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SOME CLASSES OF STRONGLY QUASICONTINUOUS FUNCTIONS

Abstract

Some classes of strongly quasicontinuous functions are investigated.

Let $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ be the set of all real, rational and positive integer numbers, respectively. For a set $A \subset \mathbb{R}$ denote by $\operatorname{Int} A$ and $\operatorname{Cl} A$ the interior and the closure of A, respectively. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous at a point $x \in \mathbb{R}$ if for each $\varepsilon > 0$ and for each neighborhood U of x there is a nonempty open set $G \subset U$ such that $f(G) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$ ([3]). Denote by $Q(f)$ ($C(f)$) the set of all quasi-continuity (continuity) points of f. It is well-known that the set $Q(f) \setminus C(f)$ is of the first category but it need not be measurable or of measure zero (e.g., if T is a closed nowhere dense set of positive measure and $S \subset T$ is dense in T and nonmeasurable (of measure zero), then for its characteristic function χ_S the set $Q(\chi_S) \setminus C(\chi_S)$ is nonmeasurable (of positive measure).

Let ℓ_e (ℓ) denote the outer Lebesgue measure (Lebesgue measure) in $\mathbb{R}$. Let

$$d_u(A, x) = \limsup_{h \to 0^+} \ell_e(A \cap (x - h, x + h))/2h$$
$$(d_l(A, x) = \liminf_{h \to 0^+} \ell_e(A \cap (x - h, x + h)/2h)$$

the upper (lower) outer density of $A \subset \mathbb{R}$ at a point $x \in \mathbb{R}$.

Z. Grande in [1] introduced properties $A(x)$ and $B(x)$ of functions:

A function $f : \mathbb{R} \to \mathbb{R}$ has property $A(x)$ at a point $x \in \mathbb{R}$ if there exists an open set U such that $d_u(U, x) > 0$ and the restricted function $f \restriction (U \cup \{x\})$ is continuous at x. We will write $f \in A(x)$ if f has the property $A(x)$ at a point x.

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A function $f:\mathbb{R}\to\mathbb{R}$ has property $B(x)$ at $x\in\mathbb{R}$ (abbreviated $f\in B(x)$) if for $\varepsilon>0$ we have $d_u(\mathrm{Int}\{y:|f(y)-f(x)|<\varepsilon\},x)>0$.

Denote by $A(f)$ the set $\{x\in\mathbb{R}: f\in A(x)\}$ and by $B(f)$ the set $\{x\in\mathbb{R}: f\in B(x)\}$. Z. Grande has shown that $C(f)\subset A(f)\subset B(f)\subset Q(f)$ and that the measure of $B(f)\setminus C(f)$ is zero.

Definition 1. Let $f:\mathbb{R}\to\mathbb{R}$ be a function and let $r\in[0,1)$. We put

> $A_r(f)=\{x\in\mathbb{R}$: there is an open set U such that $d_u(U,x)>r$ and $f\restriction(U\cup\{x\})$ is continuous at $x\}$,
>
> $A_r^l(f)=\{x\in\mathbb{R}$: there is an open set U such that $d_l(U,x)>r$ and $f\restriction(U\cup\{x\})$ is continuous at $x\}$,
>
> $B_r(f)=\{x\in\mathbb{R}$: for each $\varepsilon>0$ there is an open set U such that $d_u(U,x)>r$ and $f(U)\subset(f(x)-\varepsilon,f(x)+\varepsilon)\}$,
>
> $B_r^l(f)=\{x\in\mathbb{R}$: for each $\varepsilon>0$ there is an open set U such that $d_l(U,x)>r$ and $f(U)\subset(f(x)-\varepsilon,f(x)+\varepsilon)\}$.

Evidently, $A_r^l(f)\subset A_r(f)\subset B_r(f)$ and $A_r^l(f)\subset B_r^l(f)\subset B_r(f)$ for each $r\in[0,1)$. Further, $A_r(f)\subset A_s(f)$, $B_r(f)\subset B_s(f)$, $A_r^l(f)\subset A_s^l(f)$, and $B_r^l(f)\subset B_s^l(f)$ for $0\le s<r<1$. Thus, the sets $A_r(f)\setminus C(f)$, $A_r^l(f)\setminus C(f)$, $B_r(f)\setminus C(f)$ and $B_r^l(f)\setminus C(f)$ are sets of first category and of measure zero. We shall show that $B_r(f)\subset A_s(f)$ and $B_r^l(f)\subset A_s^l(f)$ for $0\le s<r<1$.

Lemma 1. *Let $0\le\beta<1$, $a>0$, $x\in\mathbb{R}$ and let A be a measurable set. If $\ell(A\cap(x-a,x+a))>\beta$, then there is $c\in(0,a)$ such that for each $b\in(0,c)$*

$$\ell(A\cap((x-a,x-b)\cup(x+b,x+a)))>\beta.$$

Proof. Put $\ell(A\cap(x-a,x+a))=\alpha>\beta$. Then there is $c>0$ such that $\alpha-2c>\beta$. Since $2c<\alpha\le 2a$, we have $c<a$. Let $0<b<c$. Then $\alpha=\ell(A\cap(x-a,x+a))=\ell(A\cap((x-a,x-b)\cup(x+b,x+a)))+\ell(A\cap(x-b,x+b))\le\ell(A\cap((x-a,x-b)\cup(x+b,x+a)))+2b$. Therefore $\ell(A\cap((x-a,x-b)\cup(x+b,x+a)))\ge\alpha-2b>\beta$. $\square$

Theorem 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function and let $0\le s<r<1$. Then $B_r(f)\subset A_s(f)$ and $B_r^l(f)\subset A_s^l(f)$.*

Proof. The inclusion $B_r(f)\subset A_s(f)$:

Let $x\in B_r(f)$. Then for each $n\in\mathbb{N}$ there is an open set A_n such that $d_u(A_n,x)>r$ and $f(A_n)\subset(f(x)-1/n,f(x)+1/n)$. There is a sequence $(h_i^n)_i$ such that $0<h_{i+1}^n<h_i^n$, $\lim\limits_{i\to\infty}h_i^n=0$ and $\ell(A_n\cap(x-h_i^n,x+h_i^n))/(2h_i^n)>r$.

Let $v_0 = h_1^1$. Since $\ell(A_1 \cap (x - v_0, x + v_0)) > 2rv_0$, according to Lemma 1 there is $c_1 \in (0, v_0)$ such that $\ell(A_1 \cap ((x - v_0, x - b) \cup (x + b, x + v_0))) > 2rv_0$ for each $b \in (0, c_1)$. Let $j \in \mathbb{N}$ be such that $h_j^2 < c_1/2$ and let $v_1 = h_j^2$. Assume that we have positive numbers $v_0, v_1, \dots, v_n$ such that $0 < v_i < v_{i-1}/2$, $v_i \in \{h_1^{i+1}, h_2^{i+1}, \dots, h_k^{i+1}, \dots\}$ and $\ell(A_i \cap ((x - v_{i-1}, x - v_i) \cup (x + v_i, x + v_{i+1})) > 2rv_{i-1}$ for each $i \in \{1, 2, \dots, n\}$. Since $v_n = h_j^{n+1}$ for some $j \in \mathbb{N}$, so $\ell((A_{n+1} \cap (x - v_n, x + v_n)) > 2rv_n$ and according to Lemma 1 there is $c_{n+1} \in (0, v_n)$ such that $\ell((A_{n+1} \cap ((x - v_n, x - b) \cup (x + b, x + v_n))) > 2rv_n$ for each $b \in (0, c_{n+1})$ There is $k \in \mathbb{N}$ such that $h_k^{n+2} < c_{n+1}/2$ and put $v_{n+1} = h_k^{n+2}$.

Now put $V_n = A_n \cap ((x - v_{n-1}, x - v_n) \cup (x + v_n, x + v_{n-1}))$ and $V = \bigcup_{n=1}^{\infty} V_n$. Then V is an open set. We shall show that $d_u(V, x) > s$. We see that $V \cap (x - v_n, x + v_n) = \bigcup_{i=1}^{\infty}(V_i \cap (x - v_n, x + v_n)) = \bigcup_{i=n+1}^{\infty} V_i$ and therefore $\ell(V \cap (x - v_n, x + v_n)) = \sum\limits_{i=n+1}^{\infty} \ell(V_i) \geq \ell(V_{n+1}) > 2rv_n$. This yields $\dfrac{\ell(V \cap (x - v_n, x + v_n))}{2v_n} > \dfrac{2rv_n}{2v_n} = r$ and thus $d_u(V, x) \geq r > s$.

Now we shall show that $f \restriction (V \cup \{x\})$ is continuous at x. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ with $1/n < \varepsilon$. If $y \in V \cap (x - v_n, x + v_n)$, then $y \in V_j$ for some $j \geq n$. Then $f(y) \in f(V_j) \subset f(A_j) \subset (f(x) - 1/j, f(x) + 1/j) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$; i.e., $f \restriction (V \cup \{x\})$ is continuous at x.

The inclusion $B_r^l(f) \subset A_s^l(f)$:

Let $x \in B_r^l(f)$. Then for each $n \in \mathbb{N}$ there is an open set A_n such that $\beta_n = d_l(A_n, x) > r$ and $f(A_n) \subset (f(x) - 1/n, f(x) + 1/n)$. Since $r/\beta_n < 1$, for each $n \in \mathbb{N}$ there is $k_n > 2$ such that

$$\frac{k_n - 1}{k_n + 1} > \max\left\{\sqrt{\frac{r}{\beta_n}}, \sqrt{\frac{r}{\beta_{n+1}}}\right\}.$$

Put $\eta_n = \dfrac{\beta_n - r}{k_n} > 0$. Evidently $\beta_n - \eta_n > r$. Since $d_l(A_n, x) > \beta_n - \eta_n$, there is $h_n > 0$, such that $\ell(A_n \cap (x - h, x + h)) > 2(\beta_n - \eta_n)h$ for each $h \in (0, h_n]$.

We can assume that $h_{n+1} < h_n/2$. Put $p_0 = h_1$. According to Lemma 1 there is $c_1 \in (0, p_0)$ such that $\ell(A_1 \cap ((x - p_0, x - b) \cup (x + b, x + p_0))) > 2(\beta_1 - \eta_1)p_0$ for each $b \in (0, c_1)$. Further, since $d_l(A_1, x) < \beta_1 + \eta_1$, there is $p_1 < \min\{c_1, h_2\}$ such that $\ell(A_1 \cap ((x - p_1, x + p_1)) < 2(\beta_1 + \eta_1)p_1$.

Assume that we have positive numbers $p_0, p_1, \dots, p_n$ such that for each $i \in \{1, 2, \dots, n\}$

$$p_i < \min\{p_{i-1}, h_{i+1}\},$$

$$\ell((A_i \cap (x - p_i, x + p_i)) < 2(\beta_i + \eta_i)p_i \text{ and}$$

$$\ell(A_i \cap ((x - p_{i-1}, x - p_i) \cup (x + p_i, x + p_{i-1}))) > 2(\beta_i - \eta_i)p_{i-1}.$$

Since $p_n < h_{n+1}$, $\ell(A_{n+1} \cap (x-p_n, x+p_n)) > 2(\beta_{n+1} - \eta_{n+1})p_n$ and according to Lemma 1 there is $c_{n+1} \in (0, p_n)$ such that for each $b \in (0, c_{n+1})$

$$\ell(A_{n+1} \cap ((x-p_n, x-b) \cup (x+b, x+p_n)) > 2(\beta_{n+1} - \eta_{n+1})p_n.$$

Further there is $p_{n+1} < \min\{c_{n+1}, h_{n+2}\}$ such that

$$\ell(A_{n+1} \cap ((x-p_{n+1}, x+p_{n+1})) < 2(\beta_{n+1} + \eta_{n+1})p_{n+1}.$$

Then $p_{n+1} < p_n$ and $\ell(A_{n+1} \cap ((x-p_n, x-p_{n+1}) \cup (x+p_{n+1}, x+p_n)) > 2(\beta_{n+1} - \eta_{n+1})p_n$. Put $V_n = A_n \cap ((x-p_{n-1}, x-p_n) \cup (x+p_n, x+p_{n-1}))$ and $V = \bigcup_{n=1}^{\infty} V_n$. Then V is an open set. We shall show that $d_l(V, x) > s$. Let $0 < h < h_1$. Since $0 < p_{j+1} \leq p_j$ and $\lim\limits_{j\to\infty} p_j = 0$, there is $n \in \mathbb{N}$ such that $p_n < h \leq p_{n-1}$. Then $h > \frac{k_n+1}{k_n-1}p_n$ or $h \leq \frac{k_n+1}{k_n-1}p_n$.

a) Let $h > \dfrac{k_n+1}{k_n-1}p_n$. Then $k_n h - h - k_n p_n - p_n = (k_n - 1)h - (k_n+1)p_n > (k_n-1)\dfrac{k_n+1}{k_n-1}p_n - (k_n+1)p_n = 0$. Further, $h \leq p_{n-1} < h_n$ and hence

$$\begin{aligned}2(\beta_n - \eta_n)h <&\ell((A_n \cap (x-h, x+h)) = \ell(A_n \cap (x-p_n, x+p_n))\\&+ \ell(A_n \cap ((x-h, x-p_n) \cup (x+p_n, x+h))\\<&2(\beta_n + \eta_n)p_n + \ell(A_n \cap ((x-h, x-p_n) \cup (x+p_n, x+h))).\end{aligned}$$

Therefore

$$\begin{aligned}&\ell(A_n \cap ((x-h, x-p_n) \cup (x+p_n, x+h)) > 2(\beta_n - \eta_n)h - 2(\beta_n + \eta_n)p_n\\=&2r(h-p_n) + 2k_n^{-1}(k_n\beta_n h - \beta_n h + rh - k_n p_n - \beta_n p_n + rp_n - rk_n h + rk_n p_n)\\=&2r(h-p_n) + 2k_n^{-1}(\beta_n - r)(k_n h - h - k_n p_n - p_n) > 2r(h-p_n).\end{aligned}$$

Further we see that $\ell(V_{n+1}) = \ell(A_{n+1} \cap ((x-p_n, x-p_{n+1}) \cup (x+p_{n+1}, x+p_n)) > 2(\beta_{n+1} - \eta_{n+1})p_n > 2rp_n$. Therefore we obtain

$$\begin{aligned}\ell(V \cap (x-h, x+h)) &\geq \ell(V \cap ((x-h, x-p_{n+1}) \cup (x+p_{n+1}, x+h))\\&= \ell(V_{n+1}) + \ell(A_n \cap ((x-h, x-p_n) \cup (x+p_n, x+h))\\&> 2rp_n + 2r(h-p_n) = 2rh.\end{aligned}$$

b) Now let $h \leq \frac{k_n+1}{k_n-1}p_n$. We see that

$$\begin{aligned}&\frac{k_n-1}{k_n+1}(\beta_{n+1} - \eta_{n+1}) = \frac{k_n-1}{k_n+1} \cdot \frac{(k_{n+1}-1)\beta_{n+1} + r}{k_{n+1}}\\>&\frac{k_n-1}{k_n+1} \cdot \frac{k_{n+1}-1}{k_{n+1}+1}\beta_{n+1} > \sqrt{\frac{r}{\beta_{n+1}}} \cdot \sqrt{\frac{r}{\beta_{n+1}}} \cdot \beta_{n+1} = r.\end{aligned}$$

This yields

$$\begin{aligned}\ell(V \cap ((x-h, h+h)) &\geq \ell(V_{n+1}) \\ &= \ell(A_{n+1} \cap ((x-p_n, x-p_{n+1}) \cup (x+p_{n+1}, x+p_n)) \\ &> 2(\beta_{n+1}-\eta_{n+1})p_n \geq 2(\beta_{n+1}-\eta_{n+1})\frac{k_n-1}{k_n+1}h > 2rh.\end{aligned}$$

Therefore for each $h \in (0, h_1)$ we have $\ell(V \cap (x-h, x+h)) > 2rh$; i.e., $d_l(V, x) \geq r > s$. Similarly as above we can prove that $f \restriction (V \cup \{x\})$ is continuous at x. $\square$

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let $0 \leq s < r < 1$. Then*

$$\begin{tikzcd} & A_r(f) \arrow[r] & B_r(f) \arrow[r] & A_s(f) \arrow[r] & Q(f) \\ C(f) \arrow[r] & A_r^l(f) \arrow[r] \arrow[u] & B_r^l(f) \arrow[r] \arrow[u] & A_s^l(f) \arrow[u] & \end{tikzcd}$$

and each of inclusions can be proper.

Proof. The inclusions follow from previous remarks and Theorem 1. The following examples show that the inclusions can be proper. $\square$

Proposition 1. *Let $r, s \in [0, 1)$. Then there is a function $f : \mathbb{R} \to [0, 1]$ such that f is continuous at each point different from zero and $0 \in A_r(f) \setminus B_s^l(f)$.*

Proof. Put $a_n = \frac{1}{(4n)!}$, $b_n = \frac{1}{(4n-1)!}$, $c_n = \frac{1}{(4n-2)!}$ and $d_n = \frac{1}{(4n-3)!}$. Then $0 < d_{n+1} < a_n < b_n < c_n < d_n \leq 1$. Put $A = \bigcup\limits_{n=1}^{\infty} ((a_n, b_n) \cup (-b_n, -a_n))$ and $B = \bigcup\limits_{n=1}^{\infty} ((c_n, d_n) \cup (-d_n, -c_n))$. Then A and B are open disjoint sets and $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B = \{0\}$. Hence there is a continuous function $g : \mathbb{R} \setminus \{0\} \to [0, 1]$ such that $g(x) = 0$ for $x \in B$ and $g(x) = 1$ for $x \in A$. Now let $f : \mathbb{R} \to [0, 1]$ be such that $f(x) = g(x)$ for $x \neq 0$ and $f(0) = 0$. We shall show that f is our function.

We have $A \cap (0, b_n) = \bigcup_{i=n}^{\infty} (a_i, b_i)$ and therefore

$$\ell(A \cap (-b_n, 0)) = \ell(A \cap (0, b_n)) \geq \ell((a_n, b_n)) = \frac{1}{(4n-1)!} - \frac{1}{(4n)!} = \frac{4n-1}{(4n)!}$$

and

$$\frac{\ell((A \cap (-b_n, b_n))}{2b_n} \geq \frac{((4n-1)!)(4n-1)}{(4n)!} = \frac{4n-1}{4n}.$$

Since $\lim_{n\to\infty} b_n = 0$, we obtain

$$d_u(A,0) \geq \lim_{n\to\infty} \frac{\ell((A\cap(-b_n,b_n))}{2b_n} \geq \lim_{n\to\infty} \frac{4n-1}{4n} = 1.$$

Similarly we can show that $d_u(B,0)=1$. Evidently, f is continuous at each point different from zero. The set B is open, $d_u(B,0)=1>r$ and $f(x)=0$ for $x\in B\cup\{0\}$, thus $0\in A_r(f)$.

Now let U be an open set such that $d_l((U,0)>s$. Then $d_u(\mathbb{R}\setminus U,0)<1-s\leq 1$. If $A\cap U=\emptyset$ then $d_u(A,0)\leq d_u(\mathbb{R}\setminus U,0)<1$, a contradiction, Therefore $A\cap U\neq\emptyset$ and this yields $0\notin B^l_s(f)$. □

Lemma 2. *Let $0\leq\alpha<1$. Then there are disjoint closed intervals $I^n_i, J^n_i \subset (-1,0)\cup(0,1)$, $i,n\in\mathbb{N}$, such that:*

(i) $d_l(\bigcup_{n=1}^\infty \operatorname{Int} I^n_i, 0) \geq \alpha\cdot 2^{-i}$ *for each* $i\in\mathbb{N}$*,*

(ii) $d_l(\bigcup_{n=1}^\infty \operatorname{Int} J^n_i, 0) \geq (1-\alpha)2^{-i}$ *for each* $i\in\mathbb{N}$*,*

(iii) $\operatorname{Cl}(\bigcup_{n=1}^\infty\bigcup_{i=1}^\infty I^n_i)\cap \operatorname{Cl}(\bigcup_{n=1}^\infty\bigcup_{i=1}^\infty J^n_i)=\{0\}$.

Proof. There are disjoint closed intervals $K^n_i, L^n_i \subset (\frac{1}{n+1},\frac{1}{n})$, $1\leq i\leq n$, such that $\ell(K^n_i)=\frac{\alpha\cdot 2^{-i}}{n(n+1)}$ and $\ell(L^n_i)=\frac{(1-\alpha)\cdot 2^{-i}}{n(n+1)}$ for each i, $1\leq i\leq n$ (For $\alpha=0$ we require that $\ell(K^n_i)>0$ and $\ell(L^n_i)=\frac{2^{-i}}{n(n+1)}$). If $I=[a,b]$ is an interval, denote by $-I$ the interval $[-b,-a]$. Put $I^k_i=(-1)^{k+1}K_i^{i+[(k-1)/2]}$ and $J^k_i=(-1)^{k+1}L_i^{i+[(k-1)/2]}$ for $i,k\in\mathbb{N}$, where $[x]$ is the integer part of x. Then all intervals I^k_i, J^k_i are mutually disjoint. We shall show that they satisfy (i), (ii), and (iii). (i): Let $\varepsilon>0$ and $i\in\mathbb{N}$. Choose $p\in\mathbb{N}$ such that $p\geq i$ and $1/p<\varepsilon$. Let $0<h<1/p$. Then there is $n\in\mathbb{N}$, $n\geq p$, such that $1/(n+1)\leq h<1/n$. We see that

$$\bigcup_{k=1}^\infty I^k_i\cap(0,h)\supset\bigcup_{k=1}^\infty I^k_i\cap(0,1/(n+1))=\bigcup_{k=n+1}^\infty K^k_i$$

and therefore

$$\ell(\bigcup_{k=1}^\infty I^k_i\cap(0,h))\geq\ell(\bigcup_{k=n+1}^\infty K^k_i)=\sum_{k=n+1}^\infty\ell(K^k_i)=\sum_{k=n+1}^\infty\frac{\alpha\cdot 2^{-i}}{k(k+1)}$$
$$=\alpha\cdot 2^{-i}\sum_{k=n+1}^\infty\frac{1}{k(k+1)}=\frac{\alpha\cdot 2^{-i}}{n+1}.$$

Similarly we can show that $\ell(\bigcup_{k=1}^{\infty} I_i^k \cap (-h,0)) \geq \frac{\alpha\cdot 2^{-i}}{n+1}$ and hence $\ell(\bigcup_{k=1}^{\infty} I_i^k \cap (-h,h)) \geq 2\frac{\alpha\cdot 2^{-i}}{n+1}$. Since $0 < h < 1/n$, we obtain

$$\frac{\ell(\bigcup_{k=1}^{\infty} I_i^k \cap (-h,h))}{2h} \geq \frac{\alpha\cdot 2^{-i}\cdot n}{n+1} \geq \alpha\cdot 2^{-i}(1-\frac{1}{p}) > \alpha\cdot 2^{-i}(1-\varepsilon).$$

Thus $d_l(\bigcup_{n=1}^{\infty} I_i^n, 0) \geq \alpha\cdot 2^{-i}(1-\varepsilon)$ for each $\varepsilon > 0$; i.e., $d_l(\bigcup_{n=1}^{\infty} I_i^n, 0) \geq \alpha\cdot 2^{-i}$.

(ii): The proof is similar.

(iii): It follows from the construction. □

Proposition 2. *Let $0 \leq r < s < 1$. Then there is a function $f : \mathbb{R} \to [0,1]$ such that f is continuous at each point different from zero and $0 \in A_r^l(f) \setminus B_s(f)$.*

Proof. Let I_i^n, J_i^n be closed disjoint intervals from Lemma 2 for $\alpha = s$. Put $A = \bigcup_{i,n\in\mathbb{N}} \operatorname{Int} I_i^n$ and $B = \bigcup_{i,n\in\mathbb{N}} \operatorname{Int} J_i^n$. Then $\operatorname{Cl} A \cap \operatorname{Cl} B = \{0\}$ and there is a continuous function $g : \mathbb{R}\setminus\{0\} \to [0,1]$ such that $g(x) = 0$ for $x \in A$ and $g(x) = 1$ for $x \in B$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = g(x)$ for $x \neq 0$ and $f(0) = 0$. Then f is continuous at each point different from zero. The set A is open and $d_l(A,0) \geq \sum_{i=1}^{\infty} d_l(\bigcup_{n=1}^{\infty} \operatorname{Int} I_i^n, 0) \geq \sum_{i=1}^{\infty} s\cdot 2^{-i} = s > r$. Since f is constant on $A\cup\{0\}$, we have $0 \in A_r^l(f)$.

Now, let U be an open set such that $d_u(U,0) > s$. We have $d_l(B,0) \geq \sum_{i=1}^{\infty} d_l(\bigcup_{n=1}^{\infty} \operatorname{Int} J_i^n, 0) \geq \sum_{i=1}^{\infty}(1-s)2^{-i} = 1-s$. If $B\cap U = \emptyset$, then $1-s \leq d_l(B,0) \leq d_l(\mathbb{R}\setminus U, 0) < 1-s$, a contradiction. Therefore $B\cap U \neq \emptyset$ and this yields $0 \notin B_s(f)$.

□

Proposition 3. *Let $r \in [0,1)$. Then there is a function $f : \mathbb{R} \to [0,1]$ such that f is continuous at each point different from zero and $0 \in A_r^l(f) \setminus C(f)$.*

Proof. Let I_i^n, J_i^n be closed disjoint intervals from Lemma 2 for $\alpha = 0$. Put $A = \bigcup_{i,n\in\mathbb{N}} \operatorname{Int} I_i^n$ and $B = \bigcup_{i,n\in\mathbb{N}} \operatorname{Int} J_i^n$. Then there is a function $f : \mathbb{R} \to [0,1]$ such that $f(0) = 0$, $f(x) = 0$ for $x \in B$, $f(x) = 1$ for $x \in A$ and f is continuous at each point different from zero. Then B is an open set and $d_l(B,0) \geq \sum_{i=1}^{\infty} 2^{-i} = 1 > r$. Since f is constant on $B\cup\{0\}$, we have $0 \in A_r^l(f)$. Since $0 \in \operatorname{Cl} A$, we have $0 \notin C(f)$. □

Proposition 4. *There is a function $f : \mathbb{R} \to [0,1]$ such that f is continuous at each point different from zero and $0 \in Q(f) \setminus B_0(f)$.*

PROOF. Let A and B be the same as in Proposition 3. Then there is a function $f : \mathbb{R} \to [0,1]$ such that $f(0) = 0$, $f(x) = 0$ for $x \in A$, $f(x) = 1$ for $x \in B$ and f is continuous at each point different from zero. Since $0 \in \operatorname{Cl} A$, we have $0 \in Q(f)$. Now let U be an open set such that $d_u(U,0) > 0$. If $B \cap U = \emptyset$, then $1 \leq d_l(B,0) \leq d_l(\mathbb{R} \setminus U, 0) < 1$, a contradiction. Therefore $B \cap U \neq \emptyset$ and $0 \notin B_0(f)$. □

Proposition 5. *Let $r \in [0,1)$. Then there is a function $f : \mathbb{R} \to [0,1]$ such that f is continuous at each point different from zero and $0 \in B^l_r(f) \setminus A_r(f)$.*

PROOF. Let I^n_i, J^n_i be closed disjoint intervals from Lemma 2 for $\alpha = r$. Put $A = \bigcup_{i,n\in\mathbb{N}} \operatorname{Int} I^n_i$ and $B = \bigcup_{i,n\in\mathbb{N}} \operatorname{Int} J^n_i$. Then there is a function $f : \mathbb{R} \to [0,1]$ such that $f(0) = 0$, $f(x) = 0$ for $x \in A$, $f(x) = 1/i$ for $x \in J^n_i$ and f is continuous at each point different from zero.

Let $\varepsilon > 0$. Choose $i \in \mathbb{N}$ with $1/i < \varepsilon$. Then $D = A \cup \bigcup_{n=1}^\infty \operatorname{Int} J^n_i$ is an open set and $f(D) \subset (-\varepsilon, \varepsilon)$. Since $A \cap (\bigcup_{n=1}^\infty \operatorname{Int} J^n_i) = \emptyset$, we obtain

$$d_l(D,0) \geq d_l(A,0) + d_l(\bigcup_{n=1}^\infty \operatorname{Int} J^n_i, 0) \geq r + (1-r)2^{-i} > r.$$

Therefore $0 \in B^l_r(f)$.

Now let U be an open set such that $t = d_u(U,0) > r$. Let V be arbitrary open neighborhood of 0. Then $d_u(U \cap V, 0) = t$. Put $q = (t-r)/2$ and let $\eta > 0$ be such that $2\eta < q$. Let $j \in \mathbb{N}$ be such that $2^{-j} < q$ and denote by $C = \bigcup_{i=1}^j \bigcup_{n=1}^\infty \operatorname{Int} J^n_i$. Since I^n_i and J^n_i are disjoint, we obtain

$$d_l(C,0) \geq \sum_{i=1}^j d_l(\bigcup_{n=1}^\infty \operatorname{Int} J^n_i, 0) \geq \sum_{i=1}^j (1-r)2^{-i} = (1-r)(1-2^{-j}).$$

Then also $d_l(C \cap V, 0) \geq (1-r)(1-2^{-j}) > (1-r)(1-2^{-j}) - \eta$. Hence there is a $\delta > 0$ such that for each $h \in (0,\delta)$

$$\frac{\ell((-h,h) \cap V \cap C)}{2h} > (1-r)(1-2^{-j}) - \eta.$$

Since $d_u(U \cap V, 0) > t - \eta$, there is a sequence $(h_m)_m$ converging to zero such that

$$\frac{\ell((-h_m,h_m) \cap V \cap U)}{2h_m} > t - \eta.$$

We can assume that $h_m \in (0, \delta)$. Assume that $U \cap V \cap C = \emptyset$. Then

$$\begin{aligned} 1 = \frac{\ell((-h_m, h_m))}{2h_m} &\geq \frac{\ell((-h_m, h_m) \cap V \cap U)}{2h_m} + \frac{\ell((-h_m, h_m) \cap V \cap C)}{2h_m} \\ &> (1-r)(1-2^{-j}) - \eta + t - \eta > (1-r)(1-q) + t - 2\eta \\ &> 1 - r - q + t - 2\eta = 1 + q - 2\eta. \end{aligned}$$

This yields $q < 2\eta$, a contradiction. Therefore $U \cap V \cap C \neq \emptyset$. This means that each neighborhood V of 0 contains a point $z \in V \cap U$ such that $f(z) \geq 1/j$; i.e., $0 \notin A_r(f)$. $\square$

Corollary 1. *For each $s \in [0, 1)$ we have*

$$A_s(f) = \bigcup_{1>r>s} A_r(f) = \bigcup_{1>r>s} B_r(f),$$

$$A^l_s(f) = \bigcup_{1>r>s} A^l_r(f) = \bigcup_{1>r>s} B^l_r(f)$$

and for each $s \in (0, 1)$ we have

$$B_s(f) \subset \bigcap_{0 \leq r < s} A_r(f) = \bigcap_{0 \leq r < s} B_r(f),$$

$$B^l_s(f) \subset \bigcap_{0 \leq r < s} A^l_r(f) = \bigcap_{0 \leq r < s} B^l_r(f).$$

The inclusion can be proper.

PROOF. Evidently $\bigcup_{1>r>s} A_r(f) \subset \bigcup_{1>r>s} B_r(f)$ and $\bigcap_{0 \leq r < s} A_r(f) \subset \bigcap_{0 \leq r < s} B_r(f)$. From Theorem 1 we obtain $\bigcup_{1>r>s} B_r(f) \subset A_s(f)$ and $B_s(f) \subset \bigcap_{0 \leq r < s} A_r(f)$. If $x \in A_s(f)$, then there is an open set U such that $d_u(U, x) > s$ and $f \restriction (U \cup \{x\})$ is continuous at x. Now there is $r > s$ such that $d_u(U, x) > r$ and hence $x \in A_r(f) \subset \bigcup_{1>r>s} A_r(f)$. Finally, if $x \in \bigcap_{0 \leq r < s} B_r(f)$ and $0 \leq t < s$, then for $r \in (t, s)$ we have $x \in B_r(f)$ and by Theorem 1 $x \in A_t(f)$; i.e., $x \in \bigcap_{0 \leq r < s} A_r$. The function from Proposition 2 is such that $A^l_r(f) = \mathbb{R}$ for each $r \in [0, 1)$; i.e., $\bigcap_{0 \leq r < s} A_r(f) = \mathbb{R}$ but $0 \notin B_s(f)$. $\square$

Lemma 3. *Let $f, g : \mathbb{R} \to \mathbb{R}$, $r \in [0, 1)$ and $c \neq 0$. Then we have*

$C(f) \cap A_r(g) \subset A_r(f + g)$ *and* $A_r(f) = A_r(cf)$,

$C(f) \cap A^l_r(g) \subset A^l_r(f + g)$ *and* $A^l_r(f) = A^l_r(cf)$,

$$C(f) \cap B_r(g) \subset B_r(f+g) \text{ and } B_r(f) = B_r(cf),$$

$$C(f) \cap B_r^l(g) \subset B_r^l(f+g) \text{ and } B_r^l(f) = B_r^l(cf).$$

PROOF. Obvious. □

Remark 1. The set $B_r^l(f) \setminus A_r(f)$ can be dense. Let $r \in [0,1)$ and let $f : \mathbb{R} \to [0,1]$ be the function from Proposition 5 (i.e., $C(f) = \mathbb{R} \setminus \{0\}$ and $0 \in B_r^l(f) \setminus A_r(f)$). Let $D = \{d_1, d_2, \dots, d_n, \dots\}$ be a countable dense set in $\mathbb{R}$. For each $i \in \mathbb{N}$, let $f_i(x) = f(x - d_i)$. Then $C(f_i) = \mathbb{R} \setminus \{d_i\}$ and $d_i \in B_r^l(f_i) \setminus A_r(f_i)$. Put $g = \sum_{n=1}^{\infty} 2^{-n} f_n$. The function $\sum_{n \neq i} 2^{-n} f_n$ is continuous at d_i and hence by Lemma 3 we get $d_i \in B_r^l(g)$. Since $\mathbb{R} \setminus D \subset C(g)$, we get $B_r^l(g) = \mathbb{R}$. Further $d_i \notin A_r(f_i)$ and hence by Lemma 3 $d_i \notin A_r(g)$; i.e., $B_r^l(g) \setminus A_r(g) = D$.

Let us denote by $\mathcal{C}$ and $\mathcal{Q}$ the family of all continuous and quasicontinuous functions, respectively, and define the following classes of functions

Definition 2. Let $r \in [0,1)$. We put

$$\mathcal{A}_r = \{f : \mathbb{R} \to \mathbb{R}; A_r(f) = \mathbb{R}\},$$

$$\mathcal{A}_r^l = \{f : \mathbb{R} \to \mathbb{R}; A_r^l(f) = \mathbb{R}\},$$

$$\mathcal{B}_r = \{f : \mathbb{R} \to \mathbb{R}; B_r(f) = \mathbb{R}\},$$

$$\mathcal{B}_r^l = \{f : \mathbb{R} \to \mathbb{R}; B_r^l(f) = \mathbb{R}\}$$

Theorem 3. *Let $0 \le s < r < 1$. Then*

$$\begin{tikzcd}
 & \mathcal{A}_r \arrow[r] & \mathcal{B}_r \arrow[r] & \mathcal{A}_s \arrow[r] & \mathcal{Q} \\
\mathcal{C} \arrow[r] & \mathcal{A}_r^l \arrow[r] \arrow[u] & \mathcal{B}_r^l \arrow[r] \arrow[u] & \mathcal{A}_s^l \arrow[u] &
\end{tikzcd}$$

and all inclusions are proper.

PROOF. The inclusions follow from Theorem 2. Propositions 1–5 show that the inclusions are proper. □

Corollary 2. *For each $s \in [0,1)$ we have*

$$\mathcal{A}_s \supset \bigcup_{1>r>s} \mathcal{A}_r = \bigcup_{1>r>s} \mathcal{B}_r,$$

$$\mathcal{A}_s^l \supset \bigcup_{1>r>s} \mathcal{A}_r^l = \bigcup_{1>r>s} \mathcal{B}_r^l$$

and for each $s \in (0,1)$ we have

$$\mathcal{B}_s \subset \bigcap_{0\le r<s} \mathcal{A}_r = \bigcap_{0\le r<s} \mathcal{B}_r,$$

$$\mathcal{B}^l_s \subset \bigcap_{0\le r<s} \mathcal{A}^l_r = \bigcap_{0\le r<s} \mathcal{B}^l_r.$$

The inclusions are proper.

Proof. The inclusions follow from Theorem 2. The function f from Proposition 2 belongs to $\bigcap\limits_{0\le r<s} \mathcal{A}^l_r \setminus \mathcal{B}_s$. The rest follows from Theorem 4. □

Let $\rho(f,g) = \min\{1, \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$. We shall show that the inclusions in Theorem 3 and Corollary 2 mean, "is nowhere dense subset of" (with possible exception for $\mathcal{A}_r \subset \mathcal{B}_r$ and $\mathcal{A}^l_r \subset \mathcal{B}^l_r$) in the topology of the uniform convergence.

Proposition 6. *Let $s \in [0,1)$. Then the sets $\mathcal{B}_r$, $\mathcal{B}^l_r$, $\bigcup\limits_{1>r>s} \mathcal{B}_r$ and $\bigcup\limits_{1>r>s} \mathcal{B}^l_r$ are closed in the topology of the uniform convergence.*

Proof. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n \in \bigcup\limits_{1>r>s} \mathcal{B}_r$ and let $(f_n)_n$ uniformly converge to $f : \mathbb{R} \to \mathbb{R}$. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Then there is $n_0 \in \mathbb{N}$ such that $|f_n(y) - f(y)| < \varepsilon/3$ for each $n \ge n_0$ and for each $y \in \mathbb{R}$. Since $f_{n_0} \in \bigcup\limits_{1>r>s} \mathcal{B}_r$, there is $r \in (s,1)$ such that $f_{n_0} \in \mathcal{B}_r$ and there is an open set U such that $d_u(U,x) > r$ and $|f_{n_0}(y) - f_{n_0}(x)| < \varepsilon/3$ for each $y \in U$. Therefore for each $y \in U$ we have $|f(y) - f(x)| \le |f(y) - f_{n_0}(y)| + |f_{n_0}(y) - f_{n_o}(x)| + |f_{n_0}(x) - f(x)| < \varepsilon$; i.e., $f \in \mathcal{B}_r \subset \bigcup\limits_{1>r>s} \mathcal{B}_r$. Similarly we can show other cases. □

The sets $\mathcal{A}_r$ and $\mathcal{A}^l_r$ are not closed.

Proposition 7. *For each $r \in [0,1)$ there is a sequence $(f_n)_n$ of functions belonging to $\mathcal{A}^l_r$ such that its uniform limit does not belong to $\mathcal{A}_r$.*

Proof. Let I^n_i, J^n_i be closed disjoint intervals from Lemma 2 for $\alpha = r$. Define functions $f, f_k : \mathbb{R} \to \mathbb{R}$ $(k \in \mathbb{N})$ by

$$f(x) = \begin{cases} 0 & \text{for } x \in \{0\} \cup \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{n=1}^{\infty} I^n_i, \\ \frac{1}{i} & \text{for } x \in \bigcup\limits_{n=1}^{\infty} J^n_i, \\ \text{linear} & \text{on components of } \mathbb{R} \setminus (\{0\} \cup \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{n=1}^{\infty} I^n_i \cup \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{n=1}^{\infty} J^n_i), \end{cases}$$

$$f_k(x)=\begin{cases} 0 & \text{for } x\in\{0\}\cup\bigcup\limits_{i=1}^{\infty}\bigcup\limits_{n=1}^{\infty} I_i^n\cup\bigcup\limits_{i=k+1}^{\infty}\bigcup\limits_{n=1}^{\infty} J_i^n),\\ \frac{1}{i} & \text{for } x\in\bigcup\limits_{n=1}^{\infty} J_i^n \text{ and } i\le k,\\ \text{linear} & \text{on components of } \mathbb{R}\setminus(\{0\}\cup\bigcup\limits_{i=1}^{\infty}\bigcup\limits_{n=1}^{\infty} I_i^n\cup\bigcup\limits_{i=1}^{\infty}\bigcup\limits_{n=1}^{\infty} J_i^n).\end{cases}$$

We shall show that $f_k\in\mathcal{A}_r^l$ for each $k\in\mathbb{N}$, $(f_k)_k$ uniformly converges to f and $f\notin\mathcal{A}_r$. Let $k\in\mathbb{N}$. Evidently, f_k is continuous at each point different from zero. If $D=\bigcup\limits_{i=1}^{\infty}\bigcup\limits_{n=1}^{\infty}\operatorname{Int} I_i^n\cup\bigcup\limits_{n=1}^{\infty}\operatorname{Int} J_{k+1}^n$, then D is open and

$$d_l(D,0)\ge d_l(\bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\operatorname{Int} I_i^n,0)+d_l(\bigcup_{n=1}^{\infty}\operatorname{Int} J_{k+1}^n)\ge r+(1-r)2^{-k-1}>r.$$

Since f_k is constant on $\{0\}\cup D$, we have $0\in A_r^l(f_k)$ and $f_k\in\mathcal{A}_r^l$. If $x\in\{0\}\cup\bigcup\limits_{i=1}^{\infty}\bigcup\limits_{n=1}^{\infty} I_i^n$, then $f_k(x)=f(x)=0$. If $x\in\bigcup\limits_{n=1}^{\infty} J_i^n$ and $i\le k$, then $f_k(x)=f(x)=1/i$. If $x\in\bigcup\limits_{n=1}^{\infty} J_i^n$ and $j>k$, then $|f_k(x)-f(x)|=1/i<1/k$. Finally, let x belongs to a component of $\mathbb{R}\setminus(\{0\}\cup\bigcup\limits_{i=1}^{\infty}\bigcup\limits_{n=1}^{\infty} I_i^n\cup\bigcup\limits_{i=1}^{\infty}\bigcup\limits_{n=1}^{\infty} J_i^n)$. Therefore $x\in(p,q)$ for some p, q and $x=tp+(1-t)q$ for some $t\in(0,1)$. Then $f(x)=t\cdot f(p)+(1-t)f(q)$ and $f_k(x)=t\cdot f_k(p)+(1-t)f_k(q)$. From the definition of f and f_k, if $z\in\{p,q\}$ and $f(z)=0$ or $f(z)=1/i$ and $i>k$ then $f_k(z)=0$ and if $f(z)=1/i$ and $i\le k$, then $f_k(z)=1/i$. Therefore for $z\in\{p,q\}$ we have $|f_k(z)-f(z)|<1/k$. Then $|f_k(x)-f(x)|\le t|f_k(p)-f(p)|+(1-t)|f_k(q)-f(q)|<t/k+(1-t)/k=1/k$. Therefore $|f_k(x)-f(x)|<1/k$ for each $x\in\mathbb{R}$; i.e., $(f_k)_k$ uniformly converges to f. Since f is the function from Proposition 5, $f\notin\mathcal{A}_r$. □

Problem 1. Characterize uniform limits of $\mathcal{A}_r$ and $\mathcal{A}_r^l$. Is true that each function from $\mathcal{B}_r$ ($\mathcal{B}_r^l$) can be written as the uniform limit of functions from $\mathcal{A}_r$ ($\mathcal{A}_r^l$)? (Z. Grande in [1] has shown that this is true for $\mathcal{B}_0$.)

Theorem 4. *Let $s\in[0,1)$. Then $\bigcup\limits_{1>r>s}\mathcal{B}_r$ is nowhere dense set in $\mathcal{A}_s$ and $\bigcup\limits_{1>r>s}\mathcal{B}_r^l$ is nowhere dense set in $\mathcal{A}_s^l$.*

Proof. According to Proposition 6, the set $\bigcup_{1>r>s} \mathcal{B}_r$ is closed. Therefore it is sufficient to prove that its complement is dense in $\mathcal{A}_s$. Let $f \in \bigcup_{1>r>s} \mathcal{B}_r$ and let $1 > \varepsilon > 0$. Then there is $r \in (s,1)$ such that $f \in \mathcal{B}_r$. Since the set $\mathbb{R} \setminus C(f)$ is of the first category, there is a countable set $H = \{z_1, z_2, \dots, z_n, \dots\} \subset C(f)$ such that $z_{n+1} - z_n > 1$ for each $n \in \mathbb{N}$. According to Proposition 2 for each $t \in (s,1)$ there is $h_t : \mathbb{R} \to [0,1]$ such that $C(h_t) = \mathbb{R} \setminus \{0\}$ and $0 \in A_s(h_t) \setminus B_t(h_t)$. Put $n_0 = \min\{n \in \mathbb{N} : s + 1/n < 1\}$. Now define $h : \mathbb{R} \to [0,1]$ by

$$h(x) = \begin{cases} h_{s+1/n}(x - z_n) & \text{for } x \in [z_n - 1/4, z_n + 1/4] \text{ and } n \geq n_0, \\ h_{s+1/n_0}(x - z_{n_0}) & \text{for } x \leq z_{n_0} - 1/4, \\ \text{linear} & \text{on } [z_n + 1/4, z_n + 3/4] \text{ and } n \geq n_0. \end{cases}$$

Then $\mathbb{R} \setminus \{z_{n_0}, z_{n_0+1}, \dots\} \subset C(h)$ and $z_n \in A_s(h) \setminus B_{s+1/n}(h)$ for each $n \geq n_0$.

Now put $g = f + (\varepsilon/2)h$. Then $\rho(f,g) < \varepsilon$. If $n \geq n_0$, then $z_n \in C(f)$ and $z_n \in A_s(h)$. Hence by Lemma 3 we obtain $z_n \in A_s(g)$. If $x \neq z_n$ $(n \geq n_0)$, then $x \in C(h)$ and $x \in B_r(f)$. Therefore $x \in B_r(g) \subset A_s(g)$. Thus $A_s(g) = \mathbb{R}$ and $g \in \mathcal{A}_s$. Now let $t \in (s,1)$. Then there is $n \geq n_o$ such that $s + 1/n < t$. Then $z_n \notin B_{s+1/n}(h)$, $z_n \in C(f)$ and hence $z_n \notin B_{s+1/n}(g)$ and $z_n \notin B_t(g)$. Therefore $g \notin \mathcal{B}_t$ for each $t \in (s,1)$, i.e. $g \notin \bigcup_{1>r>s} \mathcal{B}_r$. □

Similarly, using Propositions 1, 3 and 4 and Lemma 3 we can show that for each $r \in [0,1)$, $\mathcal{B}_r^l \cap \mathcal{A}_r$ (and thus also $\mathcal{A}_r^l$) is nowhere dense subset of $\mathcal{A}_r$, $\mathcal{B}_r^l$ is nowhere dense subset of $\mathcal{B}_r$, $\mathcal{C}$ is nowhere dense subset of $\mathcal{A}_r^l$ and $\mathcal{B}_0$ is nowhere dense subset of $\mathcal{Q}$. Therefore, $(\mathcal{B}_r)_{r \in [0,1)}$ is the family of closed subsets of $\mathcal{Q}$ such that $\mathcal{B}_r$ is nowhere dense subset of $\mathcal{B}_s$ whenever $0 \leq s < r < 1$.

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