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CONVERGENCE OF SEQUENCES OF FUNCTIONS HAVING SOME GENERALIZED PAWLAK PROPERTIES

Abstract

A function $f : \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{M}_1$ ($\mathcal{M}_2$) if the restricted function $f \restriction D(f)$ ($f \restriction D_{ap}(f)$) is monotone. ($D(f)$ [$D_{ap}(f)$] denotes the set of all discontinuity points [the set of all approximate discontinuity points] of f.) In this article I investigate the uniform, pointwise and transfinite limits of sequences of functions with the property $\mathcal{M}_i$, $i = 1, 2$.

Let $\mathbb{R}$ be the set of all reals. Denote by μ the Lebesgue measure in $\mathbb{R}$ and by μ_e the outer Lebesgue measure in $\mathbb{R}$. For a set $A \subset \mathbb{R}$ and a point x we define the upper (lower) outer density $D_u(A, x)$ ($D_l(A, x)$) of the set A at the point x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h}$$

$$(\liminf_{h \to 0^+} \frac{\mu_e(A \cap [x-h, x+h])}{2h} \text{ respectively}).$$

A point x is said an outer density point (a density point) of a set A if $D_l(A, x) = 1$ (if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B, x) = 1$).

The family T_d of all sets A for which the implication

$$x \in A \Longrightarrow x \text{ is a density point of } A$$

holds, is a topology called the density topology ([1, 6]). The sets $A \in T_d$ are measurable ([1]).

Let T_e be the Euclidean topology in $\mathbb{R}$. A continuous function $f : (\mathbb{R}, T_d) \to (\mathbb{R}, T_e)$ is said to be approximately continuous ([1, 6]).

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For a function $f : \mathbb{R} \to \mathbb{R}$ denote by $C(f)$ the set of all continuity points of f and by $C_{ap}(f)$ the set of all approximate continuity points of f. Moreover let $D(f) = \mathbb{R} \setminus C(f)$ and $D_{ap}(f) = \mathbb{R} \setminus C_{ap}(f)$.

In [3] R. Pawlak introduced and investigated the following property of functions:

A function $f : \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{B}_1^{**}$ if the restricted function $f{\restriction}D(f)$ is continuous.

In this paper I investigate similar properties $\mathcal{M}_1$ and $\mathcal{M}_2$ defined as follows:

> a function $f : \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{M}_1$ ($f \in \mathcal{M}_1$) if the restricted function $f{\restriction}D(f)$ is monotone.
>
> a function $f : \mathbb{R} \to \mathbb{R}$ has the property $\mathcal{M}_2$ ($f \in \mathcal{M}_2$) if the restricted function $f{\restriction}D_{ap}(f)$ is monotone.

Since for arbitrary function $f : \mathbb{R} \to \mathbb{R}$ we have $C(f) \subset C_{ap}(f)$, the inclusion $\mathcal{M}_1 \subset \mathcal{M}_2$ holds.

Remark 1. *If $f \in \mathcal{M}_1$, then f is of Baire class* 1.

PROOF. Fix a real a and observe that for each point $x \in C(f)$ with $f(x) < a$ there is an open interval $I(x) \ni x$ such that $f(t) < a$ for each point $t \in I(x)$. The restricted function $f{\restriction}D(f)$ is monotone and the set $D(f)$ is an F_σ-set, so the set $\{x \in D(f); f(x) < a\}$ is an F_σ-set as the intersection of the set $D(f)$ and a straight semiline. Thus the set

$$\{x \in \mathbb{R}; f(x) < a\} = \{x \in D(f); f(x) < a\} \cup \bigcup_{x \in C(f),\ f(x)<a} I(x)$$

is an F_σ-set. In the same way we can prove that the set $\{x \in \mathbb{R}; f(x) > a\}$ is an F_σ-set. So f is of the first class of Baire.

Remark 2. *If $f \in \mathcal{M}_2$, then f is measurable (in the sense of Lebesgue).*

PROOF. Denote by $\mathbb{Z}$ the set of all integers and by $\operatorname{int}_{\mathrm{d}}(A)$ the density interior of A; i.e., the union of all subsets of A which belong to T_d. For each positive integer n and each integer $k \in \mathbb{Z}$ let

$$I_{k,n} = \Big(\frac{k-1}{2^n}, \frac{k+1}{2^n}\Big).$$

Since

$$C_{ap}(f) = \bigcap_{n \geq 1} \bigcup_{k \in Z} \operatorname{int}_{\mathrm{d}}(f^{-1}(I_{k,n})),$$

the set $C_{ap}(f)$ is measurable. For each $a \in \mathbb{R}$ and each point $x \in C_{ap}(f)$ with $f(x) < a$ there is a set $U(x) \in T_d$ such that

$$x \in U(x) \subset f^{-1}((-\infty, a)).$$

So the union

$$B(a) = \bigcup_{x \in C_{ap}(f) \cap f^{-1}((-\infty,a))} U(x) \in T_d$$

and consequently the set

$$\{x \in C_{ap}(f); f(x) < a\} = B(a) \cap C_{ap}(f)$$

is measurable. So the restricted function $f \restriction C_{ap}(f)$ is measurable. The set $D_{ap}(f) = \mathbb{R} \setminus C_{ap}(f)$ is also measurable and the restricted function $f \restriction D_{ap}(f)$ is monotone, so it is measurable. Thus f is measurable. $\square$

However there are functions $f \in \mathcal{M}_2$ which do not have the Baire property. For example, if $A \subset \mathbb{R}$ is a residual G_δ-set of measure zero, then there is a decomposition of the set A in disjoint subsets $B, C \subset A$ without the Baire property. The function

$$f(x) = x \text{ on } B \text{ and } f(x) = 0 \text{ otherwise on } \mathbb{R}$$

belongs to $\mathcal{M}_2$ but it does not have the Baire property.

For each measurable function $f : \mathbb{R} \to \mathbb{R}$ there is a function $g : \mathbb{R} \to \mathbb{R}$ of Baire class 2 such that the set $\{x; f(x) \neq g(x)\}$ is of measure zero. In the next example we show that there are functions $f \in \mathcal{M}_2$ such that for each function $h : \mathbb{R} \to \mathbb{R}$ of Baire class 1 the set $\{x; f(x) \neq h(x)\}$ is of positive measure.

Example 1. ([2]). Let (I_n) be an enumeration of all open intervals with rational endpoints and let (A_n) be a sequence of pairwise disjoint nowhere dense perfect sets of positive measure such that

$$A_{2n-1} \cup A_{2n} \subset I_n \text{ for } n = 1, 2, \ldots.$$

Put

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathrm{int_d}(A_{2n-1}),\ n \geq 1 \\ -1 & \text{for } x \in \mathrm{int_d}(A_{2n}),\ n \geq 1 \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then

$$C_{ap}(f) = \bigcup_{n \geq 1} (\mathrm{int_d}(A_{2n-1}) \cup \mathrm{int_d}(A_{2n})) \cup \mathrm{int_d}(f^{-1}(0))$$

and

$$f(x) = 0 \text{ for } x \in D_{ap}(f),$$

so $f \restriction D_{ap}(f)$ is monotone and $f \in \mathcal{M}_2$.

However for each index n and for each set A of measure zero we have

$$\mathrm{int}_{\mathrm{d}}(A_n) \setminus A \neq \emptyset, \text{ so } f^{-1}(1) \cap I_n \neq \emptyset \neq f^{-1}(-1) \cap I_n.$$

Consequently, for each function $g : \mathbb{R} \to \mathbb{R}$ with $f \restriction (\mathcal{R} \setminus A) = g \restriction (\mathbb{R} \setminus A)$ we have $C(g) = \emptyset$ and a such g is not of the first Baire class.

Theorem 1. *The classes $\mathcal{M}_1$ and $\mathcal{M}_2$ are uniformly closed.*

PROOF. Let a sequence of functions $f_n \in \mathcal{M}_1$ (resp. $f_n \in \mathcal{M}_2$) uniformly converges to a function f. Without loss of the generality we can suppose that all restricted functions $f_n \restriction D(f_n)$ (resp. $f_n \restriction D_{ap}(f_n)$) are either decreasing or increasing. Fix $x \in \mathbb{R}$ and observe that if there is a subsequence (n_k) with $x \in C(f_{n_k})$ (resp. $x \in C_{ap}(f_{n_k})$)), then from the uniform convergence of (f_n) it follows that $x \in C(f)$ (resp. $x \in C_{ap}(f)$). So if $x \in D(f)$ (resp. $x \in D_{ap}(f)$), then there is an index $n(x)$ such that $x \in D(f_n)$ (resp. $x \in D_{ap}(f_n)$) for $n \geq n(x)$. For $n \geq 1$ let

$$B_n = \bigcap_{k \geq n} D(f_k) \text{ (resp. } B_n = \bigcap_{k \geq n} D_{ap}(f_k)).$$

Then $B_n \subset B_{n+1}$ for $n \geq$ and

$$D(f) \subset \bigcup_{n \geq 1} B_n \text{ (resp. } D_{ap}(f) \subset \bigcup_{n \geq 1} B_n).$$

The restricted functions $f_n \restriction B_k$, $n \geq k$ and $k \geq 1$, are monotone and the sequence (f_n) uniformly converges to f, so the restricted functions $f \restriction B_k$ are monotone for $k \geq 1$. Now for $n \geq 1$ we put

$$A_n = B_n \cap D(f) \text{ (resp. } A_n = B_n \cap D_{ap}(f))$$

and observe that

$$D(f) = \bigcup_{n \geq 1} A_n \text{ (resp. } D_{ap}(f) = \bigcup_{n \geq 1} A_n),$$

and $f \restriction D(f)$ (resp. $f \restriction D_{ap}(f)$) are monotone. So $f \in \mathcal{M}_1$ (resp. $f \in \mathcal{M}_2$). $\square$

Theorem 2. *Suppose that a sequence of functions $f_n \in \mathcal{M}_1$ pointwise converges to a function $f : \mathbb{R} \to \mathbb{R}$. Then there are disjoint sets A_1, A_2 such that:*

(1) $A_1 \cup A_2 = \mathbb{R} \setminus \bigcup_{n\geq 1} \bigcap_{k\geq n} C(f_k)$*;*

(2) for each point $x \in A_1$ *for infinitely many indices* $n_i(x)$ *we have* $x \in C(f_{n_i(x)})$ *and for infinitely many indices* $k_j(x)$ *we have* $x \in D(f_{k_j(x)})$*;*

(3) the restricted function $f{\restriction}A_2$ *is monotone.*

PROOF. Since $f_n \in \mathcal{M}_1$, the restricted functions $f_n{\restriction}D(f_n)$, $n \geq 1$, are monotone. Without loss of the generality we can suppose that all $f_n{\restriction}D(f_n)$ are nondecreasing. For $n \geq 1$ let $B_n = \cap_{k\geq n} D(f_k)$ and let $A_2 = \cup_{n\geq 1} B_n$. Since the restricted functions $f_k{\restriction}B_n$, $k \geq n$, are nondecreasing, each function $f{\restriction}B_n$, $n \geq 1$, is also nondecreasing. Consequently, the restricted function $f{\restriction}A_2$ is also nondecreasing. If

$$A_1 = \mathbb{R} \setminus (\bigcup_{n\geq 1} \bigcap_{k\geq n} C(f_k) \cup A_2),$$

then the set A_1 satisfies all requirements. $\square$

In next examples we show that there are sequences of functions from $\mathcal{M}_1$ convergent to functions f for which $f{\restriction}A_1$ are not monotone and we show that there are sequences of functions from $\mathcal{M}_1$ convergent to functions f which are not of the first class of Baire.

Example 2. For $n \geq 1$ and $k = 1, 2, 3$ let $I_{n,k} = (k - \frac{1}{2^n}, k + \frac{1}{2^n})$ and

$$f_{3n-2}(x) = \begin{cases} 1 & \text{if } x \in \{1,3\} \\ 2 & \text{if } x = 2 \\ 0 & \text{if } x \in \mathbb{R} \setminus (\{1\} \cup I_{n,2} \cup I_{n,3}) \\ \text{linear} & \text{on the components of } I_{n,k} \setminus \{k\},\ k = 2,3, \end{cases}$$

$$f_{3n-1}(x) = \begin{cases} 1 & \text{if } x \in \{1,3\} \\ 2 & \text{if } x = 2 \\ 0 & \text{if } x \in \mathbb{R} \setminus (\{2\} \cup I_{n,1} \cup I_{n,3}) \\ \text{linear} & \text{on the components of } I_{n,k} \setminus \{k\},\ k = 1,3 \end{cases}$$

and

$$f_{3n}(x) = \begin{cases} 1 & \text{if } x \in \{1,3\} \\ 2 & \text{if } x = 2 \\ 0 & \text{if } x \in \mathbb{R} \setminus (\{3\} \cup I_{n,1} \cup I_{n,2}) \\ \text{linear} & \text{on the components of } I_{n,k} \setminus \{k\},\ k = 1,2. \end{cases}$$

Then for $n \geq 1$ we obtain

$$D(f_{3n-2}) = \{1\},\ D(f_{3n-1}) = \{2\},\ D(f_{3n}) = \{3\},$$

and consequently $f_{3n-k} \in \mathcal{M}_1$ for $k = 0, 1, 2$. Moreover the sequence (f_n) pointwise converges to

$$f(x) = \begin{cases} 0 & \text{for } x \neq 1, 2, 3 \\ 2 & \text{for } x = 2 \\ 1 & \text{for } x \in \{1, 3\}, \end{cases}$$

and for the set A_1 defined in last theorem we have $A_1 = \{1, 2, 3\}$ and $f \restriction A_1$ is not monotone.

Example 3. Enumerate all rationals in a sequence (a_n) such that $a_n \neq a_m$ for $n \neq m$. For $n \geq 1$ let

$$f_n(x) = \begin{cases} 1 & \text{if } x = a_k,\ k \leq n \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then the functions $f_n \in \mathcal{M}_1$ for $n \geq 1$ and the sequence (f_n) pointwise converges to Dirichlet's function which is not of the first Baire class.

Theorem 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If there is a residual G_δ set A such that the restricted function $f \restriction A$ is of Baire class 1 and $f \restriction (\mathbb{R} \setminus A)$ is monotone, then there is a sequence of functions $f_n \in \mathcal{M}_1$ pointwise convergent to f.*

Proof. Since $f \restriction A$ is of the first class of Baire and A is a residual G_δ-set, there is of Baire class 1 function $g : \mathbb{R} \to \mathbb{R}$ such that $f \restriction A = g \restriction A$. There are continuous functions $g_n : \mathbb{R} \to \mathbb{R}$ with $g = \lim_{n\to\infty} g_n$ and closed sets A_n such that $A_n \subset A_{n+1}$ and $\mathbb{R} \setminus A = \cup_n A_n$. For $n \geq 1$ let

$$f_n(x) = \begin{cases} f(x) & \text{for } x \in A_n \\ g_n(x) & \text{for } x \in \mathbb{R} \setminus A_n. \end{cases}$$

Then evidently $f = \lim_{n\to\infty} f_n$ and $f_n \in \mathcal{M}_1$ for $n \geq 1$.

Theorem 4. *Assume that a function $f : \mathbb{R} \to \mathbb{R}$ is the pointwise limit of a sequence of functions $f_n \in \mathcal{M}_2$. Let*

$$A = \bigcup_{n\geq 1} \bigcap_{k\geq n} C_{ap}(f_k),\ B = \bigcup_{n\geq 1} \bigcap_{k\geq n} D_{ap}(f_k) \text{ and } E = \mathbb{R} \setminus (A \cup B).$$

Then $\mu(B \cup E) = 0$, the restricted function $f{\restriction}B$ is monotone, the restricted function $f{\restriction}A$ is the limit of a sequence of approximately continuous $f_n{\restriction}A$, and for each point $x \in E$ there are infinite subsequences $(n_i(x))$ and $(k_j(x))$ of indices such that $x \in C_{ap}(f_{n_i(x)})$ and $x \in D_{ap}(f_{k_j(x)})$ for $i, j = 1, 2, \ldots$.

PROOF. The required properties of the sets A and E are evident. We will prove that the restricted function $f{\restriction}B$ is monotone. For this observe that without loss of the generality we can suppose that all restricted functions $f_n{\restriction}D_{ap}(f_n)$ are nondecreasing. For $n \geq 1$ let $B_n = \cap_{k \geq n} D_{ap}(f_k)$. Then the restricted functions $f_k{\restriction}B_n$, $k \geq n$, are nondecreasing and consequently, $f{\restriction}B_n$ and $f{\restriction}B$ are the same. □

For functions $f, g : \mathbb{R} \to \mathbb{R}$ let $E(f \neq g) = \{x; f(x) \neq g(x)\}$.

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. If there is a function $g : \mathbb{R} \to \mathbb{R}$ of the second class of Baire such that $\mu(E(f \neq g)) = 0$ and the restricted function $f{\restriction}E(f \neq g)$ is monotone, then there are functions $f_n \in \mathcal{M}_2$, $n \geq 1$, with $f = \lim_{n\to\infty} f_n$.*

PROOF. By Preiss' theorem from [4] there are approximately continuous functions $g_n : \mathbb{R} \to \mathbb{R}$ with $g = \lim_{n\to\infty} g_n$. For $n \geq 1$ let

$$f_n(x) = \begin{cases} g_n(x) & \text{for } x \in \mathbb{R} \setminus E(f \neq g) \\ f(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then $f_n \in \mathcal{M}_2$ for $n \geq 1$ and $f = \lim_{n\to\infty} f_n$. □

Now let ω_1 denote the first uncountable ordinal number and let $f_\alpha : \mathbb{R} \to \mathbb{R}$, where $\alpha < \omega_1$, be a transfinite sequence of functions. We will say that a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ converges to a function $f : \mathbb{R} \to \mathbb{R}$ ($\lim_{\alpha<\omega_1} f_\alpha = f$) if for each point $x \in \mathbb{R}$ there is a countable ordinal $\alpha(x)$ such that for each countable ordinal $\alpha > \alpha(x)$ the equality $f_\alpha(x) = f(x)$ is true ([5]).

Theorem 6. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the transfinite limit of a sequence of functions $f_\alpha \in \mathcal{M}_1$, where $\alpha < \omega_1$, then $f \in \mathcal{M}_1$.*

PROOF. First we observe that if $x \in D(f)$, then there is a countable ordinal $\beta(x)$ such that $x \in D(f_\alpha)$ for all countable ordinals $\alpha > \beta(x)$. Of course, if $x \in D(f)$, then there is a sequence (x_n) of points $x_n \neq x$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} f(x_n) \neq f(x).$$

For x and each index n there are countable ordinals $\beta(x)$ and $\beta(x_n)$ such that

$$f_\alpha(x) = f(x) \text{ for } \alpha > \beta(x) \text{ and } f_\alpha(x_n) = f(x_n) \text{ for } \alpha > \beta(x_n).$$

So, if β is a countable ordinal larger than $\beta(x)$ and $\beta(x_n)$ for $n \geq 1$, then

$$f_\beta(x) = f(x) \neq \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f_\beta(x_n),$$

and consequently $x \in D(f_\beta)$.

Now assume to the contrary that $f \notin \mathcal{M}_1$. Then there are points $x, y, z \in D(f)$ with

$$x < y < z \text{ and either } f(y) > \max(f(x), f(z)) \text{ or } f(y) < \min(f(x), f(z)).$$

There is a countable ordinal β such that for each countable ordinal $\alpha > \beta$ we have

$$x, y, z \in D(f_\alpha) \text{ and } f_\alpha(t) = f(t) \text{ for } t \in \{x, y, z\}.$$

Then for all countable ordinals $\alpha > \beta$ we obtain that $f_\alpha \notin \mathcal{M}_1$. This contradicts the hypothesis. □

Theorem 7. *Assume that the continuum hypothesis HC holds. For each function $f : \mathbb{R} \to \mathbb{R}$ there is a transfinite sequence of functions $f_\alpha : \mathbb{R} \to \mathbb{R}$ having the property $\mathcal{M}_2$ such that $\lim_{\alpha<\omega_1} f_\alpha = f$.*

PROOF. The proof of Theorem 7 is the same as the proof of Theorem 5 in [2], where functions f_α are constructed such that $f_\alpha \restriction D_{ap}(f_\alpha) = 0$ (therefore they have the property $\mathcal{M}_2$) and $\lim_{\alpha<\omega_1} f_\alpha = f$.

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