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ON THE CONVERGENCE OF SEQUENCES OF INTEGRALLY QUASICONTINUOUS FUNCTIONS

Abstract

A function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies condition $(Q_i(x))$ (resp. $(Q_s(x))$, $[Q_o(x)]$) at a point x if for each real $r > 0$ and for each set $U \ni x$ belonging to the Euclidean topology in $\mathbb{R}^n$ (resp. to the strong density topology [to the ordinary density topology]) there is an open set I such that $I \cap U \neq \emptyset$, f is Lebesgue integrable on $I \cap U$ and

$$\left| \frac{1}{\mu(U \cap I)} \int_{U \cap I} f(t)dt - f(x) \right| < r.$$

These notions are modifications of quasicontinuity or approximate quasicontinuity. In this article we investigate the limits of sequences of such functions.

Let $\mathbb{R}$ be the set of all reals and let $\mathbb{R}^n$ be the n-dimensional product space. For a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and positive reals $r_1, \dots, r_n$ put

$$I_i = (x_i - r_i, x_i + r_i) \text{ for } i = 1, 2, \dots, n,$$

and

$$P(x; r_1, \dots, r_n) = I_1 \times \dots \times I_n.$$

The symbol $Q(x, r)$ denotes the cube $P(x; r_1, \dots, r_n)$, where $r_1 = \dots = r_n = r$.

Denote Lebesgue measure in $\mathbb{R}^n$ by μ. For a Lebesgue measurable set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we define the lower strong density $D_l(A, x)$ of the set A at the point x by

$$\liminf_{h_1,\dots,h_n \to 0^+} \frac{\mu(A \cap P(x; h_1, \dots, h_n))}{\mu(P(x; h_1, \dots, h_n))}.$$

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Similarly for a Lebesgue measurable set $A \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we define the lower ordinary density $d_l(A, x)$ of the set A at the point x by

$$\liminf_{h\to 0^+} \frac{\mu(A \cap Q(x,h))}{\mu(Q(x,h))}.$$

A point x is said to be a strong density point of a measurable set A if $D_l(A, x) = 1$.

Similarly we define the notions of an ordinary density point.

The family T_{sd} (T_{od}) of all Lebesgue measurable sets A for which the implication

$$x \in A \Longrightarrow x \text{ is a strong (resp. an ordinary) density point of } A$$

is true, is a topology called the strong (resp. ordinary) density topology ([2, 3, 14]).

If T_e denotes the Euclidean topology in $\mathbb{R}^n$, then evidently $T_e \subset T_{sd} \subset T_{od}$.

The continuity of mappings f from $(\mathbb{R}^n, T_{sd})$ (resp. from $(\mathbb{R}^n, T_{od})$) to $(\mathbb{R}, T_e)$ is called the strong (ordinary) approximate continuity ([2, 3, 14]).

For an arbitrary function $f : \mathbb{R}^n \to \mathbb{R}$ denote by $C(f)$ the set of all continuity points of f. Moreover let $D(f) = \mathbb{R}^n \setminus C(f)$.

In [8, 9] the following notion is investigated.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is quasicontinuous at a point x ($f \in Q(x)$) if for each positive real r and for each set $U \in T_e$ containing x there is a nonempty open set I such that $I \subset U$ and $|f(t) - f(x)| < r$ for all points $t \in I$.

A function f is quasicontinuous, if $f \in Q(x)$ for every point $x \in \mathbb{R}^n$.

Analogously, as some particular cases of the notion of quasicontinuity of real functions on topological spaces (compare [9]) we have the following definitions.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is T_{sd}-approximately quasicontinuous (resp. T_{od}-approximately quasicontinuous) at a point x if for each positive real r and for each set $U \in T_{sd}$ (resp. $U \in T_{od}$) containing x there is a nonempty set $V \in T_{sd}$ (resp. a nonempty set $V \in T_{od}$) contained in U and such that $|f(t) - f(x)| < r$ for all points $t \in V$.

If f is T_{sd}-approximately quasicontinuous (resp. T_{od}-approximately quasicontinuous) at each point $x \in \mathbb{R}^n$, then f is said to be T_{sd}-approximately quasicontinuous (resp. T_{od}-approximately quasicontinuous).

A function $f : \mathbb{R}^n \to \mathbb{R}$ is integrally quasicontinuous at a point x ($f \in Q_i(x)$, [4]) if for each positive real r and for each bounded set $U \in T_e$ containing x there is a nonempty open set I such that $I \subset U$, f is Lebesgue integrable on I and

$$\left| \frac{\int_I f(t)dt}{\mu(I)} - f(x) \right| < r.$$

A function f is integrally quasicontinuous ($f \in Q_i$), if $f \in Q_i(x)$ for every point $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $Q_s(x)$ (resp. $f \in Q_o(x)$, [4]), if for each positive real η and for each bounded set $U \in T_{sd}$ (resp. $U \in T_{od}$) containing x there is an open set I such that $I \cap U \neq \emptyset$, the function f is Lebesgue integrable on $I \cap U$ and

$$\Big|\frac{1}{\mu(I \cap U)} \int_{I \cap U} f(t)dt - f(x)\Big| < \eta.$$

If $f \in Q_s(x)$ (resp. $f \in Q_o(x)$) for every point $x \in \mathbb{R}^n$, then we will write that $f \in Q_s$ (resp. $f \in Q_o$).

In this article I consider some kinds of the convergence of sequences of integrally quasicontinuous functions.

Observe that if a function $f : \mathbb{R}^n \to \mathbb{R}$ is integrally quasicontinuous, then the set $Z(f)$ of all points $x \in \mathbb{R}^n$ at which f is locally Lebesgue integrable is open and dense in $\mathbb{R}^n$.

We will show that there are quasicontinuous bounded functions $f : \mathbb{R} \to \mathbb{R}$ such that $Z(f) = \emptyset$.

Example 1. (see [4]).

If $A \subset \mathbb{R}$ is a nowhere dense closed set of positive measure, then we find a nonmeasurable (in the sense of Lebesgue) set $B \subset A$ such that the interior measures $\mu_i(B)$ and $\mu_i(A \setminus B)$ are zero and we put

$$f_A(x) = \begin{cases} 1 & \text{for } x \in B \\ 0 & \text{for } x \in A \setminus B, \end{cases}$$

and if (a, b) is a component of the set $\mathbb{R} \setminus A$, then for $x \in (a, b)$ we put

$$f_A(x) = \sin\Big(\frac{1}{\min(x - a, b - x)}\Big).$$

Evidently, the function f_A is quasicontinuous,

$$f_A(\mathbb{R}) = [-1, 1], \quad C(f_A) = \mathbb{R} \setminus A$$

and the restricted function $f_A \restriction A$ is not measurable (in the Lebesgue sense).

Now let $E \subset \mathbb{R}$ be a dense G_δ-set of measure zero and let $H = \mathbb{R} \setminus E$. Since H is a F_σ-set of the first category, by Sierpiński's theorem from [10] there are pairwise disjoint closed sets F_n such that $H = \cup_n F_n$. Without loss of generality we can suppose that $\mu(F_n) > 0$ for $n \geq 1$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} f_{F_n}. \qquad (*)$$

If $x \in E$, then for each integer $n \geq 1$ the point x belongs to $\mathbb{R} \setminus F_n = C(f_{F_n})$. Consequently, by the uniform convergence of the series in $(*)$, the function f is continuous at x. So, $f \in Q(x)$.

Now let $x \in H$. There is a unique integer k with $x \in F_k$. For $n \neq k$ the functions f_{F_n} are continuous at x, so the sum $\sum_{n \neq k} \frac{1}{2^n} f_{F_n}$ is also continuous at x. Since the function f_{F_k} is quasicontinuous at x, by Theorem 1 from [7] the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f_{F_n} = \sum_{n \neq k} \frac{1}{2^n} f_{F_n} + \frac{1}{2^k} f_{F_k}$$

is also quasicontinuous at x. So the function f is quasicontinuous.

Now let $K \subset \mathbb{R}$ be a Lebesgue measurable set of positive measure. Then there is an integer $j \geq 1$ with $\mu(K \cap F_j) > 0$. Since the sum $\sum_{n \neq j} \frac{1}{2^n} f_{F_n}$ is continuous on $K \cap F_j$ and the restricted function $f_{F_j \cap K}$ is not measurable, the restricted function $f \restriction K$ is not measurable. Consequently, $Z(f) = \emptyset$ and f is not integrally quasicontinuous at any point.

From the above example we obtain the following.

Remark 1. *There is a uniformly convergent sequence of functions from Q_i such that its limit is not in Q_i.*

Proof. If $\mathbb{R}^n = \mathbb{R}$, then for $m \geq 1$ let $f_m = \sum_{k \leq m} \frac{1}{2^k} f_{F_k}$. Since the functions f_m are quasicontinuous and the restrictions $f_m \restriction (\mathbb{R} \setminus E_m)$ to the complements of the nowhere dense sets $E_m = \bigcup_{k \leq m} F_k$ are continuous, the functions f_m, $m \geq 1$, are integrally quasicontinuous by Theorem 1 from [4]. Moreover the sequence (f_m) converges uniformly to $f \notin Q_i$.

If $n > 1$, then for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $m \geq 1$ we put

$$g_m(x) = f_m(x_1) \text{ and } g(x) = f(x_1),$$

and observe that $g_m \in Q_i$, the sequence (g_m) uniformly converges to g and $g \notin Q_i$. □

Remark 2. *Since there is a nonmeasurable quasicontinuous function $f : \mathbb{R}^n \to \mathbb{R}$ which is continuous on a dense open set, there is a nonmeasurable integrally quasicontinuous function.*

Proof. One can use the function $f(x_1, x_2, \ldots, x_n) = f_A(x_1)$, f_A being defined as in the above example. □

Theorem 1. *If functions $f_m : \mathbb{R}^n \to \mathbb{R}$ are integrally quasicontinuous at a point x and the sequence (f_m) converges uniformly to a function f which is locally measurable at x, then f is integrally quasicontinuous at x.*

PROOF. Fix a real $\eta > 0$ and a bounded open set $W \ni x$. Without loss of generality we can assume that the restricted function $f \restriction W$ is measurable. There is a positive integer k such that $|f_k(t) - f(t)| < \frac{\eta}{3}$ for all $t \in \mathbb{R}^n$. Since f_k is integrally quasicontinuous at x, there is a nonempty open set $U \subset W$ such that

$$\Big|\frac{\int_U f_k}{\mu(U)} - f_k(x)\Big| < \frac{\eta}{3}.$$

Observe that

$$\begin{aligned}\Big|\frac{\int_U f}{\mu(U)} - f(x)\Big| &\leq \Big|\frac{\int_U f}{\mu(U)} - \frac{\int_U f_k}{\mu(U)}\Big| + \Big|\frac{\int_U f_k}{\mu(U)} - f_k(x)\Big| + |f_k(x) - f(x)| \\ &< \frac{\eta\mu(U)}{3\mu(U)} + \frac{\eta}{3} + \frac{\eta}{3} = \eta,\end{aligned}$$

so f is integrally quasicontinuous at x. □

Theorem 2. *If for a function $f : \mathbb{R}^n \to \mathbb{R}$ the set $D(f) = \mathbb{R}^n \setminus C(f)$ is of the first category, then there is a sequence of integrally quasicontinuous functions $f_m : \mathbb{R}^n \to \mathbb{R}$ such that $f = \lim_{m\to\infty} f_m$.*

PROOF. As in [5] (for the case $f : \mathbb{R} \to \mathbb{R}$) we can prove that there is a sequence of quasicontinuous functions $f_m : \mathbb{R}^n \to \mathbb{R}$ such that the sets $D(f_m)$ are nowhere dense for $m \geq 1$ and $f = \lim_{m\to\infty} f_m$. By Theorem 1 from [4] the functions f_m are integrally quasicontinuous and the proof is completed.

The inclusions $Q_o \subset Q_s \subset Q_i$ follow immediately from the inclusions $T_e \subset T_{sd} \subset T_{od}$.

Since there are nonmeasurable integrally quasicontinuous function, by the next theorem we obtain that $Q_s \neq Q_i$. In the case $\mathbb{R}^n = \mathbb{R}$ the equality $Q_o = Q_s$ is true. In the following example I show that $Q_o \neq Q_s$ in the case $\mathbb{R}^n$, $n \geq 2$.

Example 2. Put

$$\begin{aligned}E &= \{(x, y) \in \mathbb{R}^2; x > 0 \text{ and } -x^2 \leq y \leq x^2\},\\ G &= \{(x, y) \in \mathbb{R}^2; x > 0 \text{ and } -3x^2 < y < 3x^2\},\end{aligned}$$

and

$$H = \mathbb{R}^2 \setminus G.$$

Let $f : \mathbb{R}^2 \to [0, 1]$ be a function such that

$$f(x, y) = 0 \text{ for } (x, y) \in E \cup \{(0, 0)\}, \quad f(x, y) = 1 \text{ for } (x, y) \in H \setminus \{(0, 0)\},$$

and $C(f) = \mathbb{R}^2 \setminus \{(0,0)\}$. Observe that the ordinary density

$$d_l(H,(0,0)) = \lim_{h\to 0^+} \frac{\mu(H \cap ((-h,h)\times(-h,h)))}{4h^2} = 1 - \lim_{h\to 0^+} \frac{2h^3}{4h^2} = 1.$$

Thus $(0,0) \in U = \operatorname{int}(H) \cup \{(0,0)\} \in T_{od}$. Since $f(x,y) = 1$ for $(x,y) \in \operatorname{int}(H)$ and $f(0,0) = 0$, the function $f \notin Q_o$.

On the other hand

$$\limsup_{h\to 0^+} \frac{\mu(E \cap ((-h,h)\times(-h^4,h^4)))}{4h^5} \geq \limsup_{h\to 0^+} \frac{2(h^5 - h^{12})}{4h^5} = \frac{1}{2} > 0.$$

So, for each set $U \in T_{sd}$ containing $(0,0)$ the intersection $U \cap \operatorname{int}(E)$ is nonempty and consequently $f \in Q_s((0,0))$. Since at other points of $\mathbb{R}^2$ the function f is continuous, it belongs to Q_s. Thus in the case $\mathbb{R}^n = \mathbb{R}^2$ the relation $Q_s \neq Q_o$ holds.

For the case of functions defined on $\mathbb{R}^n$, where $n > 2$ it suffices to put $g(x_1, x_2, \ldots, x_n) = f(x_1, x_2)$ and observe that $g \in Q_s \setminus Q_0$.

Remark 3. *If a function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to Q_s, then f is measurable.*

PROOF. It suffices to prove that for each nonempty open set G of finite measure the restricted function $f \restriction G$ is measurable. Fix an open set G of positive finite measure and let

$$a = \sup\{\mu(H); H \subset G \text{ is measurable and } f \restriction H \text{ is measurable}\}.$$

Assume, to a contrary, that $a < \mu(G)$. Then for each positive integer n there is a measurable set $H_n \subset G$ such that $\mu(H_n) > a - \frac{1}{n}$ and the restricted function $f \restriction H_n$ is measurable. Let $H = \bigcup_{n=1}^{\infty} H_n$. Then the set $H \subset G$ is measurable and $\mu(H) = a$ and $f \restriction H$ is measurable. So, the difference $K = G \setminus H$ is a measurable set of positive measure. By Lebesgue's density theorem the set

$$M = \{x \in K; D_l(K,x) = 1\} \text{ is measurable and } \mu(M) = \mu(K).$$

Since $M \in T_{sd}$ and $f \in Q_s(x)$ for $x \in M$, there is an open set W such that $W \cap M \neq \emptyset$ and $f \restriction (W \cap M)$ is measurable. Evidently $\mu(W \cap M) > 0$ and $W \cap M \cap H = \emptyset$. Consequently, the set $H \cup (W \cap M)$ is measurable, the restricted function $f/(H \cup (W \cap M))$ is measurable and $\mu(H \cup (W \cap M)) > a$, contrary to the definition of a. □

Theorem 3. *If a sequence of functions $f_m : \mathbb{R}^n \to \mathbb{R}$ belonging to Q_o (resp. belonging to Q_s) converges uniformly to a function $f : \mathbb{R}^n \to \mathbb{R}$, then $f \in Q_o$ (resp. $f \in Q_s$).*

PROOF. By Remark 3 f is the uniform limit of a sequence of measurable functions and thus is measurable. Now the proof is analogous to that of Theorem 1. □

A generalization of uniform convergence is Arzelà's quasi-uniform convergence.

Recall ([12]) that a sequence of functions $f_m : \mathbb{R}^n \to \mathbb{R}$ quasi-uniformly (in the sense of Arzelà) converges to a function f if it pointwise converges to f and for each real $\eta > 0$ and for each integer $m > 0$ there is a positive integer p such that for each point $x \in \mathbb{R}^n$

$$\min(|f_{m+1}(x) - f(x)|, \ldots, |f_{m+p}(x) - f(x)|) < \eta.$$

From Strońska's Theorem 2 in [13] it follows that for each measurable function $f : \mathbb{R}^n \to \mathbb{R}$ there is a sequence of T_{sd}-approximately quasicontinuous and simultaneously T_{od}-approximately quasicontinuous functions $f_m : \mathbb{R}^n \to \mathbb{R}$, which quasi-uniformly (in Arzelà's sense) converges to f. It is obvious to observe that the above functions f_m may be bounded whenever f is bounded.

Since bounded T_{od}-approximately quasicontinuous functions belong to Q_o ([4]), we obtain that the family of all quasi-uniform limits of sequences of functions from Q_o (so also from Q_s) is the family of all measurable functions on $\mathbb{R}^n$.

It is known ([9] and compare [5]) that if $f : \mathbb{R}^n \to \mathbb{R}$ is the pointwise limit of a sequence of quasicontinuous functions $f_m : \mathbb{R}^n \to \mathbb{R}$, then the set $C(f)$ of all continuity points of f is dense in $\mathbb{R}^n$.

In [1] Borsik proves that for each function $f : \mathbb{R} \to \mathbb{R}$ with dense set $C(f)$ there is a sequence of quasicontinuous functions $f_m : \mathbb{R} \to \mathbb{R}$ which quasi-uniformly converges to f in Arzelà's sense. A generalization of this theorem for functions from a pseudometrizable space X into a separable metric spaces Y is proved in Richter's article [10]. That generalization covers in particular the case $X = \mathbb{R}^n$ and $Y = \mathbb{R}$.

Now I prove the following theorem.

Theorem 4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function such that the set $C(f)$ is dense. There is a sequence of quasicontinuous and integrally quasicontinuous functions $f_m : \mathbb{R}^n \to \mathbb{R}$ which quasi-uniformly converges to f in Arzelà's sense.*

PROOF. If f is constant, then we can put $f_m = f$ for $m \geq 1$. In the contrary case put

$$a = \inf_{\mathbb{R}^n} f \quad \text{and} \quad b = \sup_{\mathbb{R}^n} f$$

and observe that $a < b$. Since the set $C(f)$ is dense, the set

$$P = \{r \in [a, b]; \mathrm{cl}(f^{-1}(r)) \ \text{ is of the second category}\}$$

is countable. For each positive integer m we find a system $P_m \subset [a,b] \setminus P$ of reals $a_1, a_2, \dots, a_{i(m)-1}$ such that for $i \in \{1, 2, \dots, i(m)\}$

$$a = a_0 < a_1 < \cdots < a_{i(m)-1} < a_{i(m)} = b \text{ and } a_i - a_{i-1} < \frac{1}{m}$$

and $P_m \subset P_{m+1}$ and $P_{m+1} \setminus P_m \neq \emptyset$ for $m \geq 1$. Let

$$g_m(x) = \begin{cases} a_{i-1} & \text{if } a_{i-1} \leq f(x) < a_i,\ 1 \leq i < i(m) \\ a_{i(m)-1} & \text{if } a_{i(m)-1} \leq f(x) \leq a_{i(m)} = b. \end{cases}$$

Then $|g_m - f| < \frac{1}{m}$.

If $x \in C(f)$ and g_m is not quasicontinuous at x and $g_m(x) = a_i$ (then evidently $i > 0$), we put $h_m(x) = a_{i-1}$. For other points $t \in \mathbb{R}^n$ we put $h_m(t) = g_m(t)$.

Observe that $|h_m - f| \leq \frac{1}{m}$ and h_m is quasicontinuous at each point $x \in C(f)$. Since the set $C(f)$ is dense and the image $h_m(\mathbb{R}^n)$ is finite, the interior $\operatorname{int}(C(h_m))$ is also dense and

$$\operatorname{int}(C(h_m)) = C(h_m) = \bigcup_{i=0}^{i(m)-1} \operatorname{int}((h_m)^{-1}(a_i)).$$

Moreover, for each point $x \in \mathbb{R}^n$ we obtain that

$$\operatorname{osc} h_m(x) \leq \operatorname{osc} f(x) + \frac{2}{m}.$$

For $i \in \{0, 1, \dots, i(m) - 1\}$ let

$$E_{m,i} = \{x; h_m(x) = a_i \text{ and } h_m \text{ is not quasicontinuous at } x\}.$$

Put $E_m = \bigcup_{i=0}^{i(m)-1} E_{m,i}$, and observe that E_m is a nowhere dense set. Assume that $E_{m,0} \neq \emptyset$. Since $C(h_m)$ is open and dense, there is a smallest integer $i_1 > 0$ such that

$$G_{m,0,i_1} = E_{m,0} \cap \operatorname{cl}(\operatorname{int}((h_m)^{-1}(a_{i_1}))) \neq \emptyset.$$

There is a family of pairwise disjoint closed balls $K_{m,0,i_1,k,j}$, $k = 1, 2$ and $j \geq 1$, such that:

(i) $$K_{m,0,i_1,k,j} \subset \mathcal{A}(G_{m,0,i_1}, \tfrac{1}{m}) \cap \operatorname{int}((h_m)^{-1}(a_{i_1})),$$
where for a set $X \neq \emptyset$ and a positive real r
$$\mathcal{A}(X, r) = \bigcup_{x \in X} K(x, r) \text{ and } K(x, r) = \{t \in \mathbb{R}^n; |t - x| \leq r\};$$

(ii) $\mathrm{cl}(\bigcup_j K_{m,0,i_1,k,j}) = \bigcup_j K_{m,0,i_1,k,j} \cup \mathrm{cl}(G_{m,0,i_1})$ for $k = 1, 2$;

(iii) $\mathrm{cl}(\bigcup_{k,j} K_{m,0,i_1,k,j}) = \bigcup_{k,j} K_{m,0,i_1,k,j} \cup \mathrm{cl}(G_{m,0,i_1})$;

(iv) the family $\{K_{m,0,i_1,k,j}; j \geq 1, k = 1, 2\}$ is locally finite at each point $x \in \mathbb{R}^n \setminus \mathrm{cl}(G_{m,0,i_1})$.

If $E_{m,0} \setminus \mathrm{cl}(G_{m,0,i_1}) \neq \emptyset$, then there is a smallest integer $i_2 > i_1$ such that

$$G_{m,0,i_2} = (E_{m,0} \setminus G_{m,0,i_1}) \cap \mathrm{cl}(\mathrm{int}((h_m)^{-1}(a_{i_2}))) \neq \emptyset.$$

There is a family of pairwise disjoint closed balls $K_{m,0,i_2,k,j}$, $k = 1, 2$ and $j \geq 1$, such that:

(i) $K_{m,0,i_2,k,j} \subset \mathcal{A}(G_{m,0,i_2}, \frac{1}{m}) \cap \mathrm{int}((h_m)^{-1}(a_{i_2}))$;

(ii) $\mathrm{cl}(\bigcup_j K_{m,0,i_2,k,j}) = \bigcup_j K_{m,0,i_2,k,j} \cup \mathrm{cl}(G_{m,0,i_2})$ for $k = 1, 2$;

(iii) $\mathrm{cl}(\bigcup_{k,j} K_{m,0,i_2,k,j}) = \bigcup_{k,j} K_{m,0,i_2,k,j} \cup \mathrm{cl}(G_{m,0,i_2})$;

(iv) the family $\{K_{m,0,i_2,k,j}; j \geq 1, k = 1, 2\}$ is locally finite at each point $x \in \mathbb{R}^n \setminus \mathrm{cl}(G_{m,0,i_2})$.

Proceeding with this reasoning we find a system $i_1 < i_2 < \cdots < i_{k_0}$ of positive integers and families of pairwise disjoint closed balls $K_{m,0,i_l,k,j}$, where $k = 1, 2$, $l \leq k_0$, $j \geq 1$, such that:

(i) for $l \leq k_0$ the difference $E_{m,0} \setminus (\mathrm{cl}(G_{m,0,i_1}) \cup \ldots \cup \mathrm{cl}(G_{m,0,i_{l-1}}))$ is nonempty and i_l is the smallest integer $i_l > i_{l-1}$ such that

$$G_{m,0,i_l} = E_{m,0} \cap \mathrm{cl}(\mathrm{int}((h_m)^{-1}(a_{i_1}))) \neq \emptyset;$$

(ii) $E_{m,0} = G_{m,0,i_1} \cup \ldots \cup G_{m,0,i_{k_0}}$;

(iii) $K_{m,0,i_l,k,j} \subset \mathcal{A}(G_{m,0,i_l}, \frac{1}{m}) \cap \mathrm{int}((h_m)^{-1}(a_{i_l}))$;

(iv) $\mathrm{cl}(\bigcup_j K_{m,0,i_l,k,j}) = \bigcup_j K_{m,0,i_l,k,j} \cup \mathrm{cl}(G_{m,0,i_l})$ for $k = 1, 2$;

(v) $\mathrm{cl}(\bigcup_{k,j} K_{m,0,i_l,k,j}) = \bigcup_{k,j} K_{m,0,i_l,k,j} \cup \mathrm{cl}(G_{m,0,i_l})$;

(vi) the family $\{K_{m,0,i_l,k,j}; j \geq 1, k = 1, 2\}$ is locally finite at each point $x \in \mathbb{R}^n \setminus \mathrm{cl}(G_{m,0,i_l})$.

Now we put

$$h_{m,0,1}(x) = \begin{cases} a_0 & \text{for } x \in K_{m,0,i_l,1,j}, \quad j \geq 1, \quad l = 1, 2, \ldots, k_0 \\ h_m(x) & \text{otherwise on } \mathbb{R}^n \end{cases}$$

and

$$h_{m,0,2}(x) = \begin{cases} a_0 & \text{for } x \in K_{m,0,i_l,2,j}, \quad j \geq 1, \quad l = 1, 2, \ldots, k_0 \\ h_m(x) & \text{otherwise on } \mathbb{R}^n. \end{cases}$$

Observe that the functions $h_{m,0,1}$ and $h_{m,0,2}$ are quasicontinuous at all points $x \in (\mathbb{R}^n \setminus E_m) \cup E_{m,0}$. If $E_{m,0} = \emptyset$, then we put $h_{m,0,1} = h_{m,0,2} = h_m$.

Proceeding with this reasoning for $E_{m,1}, \ldots, E_{m,i(m)-1}$ we define quasicontinuous functions $f_{2m-1} = h_{m,i(m)-1,1}$ and $f_{2m} = h_{m,i(m)-1,2}$ such that the interiors $\operatorname{int}(C(f_{2m-1}))$ and $\operatorname{int}(C(f_{2m}))$ are dense (so f_{2m-1} and f_{2m} are integrally quasicontinuous [4], Theorem 1),

$$\min(|f_{2m-1} - f|, |f_{2m} - f|) \leq |h_m - f| \leq \frac{1}{m} \tag{$*$}$$

and

$$\{x; f_{2m-1}(x) \neq h_m(x)\} \cup \{x; f_{2m}(x) \neq h_m(x)\} \subset \mathcal{A}(E_m, \frac{1}{m}). \tag{$**$}$$

It ought to be pointed out that for $i \geq 1$ the functions $h_{m,i,1}$ and $h_{m,i,2}$ are obtained as modifications of $h_{m,i-1,1}$ and $h_{m,i-1,2}$ respectively on closed balls $K_{m,i,i_l,1,j}$ or $K_{m,i,i_l,2,j}$, $i = 1, 2, \ldots, i(m) - 1$, $l = 1, \ldots, k_i$, $j \geq 1$, and that all the sets $K_{m,i,i_l,1,j}$ and $K_{m,i,i_l,2,j}$, $i = 1, 2, \ldots, i(m) - 1$, $l = 1, \ldots, k_i$, $j \geq 1$, are pairwise disjoint (not only for fixed i).

If $x \in D(f) = \mathbb{R}^n \setminus C(f)$, then for sufficiently large integers m we have

$$f_{2m-1}(x) = f_{2m}(x) = h_m(x), \quad \text{so} \quad \lim_{m\to\infty} f_m(x) = \lim_{m\to\infty} h_m(x) = f(x).$$

Fix a point $x \in C(f)$ and a positive real η. Let m_1 be a positive integer such that $\frac{4}{m_1} < \eta$. For $k \geq 1$ let

$$A_k = \{x; \operatorname{osc} f(x) \geq \frac{1}{k}\}.$$

Since $D(f) = \cup_k A_k$, we obtain $x \notin A_k$ for $k \geq 1$. Let $c = \inf\{|t-x|; t \in A_{m_1}\}$. Evidently, $c > 0$. There is a positive integer $k_1 > m_1$ with $\frac{1}{k_1} < c$. Let $k > k_1$ be an integer. For $t \in \mathbb{R}^n \setminus A_{m_1}$ we have $\operatorname{osc} f(t) < \frac{1}{m_1}$ and consequently,

$$\operatorname{osc} h_k(t) < \operatorname{osc} f(t) + \frac{2}{k} < \frac{1}{m_1} + \frac{2}{k}.$$

Since $x \in \mathbb{R}^n \setminus \mathcal{A}(E_m, \frac{1}{k})$, by $(**)$ we obtain that $\operatorname{osc} f(x) < \frac{1}{m_1}$ and

$$\max(|f_{2k-1}(x) - h_k(x)|, |f_{2k}(x) - h_k(x)|) < \frac{1}{m_1} + \frac{2}{k}.$$

So

$$\begin{aligned} &\max(|f_{2k-1}(x) - f(x)|, |f_{2k}(x) - f(x)|) \\ \leq &|h_k(x) - f(x)| + \max(|h_k(x) - f_{2k-1}(x)|, |h_k(x) - f_{2k}(x)|) \\ < &\frac{1}{k} + \frac{1}{m_1} + \frac{2}{k} < \frac{1}{k} + \frac{1}{m_1} + \frac{2}{k_1} < \frac{4}{m_1} < \eta, \end{aligned}$$

and $\lim_{m\to\infty} f_m(x) = f(x)$. So the sequence (f_m) converges pointwise to f. By $(*)$ it quasi-uniformly converges to f in Arzelà's sense. □

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References

[1] J. Borsik, *Quasiuniform limits of quasicontinuous functions*, Math. Slovaca, **42** (1992), 269–274.

[2] A. M. Bruckner, *Differentiation of real functions*, Lectures Notes in Math. 659, Springer-Verlag, Berlin, 1978.

[3] C. Goffman, C. Neugebauer, T. Nishiura, *Density topology and approximate continuity*, Duke Math. J., **28** (1961), 497–506.

[4] Z. Grande, E. Strońska, *On the integral quasicontinuity*, in preparation.

[5] Z. Grande, *Sur la quasi-continuité et la quasi-continuité approximative*, Fund. Math., **129** (1988), 167–172.

[6] Z. Grande, T. Natkaniec, D. Strońska, *Algebraic structures generated by d-quasicontinuous functions*, Bull. Pol. Acad. Sci., Math., **35** (1987), 717–723.

[7] Z. Grande, L. Sołtysik, *Some remarks on quasicontinuous real functions*, Problemy Matematyczne, **No. 10** (1990), 79–86.

[8] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math., **19** (1932), 184–197.

[9] T. Neubrunn, *Quasi-continuity*, Real Anal. Exch., **14, No. 2** (1988-89), 259–306.

[10] C. Richter, *Representing cliquish functions as quasiuniform limits of quasicontinuous functions*, Real Anal. Exch., **27, No. 1** (2001-02), 209–221.

[11] W. Sierpiński, *Sur une propriété des ensembles F_σ-linéaires*, Fund. Math., **14** (1929), 216–220.

[12] R. Sikorski, *Real Functions (in Polish)*, Warsaw 1957.

[13] E. Strońska, *On quasi-uniform convergence of sequences of s_1-strongly quasi-continuous functions on $\mathbb{R}^m$*, Real Anal. Exch., **30, No. 1** (2004-05), 217–234.

[14] F. D. Tall, *The density topology*, Pacific J. Math., **62** (1976), 275–284.