.3 A Finer Estimation of $\omega(x)$ When $x \to 0$.

According to Lemma 5.3 any (α,β)-function oscillates between $a_1^{\alpha-\beta}x^\beta$ and x^α. Its oscillation, when $x \to 0$, between power functions may be described in terms of finer exponents as given in the following lemma.

Lemma 5.5. *Let $0 < \alpha \le \beta < 1$. Any (α,β)-function ω satisfies the estimate*

$$x^{\mu_n} \le \omega(x) \le x^{\nu_n}, \ x \in I_{2n+1} \cup I_{2n}, \tag{5.16}$$

where $\mu_n, \nu_n \in [\alpha,\beta]$ and

$$\mu_n = \frac{\alpha \ln c_{2n+1} + \beta \ln \frac{1}{c_{2n+2}}}{\ln c_{2n+1} + \ln \frac{1}{c_{2n+2}}} \ \textit{and}\ \nu_n = \frac{\alpha \ln c_{2n+1} + \beta \ln \frac{1}{c_{2n}}}{\ln c_{2n+1} + \ln \frac{1}{c_{2n}}}. \tag{5.17}$$

PROOF. For $x \in I_{2n+1}$ the inequalities in (5.16) become $x^{\mu_n} \le c_{2n+1}x^\beta \le x^{\nu_n}$. This will certainly be valid if

$$\max_{I_{2n+1}} x^{\mu_n-\beta} \le c_{2n+1} \le \min_{I_{2n+1}} x^{\nu_n-\beta};$$

that is, $a_{2n+2}^{\mu_n-\beta} \le c_{2n+1} \le a_{2n+1}^{\nu_n-\beta}$. By formulas (5.14) we get

$$\left(\frac{c_{2n+2}}{c_{2n+1}}\right)^{\mu_n-\beta} \le c_{2n+1}^{\beta-\alpha} \le \left(\frac{c_{2n}}{c_{2n+1}}\right)^{\nu_n-\beta}$$

whence we obtain that the best exponents for (5.16) to be valid are given by (5.17).

The case when $x \in I_{2n}$ is treated similarly and gives the same values of μ_n and ν_n. $\square$

The estimate obtained in (5.16) leads to the following statement providing a more exact estimation of an (α, β)-function in terms of power function behavior near the origin

Lemma 5.6. *Let $0 < \alpha \le \beta < 1$. Suppose that for an (α, β)-function $\omega(x)$ the limits*

$$\mathcal{B} = \lim_{n\to\infty} \frac{\ln \frac{1}{c_{2n+2}}}{\ln c_{2n+1}} \text{ and } \mathcal{A} = \lim_{n\to\infty} \frac{\ln \frac{1}{c_{2n}}}{\ln c_{2n+1}} \tag{5.18}$$

exist so that $0 \le \mathcal{A} \le \mathcal{B} \le \infty$. Then for any $\varepsilon > 0$ there exists a neighborhood $[0, \delta]$ of the origin where

$$x^{\beta_1+\varepsilon} \le \omega(x) \le x^{\alpha_1-\varepsilon},\ 0 \le x \le \delta, \tag{5.19}$$

with $\alpha_1 = \frac{\alpha+\beta\mathcal{A}}{1+\mathcal{A}}$ and $\beta_1 = \frac{\alpha+\beta\mathcal{B}}{1+\mathcal{B}}$, $\alpha \le \alpha_1 \le \beta_1 \le \beta$.

Proof. The statement of the lemma follows immediately from Lemma 5.5 since $\mu_n = \alpha_1 + \xi_n$ and $\nu_n = \beta_1 + \eta_n$ where $|\xi_n| < \varepsilon$ and $|\eta_n| < \varepsilon$ for large n. $\square$

Corollary. *Under the assumption that limits (5.18) exist, a more exact estimate of the index numbers of an (α, β)-function holds than given in (5.9);* namely,

$$m_\omega \ge \frac{\alpha + \beta\mathcal{A}}{1 + \mathcal{A}} \text{ and } M_\omega \le \frac{\alpha + \beta\mathcal{B}}{1 + \mathcal{B}}. \tag{5.20}$$

5.4 Construction of an (α, β)-Characteristic $\omega \in \Phi$ with Prescribed Index Numbers $m_\omega, M_\omega \in [\alpha, \beta]$.

In general, the index numbers m_ω and M_ω of an (α, β)-function ω lie in the interval $[\alpha, \beta]$ according to (5.9). Since we are interested in constructive examples of non-equilibrated functions in Φ, we aim to show that there exist explicit examples of (α, β)-functions with $m_\omega < M_\omega$.

We give constructions which provide a variety of examples of non-equilibrated (α, β)-functions. First in Lemma 5.7 and Theorem 5.9 we give constructions of (α, β)-functions which certainly have different index numbers m_ω and M_ω, and even more, given $\varepsilon > 0$, we indicate such (α, β)-functions for which $m_\omega \in [\alpha, \alpha + \varepsilon]$ and $M_\omega \in [\beta - \varepsilon, \beta]$. Then in Theorem 5.10 we give constructions of (α, β)-functions for which $m_\omega = \alpha$ and $M_\omega = \beta$.

Lemma 5.7. *Given $0 < \alpha < \beta < 1$ and any λ_1 and λ_2 such that $0 < \alpha < \lambda_1 < \lambda_2 < \beta < 1$, there exists an (α, β)-function $\omega(x)$ such that*

$$\frac{\omega(x)}{x^{\lambda_1}} \text{ is not almost increasing and } \frac{\omega(x)}{x^{\lambda_2}} \text{ is not almost decreasing.} \tag{5.21}$$

Such a function is given by (5.4) with the choice, for instance, of

$$c_{2n} = A^{-T^n}, \ c_{2n+1} = A^{bT^n}, \ b = \frac{\beta-\lambda_1}{2(\lambda_1-\alpha)} + \frac{\beta-\lambda_2}{2(\lambda_2-\alpha)}T, \tag{5.22}$$

where $A > 1$ and $T > \frac{(\lambda_2-\alpha)(\beta-\lambda_1)}{(\lambda_1-\alpha)(\beta-\lambda_2)}$ (> 1), and the partition $\{a_n\}$ is calculated according to (5.14).

PROOF. To obtain an example of an (α, β)-function $\omega(x)$ with property (5.21), it suffices to construct such an (α, β)-function $\omega(x)$, for which there exist sequences $x_n \to 0$ and $y_n \to 0$ such that $\frac{\omega(x_n)}{x_n^{\lambda_1}} \to \infty$ and $\frac{\omega(y_n)}{y_n^{\lambda_2}} \to 0$. We shall construct such an example of an (α, β)-function ω that

$$\max_{x \in I_{2n+1} \cap I_{2n}} \frac{\omega(x)}{x^{\lambda_1}} \to \infty \text{ and } \min_{x \in I_{2n+1} \cap I_{2n}} \frac{\omega(x)}{x^{\lambda_1}} \to 0 \text{ as } n \to \infty. \tag{5.23}$$

(Recall that $I_{2n+1} = [a_{2n+2}, a_{2n+1}], I_{2n} = [a_{2n+1}, a_{2n}]$.) To this end, we observe that for any $\lambda \in (\alpha, \beta)$

$$\max_{x \in I_{2n}} \frac{\omega(x)}{x^{\lambda}} = \max_{x \in I_{2n+1}} \frac{\omega(x)}{x^{\lambda}} = c_{2n+1}^{\theta} c_{2n}^{1-\theta}, \tag{5.24}$$

$$\min_{x \in I_{2n}} \frac{\omega(x)}{x^{\lambda}} = c_{2n-1}^{\theta} c_{2n}^{1-\theta}, \ \min_{x \in I_{2n+1}} \frac{\omega(x)}{x^{\lambda}} = c_{2n+1}^{\theta} c_{2n+2}^{1-\theta}, \tag{5.25}$$

where $\theta = \frac{\lambda-\alpha}{\beta-\alpha}$ which follows from (5.4) and (5.5). Therefore, to get (5.23), we must have

$$\lim_{n\to\infty} c_{2n+1}^{\theta} c_{2n}^{1-\theta} = \lim_{n\to\infty} c_{2n+1}^{\theta} c_{2n+2}^{1-\theta} = 0, \tag{5.26}$$

where $\theta_k = \frac{\lambda_k-\alpha}{\beta-\alpha}$, $k = 1, 2;$ $0 < \theta_1 < \theta_2 < 1$. To obtain (5.26), we choose the odd coefficients c_{2n+1} via the even ones in the following way

$$c_{2n+1} = \frac{\xi_n}{c_{2n}^{\nu_1}} \text{ and at the same time } c_{2n+1} = \frac{\eta_n}{c_{2n+2}^{\nu_1}}, \tag{5.27}$$

where

$$\nu_k = \frac{1-\theta_k}{\theta_k} = \frac{\beta-\lambda_k}{\lambda_k-\alpha}, \ k = 1, 2,$$

and ξ_n and η_n are arbitrary sequences monotonically tending to ∞ and 0, respectively, $\lim\limits_{n\to\infty}\xi_n = \infty$, $\lim\limits_{n\to\infty}\eta_n = 0$. To make both the relations in (5.27) be concordant with each other, we restrict the choice of the even coefficients c_{2n} by

$$\frac{c_{2n+2}^{\nu_2}}{c_{2n}^{\nu_1}} = \frac{\eta_n}{\xi_n} \to 0. \tag{5.28}$$

(Recall that $\nu_2 < \nu_1$.) This restriction implies that the coefficients c_{2n} and consequently the partition points a_n must rapidly tend to zero as $n \to \infty$. Among examples 1)–3) given above, only example 3) satisfies (5.28). We choose $c_{2n} = A^{-T^n}$ with $A > 1, T > 1$ and then

$$\frac{c_{2n+2}^{\nu_2}}{c_{2n}^{\nu_1}} = A^{-(\nu_2 T-\nu_1)T^n} \tag{5.29}$$

where we have $\nu_2 T - \nu_1 > 0$ under the choice $T > \frac{\nu_1}{\nu_2} = \frac{\theta_2-\theta_1\theta_2}{\theta_1-\theta_1\theta_2}$. Therefore, the only restriction put on the choice of the sequences $\xi_n \to \infty$ and $\eta_n \to 0$ up to now is that $\frac{\xi_n}{\eta_n} = B^{-T^n}$, where $B = A^{\nu_2 T-\nu_1} > 1$. We may choose for example, $\xi_n = \sqrt{B^{T^n}} \to \infty$ and $\eta_n = \sqrt{-B^{T^n}} \to 0$. Then we arrive at (5.22).

We also observe that according to (5.14) the corresponding partition points a_n are easily calculated as $a_{2n} = A_1^{-T^n}$ and $a_{2n+1} = A_2^{-T^n}$, where $A_1 = A^B, A_2 - A^C$, $C = \frac{2+\nu_1+\nu_2 T}{2(\beta-\alpha)}$, the property $a_{2n+1} < a_{2n} < a_{2n-1}$ being automatically satisfied by Lemma 5.4 under the choice (5.14). $\square$

Remark 5.8. Obviously, the choice of the coefficients c_n we made in the proof of Lemma 5.7, is not unique. As can be seen from the proof, this choice is restricted only by the following three requirements:

1) c_{2n} satisfy the condition $\lim\limits_{n\to\infty}\frac{c_{2n+2}^{\nu_2}}{c_{2n}^{\nu_1}} = 0$,

2) c_{2n+1} are chosen via c_{2n} as $c_{2n+1} = \frac{\xi_n}{c_{2n}^{\nu_1}} = \frac{\eta_n}{c_{2n+2}^{\nu_2}}$, where $\xi_n \to \infty$ and $\eta_n \to 0$ are arbitrary,

3) c_{2n} and c_{2n+1} are strictly decreasing and increasing, respectively.

The following theorem is in fact just a rephrasing of Lemma 5.7.

Theorem 5.9. *Given* $0 < \alpha < \beta < 1$ *and an arbitrarily small* $\varepsilon > 0$ $\left(0 < \varepsilon < \frac{\beta-\alpha}{2}\right)$, *there may be explicitly constructed an* (α,β)*-function* $\omega(x) = \omega_\varepsilon(x)$ *such that*

$$\alpha \le m_\omega \le \alpha+\varepsilon \text{ and } \beta-\varepsilon \le M_\omega \le \beta. \tag{5.30}$$

This function is given by (5.4) with c_n *defined in (5.22) with the choice* $A > 1, T > \left(\frac{\beta-\alpha-\varepsilon}{\varepsilon}\right)^2$ *and* $b = \frac{\beta-\alpha-\varepsilon}{2\varepsilon} + \frac{\varepsilon}{2(\beta-\varepsilon)}T$.

PROOF. The statement of the theorem follows from Lemma 5.7 (with the choice $\lambda_1 = \alpha+\varepsilon, \lambda_2 = \beta-\varepsilon$) which is applicable since (α,β)-functions belong to Φ according to Lemma 5.3. $\square$

Theorem 5.10. *Given $0 < \alpha < \beta < 1$, there may be explicitly constructed an (α,β)-functions $\omega(x)$ such that*

$$m_\omega = \alpha \text{ and } M_\omega = \beta. \tag{5.31}$$

This function is given by (5.4) with the choice

$$c_{2n} = e^{-A^{u_n}},\ c_{2n+1} = e^{A^{v_n}}, \tag{5.32}$$

where $A > 1$ and u_n and v_n are arbitrary positive increasing sequences with $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} v_n = \infty$ such that

$$\lim_{n\to\infty}(v_n - u_n) = \lim_{n\to\infty}(u_n - v_{n-1}) = \infty. \tag{5.33}$$

PROOF. To prove (5.31), according to Lemma 4.2 we have to show that for any $\varepsilon > 0$ the function $\frac{\omega(x)}{x^{\alpha+\varepsilon}}$ cannot be almost increasing and the function $\frac{\omega(x)}{x^{\beta-\varepsilon}}$cannot be almost decreasing. To this end, it suffices to show that there exists sequences $x_n \to 0$ and $y_n \to 0$ such that

$$\lim_{n\to\infty}\frac{\omega(x_n)}{x_n^{\alpha+\varepsilon}} = \infty \text{ and } \lim_{n\to\infty}\frac{\omega(y_n)}{y_n^{\beta-\varepsilon}} = 0. \tag{5.34}$$

We choose $x_n = a_{2n+1}$ and $y_n = a_{2n}$ independently of ε and show that (5.34) is valid in this case for any $\varepsilon > 0$.

Direct calculation by means of formulas (5.4) and (5.14) yields

$$\lim_{n\to\infty}\frac{\omega(a_{2n+1})}{a_{2n+1}^{\alpha+\varepsilon}} = c_{2n}^{1-\varepsilon_1}c_{2n+1}^{\varepsilon_1} \text{ and } \lim_{n\to\infty}\frac{\omega(a_{2n})}{a_{2n}^{\beta-\varepsilon}} = c_{2n}^{\varepsilon_1}c_{2n-1}^{1-\varepsilon_1}$$

where $\varepsilon_1 = \frac{\varepsilon}{\beta-\alpha}$. Therefore, by (5.34) we arrive at the conditions

$$\lim_{n\to\infty}\left(\frac{\varepsilon_1}{1-\varepsilon_1}\ln c_{2n+1} - \ln\frac{1}{c_{2n}}\right) = \lim_{n\to\infty}\left(\frac{\varepsilon_1}{1-\varepsilon_1}\ln\frac{1}{c_{2n}} - \ln c_{2n-1}\right) = +\infty \tag{5.35}$$

which must hold for every $\varepsilon_1 > 0$. Obviously, there exist a large choice of increasing sequences c_{2n+1} and decreasing sequences c_{2n} for which (5.35) is valid. It's a matter of direct verification to show that for example the sequences given in (5.32)–(5.33) satisfy relations (5.35). $\square$

Corollary 5.11. *Let $0 < \alpha < \beta < 1$. Any (α, β)-function corresponding to the partition*

$$a_{2n+1} = e^{-\frac{A^{u_n}+A^{v_n}}{\beta-\alpha}},\ a_{2n} = e^{-\frac{A^{u_n}+A^{v_{n-1}}}{\beta-\alpha}} \tag{5.36}$$

where $A > 1$ and the sequences u_n and v_n are from Theorem 5.10, has the property

$$m_\omega = \alpha,\ M_\omega = \beta.$$

Indeed, it suffices to note that the values of the coefficients c_n given in (5.32) provide formulas (5.36) for partition by (5.4).

References

[1] N. K. Bari and S. B. Stechkin, *Best approximations and differential properties of two conjugate functions (in Russian)*, Proceedings of Moscow Math. Soc., **5**(1956) 483–522.

[2] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, **129** (1988) Academic Press Inc., Boston, MA, xiv+469.

[3] S. N. Bernstein, *Complete Works, Vol II. Constructive function theory* [1931–1953] (Russian), Izdat. Akad. Nauk SSSR, Moscow, (1954), 627 pages.

[4] A. Böttcher and Yu. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Birkhäuser Verlag, Basel, Boston, Berlin, (1997), 397 pages.

[5] A. I. Guseinov and H. Sh. Mukhtarov, *Introduction to the theory of nonlinear singular integral equations (in Russian)*, Moscow, Nauka, (1980), 416 pages.

[6] N. K. Karapetiants and N. G. Samko, *Weighted theorems on fractional integrals in the generalized Hölder spaces $H0^\omega(\rho)$ via the indices $m\omega$ and $M\omega$*, Fract. Calc. Appl. Anal., (2005), to appear.

[7] N. K. Karapetyants, Kh. M. Murdaev and Ya. Yakubov, *The isomorphism realized by fractional integrals in generalized Hölder classes*, Dokl. Akad. Nauk SSSR, Doklady Akademii Nauk SSSR, **314, No. 2** (1990), 288–291.

[8] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of linear operators*, Moscow, Nauka, (1978), 499 pages.

[9] ———, *Interpolation of linear operators*, Translations of Mathematical Monographs, **54** American Mathematical Society, Providence, R.I., (1982).

[10] L. Maligranda, *Indices and interpolation*, Dissertationes Math. (Rozprawy Mat.), **234** (1985), 49 pages.

[11] ———, *Orlicz spaces and interpolation*, Departamento de Matemática, Universidade Estadual de Campinas, Campinas SP, Brazil, (1989).

[12] N. G. Samko, *On boundedness of singular operator in weighted generalized Hölder spaces $H0^{\omega}(\Gamma, \rho)$ in terms of upper and lower indices of these spaces (in Russian)*, DepInVINITI, (1991), no. 349-B91, 28 pages.

[13] ———, *Singular Integral Operators with Discontinuous Coefficients in the Generalized Hölder Spaces* (in Russian), Voronezh State University, Voronezh, 150 pages, (1991).

[14] ———, *Singular integral operators in weighted spaces with generalized Hölder condition*, Proc. A. Razmadze Math. Inst, **12** (1999), 107–134.

[15] ———, *Criterion of Fredholmness of singular operators with piece-wise continuous coefficients in the generalized Hölder spaces with weight*, Birkhäuser, In: "Operator Theory: Advances and Applications", **142**, (2002) *Proceedings of IWOTA 2000*, Setembro 12-15, Faro, Portugal, 363–376.

[16] ———, *On compactness of Integral Operators with a Generalized Weakthebi Singularity in Weighted Spaces of Continuous Functions with a Given Continuity Modulus*, Proc. A. Razmadze Math. Inst, (2004) **136**, 91–113.

[17] ———, *Singular integral operators with Carleman shift and discontinuous coefficients in the spaces $H0^{\omega}(\Gamma, \rho)$*, Integr. Equat. Operator Theory, **51, No. 3** (2004), 417– 433.

[18] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon & Breach Sci. Publ., London, New York, (Russian edition - *Fractional Integrals and Derivatives and some of their Applications*, Minsk, Nauka i Tekhnika, 1987.), (1993) 1012 pages.

[19] S. G. Samko and Kh. M. Murdaev, *Weighted Zygmund estimates for fractional differentiation and integration, and their applications*, Trudy

Matem. Ibst. Steklov, **180** (1987), 197-198, translated in Proc. Steklov Inst. Math., Amer. Math. Soc., (1989) Issue 3, 233–235.

[20] S. G. Samko and A. Ya. Yakubov, *Zygmund's estimate for a singular integral with a rapidly decreasing power weight*, Izv. Vyssh. Uchebn. Zaved. Mat., Izvestiya Vysshikh Uchebnykh Zavedeniĭ Matematika, (1984) No. 12, 42–51.

[21] ———, *Zygmund estimate for moduli of continuity of fractional order of a conjugate function*, Izv. Vyssh. Uchebn. Zaved. Mat., **12** (1985), 49–53, 77

[22] A. Zygmund, *O module ciaglosci sumi szeregu sprzezonego z szeregiem Fouriera (Sur le module de continuité de la somme de la serie conjugée de la serie de Fourier)*, Prace Mat. - fiz., **33** (1924), 125–132.

[23] A. Zygmund, *Trigonometric series, v. 2*, (Russian), Moscow, Mir, (1965) 624 pages.