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# DIRICHLET FORMS ON FRACTAL SUBSETS OF THE REAL LINE

**Abstract**

Measure theoretic Dirichlet forms on compact subsets of the real line are introduced. Using the technique of Dirichlet–Neumann–bracketing, estimates of the eigenvalue counting functions of the associated measure geometric Laplacians are obtained.

## 1 Introduction.

In [2], a class of generalized second order differential operators of the form $\Delta^{\mu,\nu} = \frac{d}{d\mu}\frac{d}{d\nu}$ is introduced. These operators are given as the second derivative w.r.t. two atomless finite Borel measures $\mu$ and $\nu$ with compact supports $L := \text{supp } \mu$ and $K := \text{supp } \nu$, such that $L \subseteq K \subseteq \mathbb{R}$. This means that the functions in the domain of these operators are defined on the set $K$ (which also can be a closed interval; i.e., a "fractal" of Hausdorff dimension equal to 1) while the function driving the diffusion is given only on a subset $L \subseteq K$ (which, of course, also can be all of $K$). Thus, the operator $\Delta^{\mu,\nu}$ has an interpretation as a measure geometric Laplacian on $L_2(K, \mu)$. Moreover, this approach generalizes the well-known notion of the Sturm–Liouville– (or, Krein–Feller–) operator of the form $\frac{d}{d\mu}\frac{d}{dx}$ which is introduced for example in [7].

In the present paper, the Dirichlet form, which is associated with the operator $\Delta^{\mu,\nu}$, is constructed. To this end, in Section 2, we recall the definition and

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some fundamental properties of $\Delta^{\mu,\nu}$ which can be found in [2]. We introduce the first derivative $\frac{d}{d\nu}$ on the space

$$\mathcal{D}_1^\nu := \{f : K \to \mathbb{R} : \exists f' \in L_2(K,\nu) : f(x) = f(a) + \int_a^x f'(y)d\nu(y),\ x \in K\}.$$

Iterating this procedure w.r.t. a second measure $\mu$, the operator $\Delta^{\mu,\nu} = \frac{d}{d\mu}\frac{d}{d\nu}$ is introduced on $L_2(K,\mu)$. We restrict ourselves to the case where homogeneous Dirichlet– or, Neumann–boundary–conditions are satisfied, and we define the corresponding eigenvalue counting functions $N_D^{\mu,\nu}(\cdot)$ and $N_N^{\mu,\nu}(\cdot)$. The asymptotic behavior of these eigenvalue counting functions is determined in [3].

In Section 3, we recall the definition of a Dirichlet form, and we introduce the eigenvalues of a Dirichlet form. Following [8], we present the technique of the Dirichlet–Neumann–bracketing which gives a relation between the eigenvalue counting functions of two Dirichlet forms with domains which are related by a directed inclusion; i.e., the domain of one form has to be a closed subspace of the domain of the other form.

In Section 4, we prove that

$$\mathcal{E}^\nu(f,g) := \int_a^b \nabla^\nu f(x)\nabla^\nu g(x)d\nu(x) = \langle \nabla^\nu f, \nabla^\nu g\rangle_\nu,\ f,g \in \mathcal{D}_1^\nu,$$

defines a Dirichlet form on $L_2(K,\mu)$.

In Section 5, we show that the Dirichlet form $(\mathcal{E}^\nu, \mathcal{D}_1^\nu)$ has the same eigenvalues as the measure geometric Neumann Laplacian $\Delta_N^{\mu,\nu}$. Moreover, we construct a second Dirichlet form, which is in the same correspondence with the Dirichlet Laplacian $\Delta_D^{\mu,\nu}$. Applying the techniques introduced in Section 3, we obtain estimations of the eigenvalue counting functions $N_D^{\mu,\nu}(\cdot)$ and $N_N^{\mu,\nu}(\cdot)$.

In Section 6, we restrict ourselves to the case where $\nu$ and $\mu$ are the same and, in addition, self similar measures. In this special case, we can extend the notion of a "variational fractal", which has been introduced in [13] for certain connected fractals, to generalized Cantor sets, which are disconnected fractals. In particular, we obtain that the eigenvalue counting function behaves in this case asymptotically like $x^{1/2}$. Using other methods, this result was also obtained in [4].

## 2 Definition and Fundamental Properties of the Measure Geometric Laplacian.

Let $[a,b] \subset \mathbb{R}^1$ be a closed interval and $\nu$ be an atomless finite Borel measure on $[a,b]$ with compact support $K := \operatorname{supp}\nu$ and $a,b \in K$. Further, let $L_2 :=$

$L_2(K,\nu)$ be the separable Hilbert space with scalar product $\langle f,g\rangle := \int_a^b fg\, d\nu$. Without loss of generality we assume that $\nu(K) = 1$.

Let

$$\mathcal{D}_1^\nu := \{f : K \to \mathbb{R} : \exists f' \in L_2(K,\nu) : f(x) = f(a) + \int_a^x f'(y)d\nu(y),\ x \in K\}. \quad (1)$$

By standard measure theoretic arguments, it follows that $\mathcal{D}_1^\nu \subset \mathcal{C}(K) \subset L_2(K,\nu)$; i.e., every function $f$ in $\mathcal{D}_1^\nu$ is continuous on $K$. Moreover, the function $f'$ defined in (1) is unique in $L_2(K,\nu)$. Thus, for any $f \in \mathcal{D}_1^\nu$, we can define the $\nu$-*derivative* of $f$ by setting

$$\nabla^\nu f = \frac{df}{d\nu} := f'.$$

Note that in the case $K = [a,b]$ and $\nu = \lambda$, where $\lambda$ denotes the normalized Lebesgue measure on $[a,b]$, $\mathcal{D}_1^\nu$ coincides with the Sobolev space $W^{1,2}$.

In order to define the second derivative, we repeat the above construction with respect to another measure. Let $K$ and $\nu$ be as above.

Now let $\mu$ be a second atomless, normalized Borel measure on $[a,b]$ with compact support $L := \operatorname{supp}\mu$ and $a,b \in L$. Furthermore, we assume that $L \subset K$ and, if $K \setminus L \neq \emptyset$, we agree upon the following notation.
$L^C := [a,b]\setminus L$ is open in $\mathbb{R}$ and therefore a countable union of pairwise disjoint open intervals with endpoints in $L$. From $L = L \cap K = K \setminus L^C$ we obtain for some $c_i$ and $d_i$, $i = 1,2,\ldots$

$$L = K \setminus \Big(\sum_{i=1}^{\infty}(c_i,d_i)\Big) \text{ with } a < c_i < d_i < b,\ c_i,d_i \in L,\ i = 1,2,\ldots. \quad (2)$$

Furthermore, let $L_2(L,\mu)$ (and $L_2(K,\mu)$, resp.) denote the separable Hilbert space of all square $\mu$–integrable functions on $L$ (and $K$, resp.), both equipped with the scalar product $\langle f,g\rangle_\mu := \int_a^b fg\, d\mu$. Setting

$$\mathcal{D}_2^{\mu,\nu} := \{f \in \mathcal{D}_1^\nu : \exists f'' \in L_2(L,\mu) : \nabla^\nu f(x) = \nabla^\nu f(a) + \int_a^x f''(y)d\mu(y),\ x \in K\}, \quad (3)$$

the following properties are easy to show:

**Proposition 2.1.** *(i)* $\mathcal{D}_2^{\mu,\nu} \subset \mathcal{D}_1^\nu \subset \mathcal{C}(K) \subset L_2(K,\mu) \cap L_2(K,\nu)$.

*(ii) If* $L \neq K$*, then according to the notation of (2), for any* $f \in \mathcal{D}_2^{\mu,\nu}$

$$\nabla^\nu f(x) \equiv \nabla^\nu f(c_i),\ x \in (c_i,d_i) \cap K,\ i = 1,2,\ldots;$$

*i.e., for any function* $f \in \mathcal{D}_2^{\mu,\nu}$ *the* $\nu$*–derivative* $\nabla^\nu f$ *is uniquely determined on all of* $K$ *by its values on the subset* $L$*.*

*(iii) Under the same assumptions as made in (ii), we have for any* $f \in \mathcal{D}_2^{\mu,\nu}$*:*

$$f(x) = f(c_i) + \nabla^\nu f(c_i) \cdot \nu([c_i, x)),\ x \in (c_i, d_i) \cap K,\ i = 1, 2, \dots;$$

*i.e., every function* $f \in \mathcal{D}_2^{\mu,\nu}$ *itself is uniquely determined on all of* $K$ *by its values on* $L$.

*(iv) The function* $f''$ *defined by (3) is unique in* $L_2(L, \mu)$.

We define the *$\mu$-$\nu$-Laplacian* of $f \in \mathcal{D}_2^{\mu,\nu}$ by

$$\Delta^{\mu,\nu} f = \nabla^\mu\Big(\nabla^\nu f\Big) = \frac{d}{d\mu}\Big(\frac{df}{d\nu}\Big) := \begin{cases} f'' & \text{on } L \\ 0 & \text{on } K \setminus L \end{cases}$$

where $f''$ is given by (3). Note that for $f \in \mathcal{D}_2^{\mu,\nu}$ the function $\nabla^\nu f$ is $\nu$-unique and continuous on $K$ and therefore unique on $K$. From Proposition 2.1, (iv) it follows that

$$\Delta^{\mu,\nu} : \mathcal{D}_2^{\mu,\nu} \subseteq L_2(K, \mu) \to L_2(K, \mu)$$

is well defined.

**Remark 2.2.** As $\mathcal{D}_2^{\mu,\nu}$ is the set of all functions $f : K \to \mathbb{R}$ such there exist functions $f' \in L_2(K, \nu)$ and $f'' \in L_2(L, \mu)$ with $f(x) = f(a) + \int_a^x f'(y)\, d\nu(y)$, $x \in K$, and $f'(y) = f'(a) + \int_a^y f''(z)\, d\mu(z)$, $y \in K$, we infer by Fubini's theorem the following representation of $f \in \mathcal{D}_2^{\mu,\nu}$.

$$f(x) = f(a) + \nabla^\nu f(a) \cdot \nu([a, x)) + \int_a^x \nu([y, x)) \Delta^{\mu,\nu} f(y)\, d\mu(y),\ x \in K.$$

We now introduce *Dirichlet* and *Neumann boundary conditions*, respectively:

$$\mathcal{D}_{2,D}^{\mu,\nu} := \{f \in \mathcal{D}_2^{\mu,\nu} : f(a) = f(b) = 0\} \tag{4}$$

and

$$\mathcal{D}_{2,N}^{\mu,\nu} := \{f \in \mathcal{D}_2^{\mu,\nu} : \nabla^\nu f(a) = \nabla^\nu f(b) = 0\}. \tag{5}$$

The restriction of $\Delta^{\mu,\nu}$ on $\mathcal{D}_{2,D}^{\mu,\nu}$ (or $\mathcal{D}_{2,N}^{\mu,\nu}$, resp.) is called *Dirichlet-$\mu$-$\nu$-Laplacian* (or *Neumann-$\mu$-$\nu$-Laplacian,* resp.) and we denote it by $\Delta_D^{\mu,\nu}$ (or $\Delta_N^{\mu,\nu}$, resp.). In [2] is shown that $\Delta_D^{\mu,\nu}$ and $\Delta_N^{\mu,\nu}$ are negative symmetric operators on $L_2(K, \mu)$. Moreover, the eigenvalues of $\Delta_D^{\mu,\nu}$ (or $\Delta_N^{\mu,\nu}$, resp.) have finite multiplicities. They form a countable sequence which has no accumulation point except $-\infty$. Thus, we are allowed to define the eigenvalue counting function of $-\Delta_{D/N}^{\mu,\nu}$ given by

$$N_{D/N}^{\mu,\nu}(x) := \#\Big\{\kappa_k \le x : \kappa_k \text{ is eigenvalue of } -\Delta_{D/N}^{\mu,\nu}\Big\} \tag{6}$$

– counting according to multiplicities. Further, in [2] is obtained that the domains $\mathcal{D}_1^{\nu}$, $\mathcal{D}_2^{\mu,\nu}$, $\mathcal{D}_{2/D}^{\mu,\nu}$ and $\mathcal{D}_{2/N}^{\mu,\nu}$ defined by (1), (3), (4) and (5) are dense subspaces of $L_2(K,\mu)$.

**Remark 2.3.** By $\mathcal{H}^{\mu,\nu}$, we denote the space of the $\Delta^{\mu,\nu}$-harmonic functions; i.e.,

$$\mathcal{H}^{\mu,\nu} := \{f \in \mathcal{D}_2^{\mu,\nu} : \Delta^{\mu,\nu} f \equiv 0\}.$$

It is easy to see that $\dim_{\mathbb{R}} \mathcal{H}^{\mu,\nu} = 2$ and $\mathcal{D}_2^{\mu,\nu} = \mathcal{D}_{2,D}^{\mu,\nu} \oplus \mathcal{H}^{\mu,\nu}$.

In the following theorem, we state that every boundary value problem has a unique solution. This solution is given with respect to a kernel, which is given in terms of the measure $\nu$.

**Theorem 2.4 (see [2]).** *For any function $f \in L_2(K,\mu)$ and for any boundary values $u(a)$ and $u(b)$, the equation*

$$\Delta^{\mu,\nu} u = f$$

*has a solution $u \in \mathcal{D}_2^{\mu,\nu}$. Further, $u$ is unique in $L_2(K,\mu)$ and has the representation*

$$u(x) = u(a)\nu([x,b)) + u(b)\nu([a,x)) - \int_a^b g^{\nu}(x,y)f(y)\,d\mu(y),\ x \in K,$$

*where $g^{\nu}(.,.)$ denotes the $\nu$-Green function, which is given on $K \times K$ by*

$$g^{\nu}(y,x) = g^{\nu}(x,y) := \begin{cases} \nu([a,x))\nu([y,b)) & \text{for } x \le y \\ \nu([a,y))\nu([x,b)) & \text{for } x > y. \end{cases}$$

## 3 Dirichlet Forms and Dirichlet-Neumann-Bracketing.

In this section, we recall the definition of a Dirichlet form and present the technique of the so-called Dirichlet-Neumann-bracketing which goes back to Métivier [12] and Lapidus [10]. A general survey on Dirichlet forms can be found, for example, in [5] or [11].

Let $X$ be a compact set, and let $\tau$ be a Borel measure on $X$.

**Definition 3.1.** Let $\mathcal{F}$ be a dense subspace of the Hilbert space $L_2(X,\tau)$ equipped with the scalar product

$$\langle u,v\rangle_{L_2(X,\tau)} := \int_X uv\,d\tau,$$

and let $\mathcal{E}$ be a positive definite symmetric bilinear form on $\mathcal{F}$.
Then we call the pair $(\mathcal{E}, \mathcal{F})$ a Dirichlet form on $L_2(X, \tau)$ if the following properties hold:
(i) For any $\alpha > 0$, we define

$$\mathcal{E}_\alpha(u, v) := \mathcal{E}(u, v) + \alpha \langle u, v \rangle_{L_2(X,\tau)}.$$

Then $(\mathcal{F}, \mathcal{E}_\alpha)$ has to be a Hilbert space for any $\alpha > 0$.
(ii) (Markov property:) For any $u \in \mathcal{F}$ we define the function $\bar{u}$ by

$$\bar{u}(x) := \begin{cases} 1 & \text{if } u(x) > 1, \\ 0 & \text{if } u(x) < 0, \\ u(x) & \text{otherwise.} \end{cases}$$

Then $\bar{u}$ has to be in $\mathcal{F}$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$.

**Remark 3.2.** If $(\mathcal{F}, \mathcal{E}_1)$ is a Hilbert space, it is every $(\mathcal{F}, \mathcal{E}_\alpha)$, $\alpha > 0$ (see, for example, [11]).

Now we formulate the eigenvalue problem associated with a Dirichlet form.

**Definition 3.3.** Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L_2(X, \tau)$. If for a function $u \in \mathcal{F}$

$$\mathcal{E}(u, v) = \lambda \langle u, v \rangle_{L_2(X,\tau)}, \ \forall \ v \in \mathcal{F},$$

then we call $\lambda$ an eigenvalue of the form $(\mathcal{E}, \mathcal{F})$, and $u$ is a corresponding eigenfunction.

Following [8], we introduce the technique of of the Dirichlet-Neumann-bracketing.

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L_2(X, \tau)$ such that the eigenvalues of $(\mathcal{E}, \mathcal{F})$ form a sequence of real, nonnegative numbers with finite multiplicities which have no accumulation point except $+\infty$. (For example, it is sufficient that for some fixed $\alpha > 0$ the natural inclusion $(\mathcal{F}, \mathcal{E}_\alpha) \hookrightarrow L_2(X, \tau)$ is a compact operator, see [8].) Then, the eigenvalue counting function $N(x; \mathcal{E}, \mathcal{F})$ of $(\mathcal{E}, \mathcal{F})$

$$N(x; \mathcal{E}, \mathcal{F}) := \#\{i \geq 1 : \lambda_i \leq x\}$$

is well defined, where $\{\lambda_i\}_{i=1}^\infty$ denotes the increasing sequence of the eigenvalues of $(\mathcal{E}, \mathcal{F})$, according to multiplicities. These eigenvalues are given by the following Maximum–Minimum–principle (see Reed and Simon, [15]).

**Proposition 3.4.** *Let* $\{\lambda_i\}_{i=1}^{\infty}$ *denote the sequence of the eigenvalues of* $(\mathcal{E}, \mathcal{F})$ *as introduced above and fix* $\alpha > 0$. *Then*

$$(\lambda_i + \alpha)^{-1/2} = d_{i-1}\left(S_\alpha(\mathcal{F})\right),$$

*where* $S_\alpha(\mathcal{B})$ *is defined by*

$$S_\alpha(\mathcal{B}) := \{u \in \mathcal{B} \cap \mathcal{F} : \mathcal{E}_\alpha(u,u) \leq 1\},\ \mathcal{B} \subset L_2(X,\tau),$$

*and* $d_i = d_i(S_\alpha(\mathcal{B})), i \geq 0$ *is given by*

$$d_i := \inf\Big\{ \sup_{x \in S_\alpha(\mathcal{B})} \inf_{y \in \mathcal{Y}} ||x-y||_{L_2(X,\tau)} / \mathcal{Y} \subseteq L_2(X,\tau) \textit{ is a subspace } \textit{with } \dim \mathcal{Y} = i \Big\}.$$

**Remark 3.5.** If $\mathcal{B} \subseteq \mathcal{Y}$, where $\mathcal{Y}$ is a $n$-dimensional subspace of $L_2(X,\tau)$, then it follows that $d_i(S_\alpha(\mathcal{B})) = 0$ for $i \geq n$.

From Proposition 3.4 we conclude that

$$N(x;\mathcal{E},\mathcal{F}) = \#\{i \geq 0 : d_i\left(S_\alpha(\mathcal{F})\right) \geq (x+\alpha)^{-1/2}\}.$$

Now we introduce the technique of the Dirichlet-Neumann-bracketing. Let $(\mathcal{E}', \mathcal{F}')$ be another Dirichlet form on $L_2(X,\tau)$ such that $\mathcal{F}' \subseteq \mathcal{F}$ is a closed subspace, and $\mathcal{E}'$ is given by $\mathcal{E}' := \mathcal{E}_{|\mathcal{F}' \times \mathcal{F}'}$. The following property gives a relation between the eigenvalue counting functions $N(x;\mathcal{E},\mathcal{F})$ and $N(x;\mathcal{E}',\mathcal{F}')$ (for the proof we refer to [8]):

**Proposition 3.6.** *If* $\dim(\mathcal{F}/\mathcal{F}') < \infty$, *then for any* $x \geq 0$.

$$N(x;\mathcal{E}',\mathcal{F}') \leq N(x;\mathcal{E},\mathcal{F}) \leq N(x;\mathcal{E}',\mathcal{F}') + dim\mathcal{F}/\mathcal{F}'.$$

## 4 The $\nu$-Dirichlet Form.

Suppose we are given two measures $\nu$ and $\mu$ with supp $\mu = L \subset K =$ supp $\nu$ as in Section 2. Furthermore, as above, $\mathcal{D}_1^\nu$ denotes the space of all functions possessing a $\nu$-derivative in $L_2(K,\nu)$. We define the following nonnegative symmetric bilinear form $\mathcal{E}^\nu$ on $\mathcal{D}_1^\nu$.

$$\mathcal{E}^\nu(f,g) := \int_a^b \nabla^\nu f(x) \nabla^\nu g(x) d\nu(x) = \langle \nabla^\nu f, \nabla^\nu g \rangle_\nu,\ f,g \in \mathcal{D}_1^\nu.$$

Then the following holds.

**Theorem 4.1.** *$(\mathcal{E}^\nu, \mathcal{D}_1^\nu)$ is a Dirichlet form on $L_2(K,\mu)$.*

PROOF. We have to show:

(i) $\mathcal{D}_1^\nu \subset L_2(K,\mu)$ is a dense subspace.

(ii) $(\mathcal{D}_1^\nu, \mathcal{E}_1^\nu)$ is a Hilbert space.

(iii) The Markov property holds.

(i) The density of $\mathcal{D}_1^\nu$ in $L_2(K,\mu)$ is proved in [2], Corollary 6.4.
(ii) Obviously, $\mathcal{E}_1^\nu$ defines a scalar product on $\mathcal{D}_1^\nu$, therefore $(\mathcal{D}_1^\nu, \mathcal{E}_1^\nu)$ is a pre-Hilbert space. It remains to show that $\mathcal{D}_1^\nu$ is complete w.r.t. to the norm $\sqrt{\mathcal{E}_1^\nu}$. Let $(u_n) \subset \mathcal{D}_1^\nu$ be a Cauchy sequence w.r.t. $\sqrt{\mathcal{E}_1^\nu}$; i.e.,

$$||\nabla^\nu u_n - \nabla^\nu u_m||^2_{|L_2(K,\nu)} + ||u_n - u_m||^2_{|L_2(K,\mu)} \to 0, \quad n,m \to \infty.$$

As $L_2(K,\nu)$ and $L_2(K,\mu)$ are Hilbert spaces, there exist functions $f \in L_2(K,\nu)$ and $u \in L_2(K,\mu)$ such that

$$||\nabla^\nu u_n - f||^2_{|L_2(K,\nu)} \to 0, \quad n \to \infty,$$

and

$$||u_n - u||^2_{|L_2(K,\mu)} \to 0, \; n \to \infty.$$

Because of

$$\begin{aligned} &\int_K \left| \int_c^x (f(z) - \nabla^\nu u_n(z))\, d\nu(z) \right| d\mu(x) \\ &\le \int_K \int_K |f(z) - \nabla^\nu u_n(z)|\, d\mu(x) d\nu(z) \to 0, \; n \to \infty, \end{aligned}$$

we obtain

$$\begin{aligned} &\int_K \left| u(x) - u(c) - \int_c^x f(z) d\nu(z) \right| d\mu(x) \\ &= \lim_{k\to\infty} \int_K \left| u_{n_k}(x) - u_{n_k}(c) - \int_c^x \nabla^\nu u_{n_k}(z) d\nu(z) \right| d\mu(x) = 0. \end{aligned}$$

Hence,

$$u(x) = u(c) + \int_c^x f(z) d\nu(z) \tag{7}$$

holds for $\mu$-almost every $x$ and for $\mu$-almost every $c$ in $K$; i.e., (7) holds in $L_2(K,\mu)$. Thus, we conclude that $u$ is in $\mathcal{D}_1^\nu$ with $\nabla^\nu u = f$ and therefore

$\mathcal{E}_1^\nu(u_n, u) \to 0$ as $n \to \infty$.

(iii) Choose $u \in \mathcal{D}_1^\nu$; i.e., there exists a function $\nabla^\nu u \in L_2(K,\nu)$ with

$$u(x) = u(a) + \int_a^x \nabla^\nu u(y) d\nu(y), \quad x \in K.$$

Setting $\bar{u} = 0 \vee u \wedge 1$, we define the function $\nabla^\nu \bar{u}$ in $L_2(K,\nu)$ by

$$\nabla^\nu \bar{u}(y) := \begin{cases} \nabla^\nu u(y) & y \in A := K \cap \{0 \le u(y) \le 1\} \\ 0 & y \in B := K \setminus A \end{cases}$$

Obviously,

$$\bar{u}(x) = \bar{u}(a) + \int_a^x \nabla^\nu \bar{u}(y) d\nu(y), \; x \in K,$$

and therefore we infer $\bar{u} \in \mathcal{D}_1^\nu$. The definition of $\nabla^\nu \bar{u}$ yields immediately $\mathcal{E}^\nu(\bar{u},\bar{u}) \le \mathcal{E}^\nu(u,u)$. □

**Remark 4.2.** As in the classical Lebesgue case we have the Gauß-Green-formula (see [2], Proposition 3.1.):

$$\int_a^b (\Delta^{\mu,\nu} f, g)\, d\mu = (\nabla^\nu f)\, g\Big|_a^b - \mathcal{E}^\nu(f,g) \quad f \in \mathcal{D}_2^{\mu,\nu}, g \in \mathcal{D}_1^\nu. \tag{8}$$

**Remark 4.3.** From $\mathcal{D}_1^\nu \subseteq \mathcal{C}(K)$ we obtain that $\mathcal{C}_0(K) \cap \mathcal{D}_1^\nu$ is dense in $\mathcal{D}_1^\nu$ w.r.t. the norm $\sqrt{\mathcal{E}_1^\nu}$ and dense in $\mathcal{C}_0(K)$ w.r.t. the norm $||\cdot||_\infty$. Hence, the form $(\mathcal{E}^\nu, \mathcal{D}_1^\nu)$ is a regular Dirichlet form on $L_2(K,\mu)$. Moreover, it is easy to see that $(\mathcal{E}^\nu, \mathcal{D}_1^\nu)$ is local. From the theory of Dirichlet forms it follows (see, for example, [11]) that there exists an associated strong Markovian process with almost surely continuous paths on $L$. In the special case of the operator $\frac{d}{d\mu}\frac{d}{dx}$; i.e., $\nu$ is just given by Lebesgue measure, these operators have already been studied. The corresponding stochastic processes are the so-called quasi-, or gap-diffusions (see, for exp, [7], [9]).

**Remark 4.4.** Obviously, the functions $\phi_C^\nu$, $C \in \mathbb{R}$, defined by $\phi_C^\nu(x) := C \cdot \nu([a,x))$, $x \in K$, are in $\mathcal{D}_1^\nu$, their $\nu$-derivative is given by $\nabla^\nu \phi_C^\nu \equiv C$. Hence, $\mathcal{E}^\nu(\phi_C^\nu, \phi_C^\nu) = C^2$; i.e., the form $(\mathcal{E}^\nu, \mathcal{D}_1^\nu)$ is non vanishing.

## 5 Application of the Dirichlet-Neumann-Bracketing.

In the present section, we show that the eigenvalue problem for the $\nu$-Dirichlet form $(\mathcal{E}^\nu, \mathcal{D}_1^\nu)$ and the eigenvalue problem for the negative Neumann-$\mu$-$\nu$-Laplacian $-\Delta_N^{\mu,\nu}$ are equivalent. Moreover, we define another Dirichlet form

which is in the same correspondence with the Dirichlet-$\mu$-$\nu$-Laplacian $\Delta_D^{\mu,\nu}$. The aim of this construction is to get from Proposition 3.6 estimations for the eigenvalue counting functions of the Laplacian $N_N^{\mu,\nu}(\cdot)$ and $N_D^{\mu,\nu}(\cdot)$ introduced in (6).
Let $(\mathcal{E}^\nu, \mathcal{D}_1^\nu)$ be the $\nu$–Dirichlet form defined in Section 4.

**Proposition 5.1.** *For any $\lambda \in \mathbb{R}$ and any function $u \in \mathcal{D}_1^\nu$*

$$\mathcal{E}^\nu(u,v) = \lambda\langle u,v\rangle_\mu, \text{ for every } v \in \mathcal{D}_1^\nu,$$

*(i.e., $u$ is an eigenfunction of $(\mathcal{E}^\nu, \mathcal{D}_1^\nu)$ to the eigenvalue $\lambda$) if and only if*

$$u \in \mathcal{D}_{2,N}^{\mu,\nu} \text{ and } \Delta^{\mu,\nu}u = -\lambda u$$

*(i.e., $u$ is a Neumann-eigenfunction of $\Delta^{\mu,\nu}$ to the eigenvalue $-\lambda$).*

In order to prove this proposition we make use of the following lemma.

**Lemma 5.2.** *For any $x \in K$ let $g^{\nu,x}(y) := g^\nu(x,y)$, where $g^\nu(x,y)$ is the $\nu$-Green function given in Theorem 2.4. Then:*

*(i)* $g^{\nu,a} \equiv g^{\nu,b} \equiv 0$

*(ii) For any $x \in K$, $g^{\nu,x}$ is in $\mathcal{D}_1^\nu$ and*

$$\mathcal{E}^\nu(g^{\nu,x}, f) = f(x) - f(a)\phi_a^\nu(x) - f(b)\phi_b^\nu(x), \ f \in \mathcal{D}_1^\nu,$$

*where $\phi_a^\nu$ and $\phi_b^\nu$ are special $\mu$-$\nu$-harmonic functions given by*

$$\phi_a^\nu(x) := \nu([x,b)) \text{ and } \phi_b^\nu(x) := \nu([a,x)).$$

*(iii)* $\mathcal{E}^\nu(g^{\nu,x}, g^{\nu,x}) = g^\nu(x,x), \ x \in K.$

PROOF. (i) This assertion follows immediately from the definition of $g^\nu(x,y)$.

(ii) Obviously, it holds that $\nabla^\nu\nu([a,\cdot)) \equiv 1$ and $\nabla^\nu\nu([\cdot,b)) \equiv -1$, therefore we have for fixed $x \in K$: $g^{\nu,x} \in \mathcal{D}_1^\nu$ and

$$\begin{aligned}\mathcal{E}^\nu(g^{\nu,x}, f) &= \int_a^b \nabla^\nu g^{\nu,x}(y) \nabla^\nu f(y)\, d\nu(y)\\ &= \int_a^x \nabla^\nu \Big[\nu([a,y))\nu([x,b))\Big] \nabla^\nu f(y)\, d\nu(y)\\ &\quad + \int_x^b \nabla^\nu \Big[\nu([a,x))\nu([y,b))\Big] \nabla^\nu f(y)\, d\nu(y)\\ &=\nu([x,b)) \int_a^x \nabla^\nu f(y) d\nu(y) - \nu([a,x)) \int_x^b \nabla^\nu f(y)\, d\nu(y)\\ &=\nu([x,b))\Big[f(x) - f(a)\Big] - \nu([a,x))\Big[f(b) - f(x)\Big]\\ &=f(x) - f(a)\phi_a^\nu(x) - f(b)\phi_b^\nu(x).\end{aligned}$$

(iii) From (i), (ii) and the symmetry of $g^\nu(x,y)$ we obtain

$$\begin{aligned}\mathcal{E}^\nu(g^{\nu,x}, g^{\nu,x}) &=g^{\nu,x}(x) - g^{\nu,x}(a)\phi_a^\nu(x) - g^{\nu,x}(b)\phi_b^\nu(x)\\ &=g^{\nu,x}(x) - g^{\nu,a}(x)\phi_a^\nu(x) - g^{\nu,b}(x)\phi_b^\nu(x)\\ &=g^{\nu,x}(x) = g^\nu(x,x).\end{aligned}$$ □

Proof of Proposition 5.1. First, we assume that $\mathcal{E}^\nu(u,v) = \lambda\langle u,v\rangle_\mu$ for every $v \in \mathcal{D}_1^\nu$, and we choose $v = g^{\nu,x}$, $x \in K$ fixed. According to Lemma 5.2 (ii) the function $g^{\nu,x}$ is in $\mathcal{D}_1^\nu$, and

$$\begin{aligned}u(x) - u(a)\nu([x,b)) - u(b)\nu([a,x)) &=\mathcal{E}^\nu(g^{\nu,x}, u) = \lambda\langle u, g^{\nu,x}\rangle_\mu,\\ &=\lambda \int_a^b u(y) g^\nu(x,y) d\mu(y).\end{aligned}$$

This is true for any $x \in K$. Hence we infer from Theorem 2.4 that $u$ is in $\mathcal{D}_2^{\mu,\nu}$ and $\Delta^{\mu,\nu} u = -\lambda u$ on $L_2(K,\mu)$. Moreover, (8) yields

$$\begin{aligned}\lambda\langle u,v\rangle_\mu = \mathcal{E}^\nu(u,v) &= (\nabla^\nu u)\, v|_a^b - \langle \Delta^{\mu,\nu} u, v\rangle_\mu\\ &= (\nabla^\nu u)\, v|_a^b + \lambda\langle u,v\rangle_\mu, \quad \forall v \in \mathcal{D}_1^\nu.\end{aligned}$$

From this we obtain $\nabla^\nu u(a) = \nabla^\nu u(b) = 0$; i.e., $u \in \mathcal{D}_{2,N}^{\mu,\nu}$.

The converse is an immediate consequence of formula (8). □

Now we define the closed subspace of $\mathcal{D}_1^\nu$ by

$$\mathcal{D}_{1,D}^\nu := \{f \in \mathcal{D}_1^\nu : f(a) = f(b) = 0\},$$

and we consider the restriction of $\mathcal{E}^\nu$ to this subspace $\mathcal{E}_0^\nu := \mathcal{E}^\nu_{|\mathcal{D}^\nu_{1,D}\times\mathcal{D}^\nu_{1,D}}$. Analogous to Proposition 5.1, we have the following.

**Proposition 5.3.** *(i) $(\mathcal{E}_0^\nu, \mathcal{D}^\nu_{1,D})$ is a Dirichlet form on $L_2(K,\mu)$.*

*(ii) For any $\lambda \in \mathbb{R}, u \in \mathcal{D}^\nu_{1,D}$*

$$\mathcal{E}_0^\nu(u,v) = \lambda\langle u,v\rangle_\mu, \text{ for every } v \in \mathcal{D}^\nu_{1,D},$$

*(i.e., $u$ is an eigenfunction of $(\mathcal{E}_0^\nu, \mathcal{D}^\nu_{1,D})$ to the eigenvalue $\lambda$) if and only if*

$$u \in \mathcal{D}^{\mu,\nu}_{2,D} \text{ and } \Delta^{\mu,\nu}u = -\lambda u$$

*(i.e., $u$ is a Dirichlet–eigenfunction of $\Delta^{\mu,\nu}$ to the eigenvalue $-\lambda$).*

PROOF. Note that $\mathcal{D}^\nu_1 = \mathcal{D}^\nu_{1,D} \oplus \mathcal{H}^{\mu,\nu}$, where $\mathcal{H}^{\mu,\nu}$ is the space of $\mu$-$\nu$-harmonic functions introduced in Remark 2.3. From $\dim \mathcal{H}^{\mu,\nu} = 2$ we obtain that $\mathcal{D}^\nu_{1,D}$ is dense in $L_2(K,\mu)$ because $\mathcal{D}^\nu_1$ is dense in $L_2(K,\mu)$. The rest of the proof is a simple modification of the proofs of Theorem 4.1 and Proposition 5.1. □

From Proposition 5.1 and Proposition 5.3, we obtain for any $x \geq 0$:

$$N_N^{\mu,\nu}(x) = N(x; \mathcal{E}^\nu, \mathcal{D}^\nu_1) \tag{9}$$

and

$$N_D^{\mu,\nu}(x) = N(x; \mathcal{E}_0^\nu, \mathcal{D}^\nu_{1,D}), \tag{10}$$

where $N(x;\mathcal{E},\mathcal{F})$ denotes the eigenvalue counting function of the Dirichlet form $(\mathcal{E},\mathcal{F})$. In particular, we conclude the following.

**Corollar 5.4.** *The eigenvalues of $-\Delta_N^{\mu,\nu}$ (or $-\Delta_D^{\mu,\nu}$, respectively) are given by the Maximum-Minimum-principle stated in Proposition 3.4 where one has to set $\mathcal{F} = \mathcal{D}^\nu_1$ (or $\mathcal{F} = \mathcal{D}^\nu_{1,D}$, respectively) and $\mathcal{E}_\alpha = \mathcal{E}^\nu_\alpha$.*

Finally, we get from Proposition 3.6, Remark 2.3 and the equalities (9) and (10) the following estimation.

**Proposition 5.5.** *For any real number $x \geq 0$*

$$N_D^{\mu,\nu}(x) \leq N_N^{\mu,\nu}(x) \leq N_D^{\mu,\nu}(x) + 2.$$

**Remark 5.6.** The last proposition is a crucial tool in determining the spectral asymptotics of the Dirichlet-and Neumann-Laplacians in the case of self similar measures (see [3]).

## 6 Self-Similar Measures and Variational Fractals.

In this section we restrict ourselves to the case where $\nu$ and $\mu$ are the same and, in addition, self similar measures. This makes it possible to extend the notion of a "variational fractal" to a certain class of disconnected fractal subsets of the real line. For the definition of self-similar sets and self-similar measures, we refer the reader to [1] and [6].

Let $K$ be the unique self similar set with respect to a finite family of affine contractions $S = \{S_1, \ldots, S_N\}$ from $[a, b]$ to $[a, b]$ with contraction ratios $r_1, \ldots, r_N$ such that the images $S_i[a, b]$ and $S_j[a, b]$ intersect in at most one point for $i \neq j$. Without loss of generality, we assume that $a, b \in K$. Furthermore, we are given a vector of weights $\rho = (\rho_1, \ldots, \rho_N)$; i.e., $\rho_i \in (0, 1), i = 1 \ldots N$, $\sum_{i=1}^{N} \rho_i = 1$. Then there exists a unique self similar measure $\mu = \mu(S, \rho)$ with respect to $S$ and $\rho$; i.e.,

$$\mu(A) = \sum_{i=1}^{N} \rho_i \mu(S_i^{-1} A)$$

for any Borel set $A$ in $[a, b]$.

Now we assume that $\nu = \mu = \mu(S, \rho)$; i.e, the measures are equal and given as a self similar measure w.r.t. a family of contractions $S$ and weights $\rho$ as described above. In this case, the eigenvalue counting function behaves asymptotically under both, Dirichlet and Neumann boundary conditions, as follows (see [4] for the proof):

$$N_{D/N}^{\mu,\mu}(x) \asymp x^{1/2},\ x \to \infty; \tag{11}$$

i.e., there exist positive constants $C_1$, $C_2$ and $x_0$, such that

$$C_1 x^{1/2} \leq N_{D/N}^{\mu,\mu}(x) \leq C_2 x^{1/2},\ x \geq x_0.$$

Now we introduce the notion of a variational fractal which goes back to Mosco (see [13]).

**Definition 6.1.** A triple $(K, \mu, \mathcal{E})$ is called a variational fractal if the following is satisfied.

(i) $\mathcal{E}$ is a strongly local, regular, non vanishing Dirichlet form on $L_2(K, \mu)$ with domain $\mathcal{F}$.

(ii) $\mathcal{E}$ satisfies the scaling property; i.e., there exists a constant $\sigma < 1$, s.t.

$$\mathcal{E}(u, u) = \sum_{i=1}^{N} [\mu(S_i K)]^{\sigma}\, \mathcal{E}(u \circ S_i, u \circ S_i),\ u \in \mathcal{F}. \tag{12}$$

**Proposition 6.2.** $(K, \mu, \mathcal{E}^\mu)$ *with the previous properties is a variational fractal with* $\sigma = -1$.

PROOF. According to Theorem 4.1, Remark 4.3 and Remark 4.4 we only have to show the scaling property. By the self similarity of the measure $\mu$ we have $\mu \llcorner S_i[a,b] = \rho_i \mu \circ S_i^{-1} \llcorner S_i[a,b]$, $i = 1 \dots N$, and therefore we obtain

$$\begin{aligned}\mathcal{E}^\mu(f,g) &= \int_a^b (\nabla^\mu f)(\nabla^\mu g) d\mu = \sum_{i=1}^N \int_{S_i a}^{S_i b} (\nabla^\mu f)\,(\nabla^\mu g)\, d\mu \\ &= \sum_{i=1}^N \rho_i \int_{S_i a}^{S_i b} (\nabla^\mu f)\,(y)\,(\nabla^\mu g)\,(y) dS_i\mu(y) \\ &= \sum_{i=1}^N \rho_i \int_a^b (\nabla^\mu f)\,(S_i y)\,(\nabla^\mu g)\,(S_i y) d\mu(y).\end{aligned}$$

Note that $\nabla^\mu f(S_i(y)) = \rho_i^{-1} \nabla^\mu (f \circ S_i)(y), i = 1 \dots N$. Hence, we obtain

$$\mathcal{E}^\mu(f,g) \quad = \quad \sum_{i=1}^N \rho_i^{-1} \int_a^b \nabla^\mu\,(f \circ S_i)\,(y) \nabla^\mu\,(g \circ S_i)\,(y) d\mu(y),$$

which yields the assertion. □

**Remark 6.3.** In [14], Posta proved that the eigenvalue counting function of the Laplacian which is associated with the Dirichlet form of a variational fractal behaves asymptotically like $x^{\nu/2}$, where $\nu = \frac{2}{1-\sigma}$ and $\sigma$ is the exponent in the scaling property (12). This coincides with our result (11).

**Remark 6.4.** Obviously, the notion of a variational fractal does not make sense if $\mu$ and $\nu$ are different measures, even not if supp $\mu =$ supp $\nu$.

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