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AN L^p DIFFERENTIABLE NON-DIFFERENTIABLE FUNCTION

Abstract

There is a set E of positive Lebesgue measure and a function nowhere differentiable on E which is differentiable in the L^p sense for every positive p at each point of E . For every $p \in (0, \infty]$ and every positive integer k there is a set $E = E(k, p)$ of positive measure and a function which for every $q < p$ has k L^q Peano derivatives at every point of E despite not having an L^p k th derivative at any point of E .

A real-valued function f of a real variable is *differentiable at x* if there is a real number $f'(x)$ such that

$$|f(x+h) - f(x) - f'(x)h| = o(h) \text{ as } h \rightarrow 0.$$

Fix $p \in (0, \infty)$. A function is *differentiable in the L^p sense at x* if there is a real number $f'_p(x)$ such that

$$\|f(x+h) - f(x) - f'_p(x)h\|_p = o(h) \text{ as } h \rightarrow 0,$$

where $\|g(h)\|_p = \left(\frac{1}{h} \int_{-h}^h |g(t)|^p dt\right)^{1/p}$.

We have an infinite family of generalized first derivatives indexed by the parameter p . Most generalized derivatives are not equivalent to the ordinary derivative at a single point, but many are equivalent on an almost everywhere basis. For example, the symmetric derivative, defined by $f'_s(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}$, is zero for the absolute value function at $x = 0$ even though that function is not differentiable at $x = 0$, but this phenomenon

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which occurs at the single point $x = 0$ never occurs on a set of positive measure; there cannot exist a set of positive measure E and a function g so that $g'_s(x)$ exists at all points of E and $g'(x)$ exists at no points of E [K, page 217]. In this sense the symmetric derivative is equivalent to ordinary differentiation. So a natural question to ask here is whether in this sense the various L^p derivatives are different from ordinary differentiation and from one another. The point of this paper is to answer “yes” to this question.

If $p_1 < p_2$ and f is L^{p_2} differentiable at x , then f is L^{p_1} differentiable at x since by Holder’s inequality,

$$\begin{aligned} \|f(x+h) - f(x) - f'_{p_2}(x)h\|_{p_1} &\leq 2^{\frac{1}{p_1} - \frac{1}{p_2}} \|f(x+h) - f(x) - f'_{p_2}(x)h\|_{p_2} \\ &= o(h), \end{aligned}$$

so that $f'_{p_1}(x)$ exists and equals $f'_{p_2}(x)$. It may be useful to think of a scale of derivatives indexed by p , the higher the value of p , the better the behavior. The best behavior, ordinary differentiability, occurs when $p = \infty$. Sometimes the scale is extended by placing the approximate derivative at $p = 0$.

A function f has a k th Peano derivative at x if there are real numbers $f^i(x)$, $i = 0, 1, 2, \dots, k$, such that

$$\left| f(x+h) - f^0(x) - f^1(x)h - \dots - f^k(x)\frac{h^k}{k!} \right| = o(h^k) \text{ as } h \rightarrow 0.$$

Fix $p \in (0, \infty)$. A function f has a k th Peano derivative in the L^p sense at x if there are real numbers $f_p^i(x)$, $i = 0, 1, 2, \dots, k$, such that

$$\left\| f(x+h) - f_p^0(x) - f_p^1(x)h - \dots - f_p^k(x)\frac{h^k}{k!} \right\|_p = o(h^k) \text{ as } h \rightarrow 0.$$

The same p -scale mentioned for first derivatives also holds for k th Peano ones as well. Whatever the value of k , when $p \neq q$, L^p k th order Peano differentiability is not a.e. equivalent to L^q k th order Peano differentiability; this is the content of Theorems 2 and 3 below.

The first extensive discussion of the L^p Peano derivative that I am aware of appeared in reference [CZ]. Differentiation in the L^p sense for the characteristic function of a set is very closely related to the concept of super density, which is discussed in reference [LMZ].

Theorem 1. *There is a set E of positive Lebesgue measure and a function nowhere differentiable on E which is differentiable in the L^p sense for every positive p at each point of E .*

PROOF. Note that the characteristic function of the rational numbers provides a trivial example since it is nowhere differentiable, but is L^p differentiable to 0 at every irrational point. To avoid such a triviality, we further specify that every element of the equivalence class defining the L^p function should also fail to be differentiable on E ; i.e., changing the function on a set of measure 0 should not improve the differentiability of the function.

Order the rational numbers into a sequence and for $n = 1, 2, \dots$, let G_n be an open interval centered at the n th rational of length 2^{-n^2} . Let C be the complement of $\cup_i G_i$. Since $|\cup_i G_i| \leq \sum 2^{-n^2} < \infty$, $|C| = \infty$. Let χ be the characteristic function of C . Let $I(x, h) = [x - h, x + h]$.

1. χ is not differentiable at almost every point of C . Let $C_1 = \{x \in C : x \text{ is a point of density of } C\}$. Note that $|C \setminus C_1| = 0$. Let $x \in C_1$. If h is sufficiently small, $|I(x, h) \cap C| > h/2$ so the essential lim sup of χ is 1. On the other hand, since for any $h > 0$, the interval $I(x, h)$ contains a rational number and hence a subinterval on which $\chi = 0$, the essential lim inf of χ is 0. Thus χ has no limiting value at x and so all the more is not differentiable there.

2. χ does have a zero L^p derivative for every positive p at almost every point of C_1 . This full measured subset of C_1 will be a set of positive measure and is the set promised in the statement of the theorem. Suppose that for each $p > 0$, χ is L^p differentiable on C^p , a full-measured subset of C_1 . Then letting $A_p = C_1 \setminus C^p$, $|A_p| = 0$. Let $A = \cup A_n$ and $C_2 = C_1 \setminus A$. Then χ is not differentiable on C_2 , but is L^p differentiable on C_2 for every $p > 0$, since by definition χ is $L^{\lceil p \rceil}$ differentiable and Holder's inequality implies L^p differentiability since $p \leq \lceil p \rceil$. Thus it is sufficient to fix p and show that A_p has measure 0.

On C^p we have

$$\left(\frac{1}{h} \int_{-h}^h |\chi(x+t) - \chi(x) - 0 \cdot t|^p dt \right)^{1/p} = o(h),$$

or, equivalently,

$$\int_{-h}^h |\chi(x+t) - \chi(x) - 0 \cdot t|^p dt = o(h^{p+1}), \quad (1)$$

as $h \rightarrow 0$. To show that $|A_p| = 0$, it suffices to show that for each $\epsilon > 0$, $|A_p| < \epsilon$. Fix such an ϵ and pick n so large that $n > p + 1$ and so large that $(n + 1)2^{-n+1} < \epsilon$. Let

$$B_p = \cup_{i=1}^n \{x \in C_1 : \text{dist}(x, G_i) < 2^{-n}\} \cup (\cup_{j>n} \{x \in C_1 : \text{dist}(x, G_j) < 2^{-j}\}).$$

Then $|B_p| \leq (2 \cdot 2^{-n})n + \sum_{j>n} 2 \cdot 2^{-j} = (n + 1)2^{-n+1} < \epsilon$, so it remains to show that (1) holds for $x \in C_1 \setminus B_p$ so that $A_p \subset B_p$. Since $x \in C$, $\chi(x) = 1$ and the absolute value of the left hand side is

$$\ell = \int_{x-h}^{x+h} |\chi(s) - 1|^p ds = |C^c \cap I|,$$

where $I = [x - h, x + h]$. Assume $h < 2^{-n}$. Let G_j be the first complementary interval that meets I . Since $x \notin B_p$, $j > n$. Since $2^{-(i+1)^2} \leq \frac{1}{2}2^{-i^2}$ and $1 + 2^{-1} + 2^{-2} + \dots = 2$,

$$\ell \leq |\cup_{i \geq j} G_i| \leq \sum_{i \geq j} 2^{-i^2} \leq 2 \cdot 2^{-j^2} = 2(2^{-j})^j \leq 2h^j.$$

The last inequality holds because $x \notin B_p$ implies $2^{-j} \leq \text{dist}(x, G_j)$ and $G_j \cap I(x, h) \neq \emptyset$ implies $\text{dist}(x, G_j) \leq h$. Since $j > n > p + 1$, h^j is $o(h^{p+1})$ and relation (1) follows. \square

This example splits ordinary differentiation from all finite L^p differentiation. Given any $p > 0$, we can also create a function f_p for which there is a set E of positive measure on which f_p is differentiable in the L^q sense for every $q < p$; but f_p is not differentiable at any point of E in the L^p sense. We do this by making a “fat Cantor set” the i th stage complementary open intervals being centered at all $(2j + 1)/2^n$ and having measure $2^{-i(p+1)}$. The details are slightly more complicated. Theorem 3 below does this and a little bit more.

Note that the following theorem in particular separates the k th Peano derivative from all L^p k th Peano derivatives, $0 < p < \infty$.

Theorem 2. *There is a set E of positive Lebesgue measure and a function having no limit at each point of E which has a k th Peano derivative in the L^p sense for every natural number k and every positive p at each point of E .*

PROOF. The function χ and the subset of C of full measure appearing in the proof of the previous theorem are sufficient for this theorem also. In fact, for $x \in C$ set $f_p^0(x) = f(x) = 1$ for $p \in (0, \infty)$; and set $f_p^i(x) = 0$, for $i = 1, 2, \dots$ and $p \in (0, \infty)$. The defining condition for having a k th L^p Peano derivative at such an x is

$$\left(\frac{1}{h} \int_{-h}^h \left| f(x+t) - 1 - 0t - \dots - 0 \frac{t^k}{k!} \right|^p dt \right)^{1/p} = o(h^k)$$

or

$$\int_{-h}^h |f(x+t) - 1|^p dt = o(h^{kp+1}).$$

The reasoning and calculations above remain unchanged, except that n must be chosen larger than $kp + 1$ instead of larger than $p + 1$. \square

Theorem 3. *Let $p > 0$ and k be a positive integer. There is a set E of positive Lebesgue measure and a bounded function nowhere Peano differentiable of order k in the L^p sense on E which is Peano differentiable of order k in the L^q sense for every positive $q < p$ at each point of E .*

PROOF. The example which worked for $p = \infty$ and $k = 1$ (Theorem 1) and also for $p = \infty$ and general k (Theorem 2) is not delicate enough to adapt here. Instead, as an example for finite p we offer the characteristic function of a “fat Cantor set” with the n th stage complementary open intervals being centered at all $(2j + 1)/2^n$ and having measure $c_{kp}2^{-n(kp+1)}$, where $c_{kp} = 2^{kp} - 1$. The details follow.

For $N = 1, 2, 3, \dots$, the complementary intervals of rank N will be the open intervals G_{iN} , $i = 1, 2, \dots, 2^{N-1}$, where the center of G_{iN} is centered at $(2i - 1)/2^N$ and $|G_{iN}| = c_{kp}2^{-N(kp+1)}$. The center to center distance between contiguous intervals of rank N is $2 \cdot \frac{1}{2^N} = 2^{1-N}$. It will be convenient to work on $[0, 1]$ thought of as a torus so that in particular G_{1N} and $G_{(2^N-1)N}$ are contiguous.

Let $C = \left(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} G_{in}\right)^c$, χ = characteristic function of C , $x \in C$, and $h > 0$. Note $|C| = 1 - |C^c|$ and $|C^c| \leq \sum_{n=1}^{\infty} 2^{n-1}c_{kp}2^{-n(kp+1)} = 1/2$, so $|C| > 0$.

Then for any $p > 0$,

$$\begin{aligned} \int_{-h}^h \left| \chi(x+t) - \chi(x) - 0 \cdot t - 0 \frac{t^2}{2} - \dots - 0 \frac{t^k}{k!} \right|^p dt &= \int_{-h}^h |\chi(x+t) - 1|^p dt \\ &= |I \cap C^c|, \end{aligned} \tag{2}$$

where $I = [x - h, x + h]$. Find m so that $2^{-m} \leq h < 2^{-m+1}$. We have for some j , $\frac{j}{2^m} \leq x < \frac{j+1}{2^m}$. The complementary interval G centered at the element of $\left\{ \frac{j}{2^m}, \frac{j+1}{2^m} \right\}$ having even numerator has rank at most $m - 1$ so that the

half of G interior to $\left[\frac{j}{2^m}, \frac{j+1}{2^m} \right]$ has measure at least $\frac{1}{2} \frac{c_{kp}}{2^{(kp+1)(m-1)}}$. Thus

$$|I \cap C^c| \geq \frac{c_{kp}}{2} \left(\frac{1}{2^{m-1}} \right)^{kp+1} \geq \frac{c_{kp}}{2} h^{kp+1}.$$

We show below that when $q < p$, the first k Peano L^q derivatives of χ are 0 at a.e. $x \in C$, so by Holder’s inequality, if the L^p Peano derivatives exist at all, they must be zero. However, combining this inequality with equation (2) shows that

$$\left(\frac{1}{h} \int_{-h}^h \left| \chi(x+t) - \chi(x) - 0 \cdot t - 0 \frac{t^2}{2!} - \dots - 0 \frac{t^k}{k!} \right|^p dt \right)^{\frac{1}{p}} > \left(\frac{c_{kp}}{2} \right)^{\frac{1}{p}} h^k$$

which is not $o(h^k)$ so χ does not have a k th L^p Peano derivative at a.e. $x \in C$.

By the same reasoning as in the L^∞ case above, it is enough to prove that if $q < p$ are fixed, and if $\epsilon > 0$ is fixed, then there is a set $A = A(p, q, \epsilon)$, $A \subset C$ such that $|A| < \epsilon$ and for every $x \in C \setminus A$,

$$|[x - h, x + h] \cap C^c| = o(h^{kq+1}).$$

(In the reduction to the sufficiency of this assertion, one needs to establish this estimate directly for a countable set of q ’s that belong to $(0, p)$ and approach p .)

Pick n such that $\frac{3}{n} < \epsilon$. Then for each positive integer i , let A_i be the points of C which are “close” to the complementary intervals of rank i ; explicitly, for rank i , $i \leq n$ let $A_i = \cup_{k=1}^{2^{i-1}} \left\{ x \in C : \text{dist}(x, G_{ki}) < \frac{1}{n^2} \frac{1}{2^n} \right\}$ and for rank j , $j > n$ let $A_j = \cup_{k=1}^{2^{j-1}} \left\{ x \in C : \text{dist}(x, G_{kj}) < \frac{1}{j^2} \frac{1}{2^j} \right\}$. Let $A = \cup_{i=1}^\infty A_i$. Then

$$\begin{aligned} |A| &\leq \sum_{i=1}^n |A_i| + \sum_{i=n+1}^\infty |A_i| = \frac{2}{n^2} \frac{1}{2^n} \left(\sum_{j=1}^n 2^{j-1} \right) + \sum_{i=n+1}^\infty \frac{2}{i^2} \frac{1}{2^i} 2^{i-1} \\ &= \frac{2}{n^2} \frac{1}{2^n} (2^n - 1) + \sum_{i=n+1}^\infty \frac{1}{i^2} \leq \frac{2}{n^2} + \int_n^\infty x^{-2} dx = \frac{2}{n^2} + \frac{1}{n} < \frac{3}{n} < \epsilon. \end{aligned}$$

Let $x \in C \setminus A$ and fix $h > 0$ so small that $h < \frac{1}{n^2} \frac{1}{2^n}$. Let $I = [x - h, x + h]$. Let G be the first complementary interval intersecting I and let ℓ be the rank of G so that $|G| = \frac{c_{kp}}{2^{(kp+1)\ell}}$. Note that $\ell \geq n + 1$ since h is too small to allow

any G of rank $\leq n$ to intersect I . Since G intersects I ,

$$h > \frac{1}{\ell^2} \frac{1}{2^\ell}. \tag{3}$$

Let $m = \lfloor \log_2 h \rfloor$ so that $2^{-m} \leq h < 2^{-m+1}$,

$$m \lesssim \log(1/h). \tag{4}$$

Let $a(s)$ be the number of elements of rank s that intersect I . Excluding the left-most and right-most elements, $a(s) - 2$ centers of rank s intervals are in I and each of the $a(s) - 3$ distances between these centers is $2^{\frac{1}{2^s}}$, whence $(a(s) - 3) 2^{-s+1} \leq 2h$, so

$$a(s) \leq 3 + 2^s h. \tag{5}$$

Since $h < 2^{1-m}$, it follows that

$$\text{if } s < m, \text{ then } a(s) \leq 4. \tag{6}$$

If $\ell < m$, use inequalities (6) and (5) to obtain

$$\begin{aligned} |I \cap C^c| &\leq \sum_{s=\ell}^{\infty} a(s) c_{kp} 2^{-(kp+1)s} \\ &\leq \sum_{s=\ell}^{m-1} 4 \cdot c_{kp} 2^{-(kp+1)s} + 3 \sum_{s=m}^{\infty} c_{kp} 2^{-(kp+1)s} + h \sum_{s=m}^{\infty} c_{kp} 2^{-kps} \tag{7} \\ &\lesssim 2^{-(kp+1)\ell} + h 2^{-kpm}, \end{aligned}$$

where $A \lesssim B$ means that for some constant $C(k, p)$, $A \leq C(k, p) B$. From this and inequalities (3) and (4) we have

$$\begin{aligned} |I \cap C^c| &\lesssim \ell^{2kp+2} \left(\frac{1}{\ell^2 2^\ell}\right)^{kp+1} + h \left(\frac{1}{2^{m-1}}\right)^{kp} \\ &\leq m^{2kp+2} h^{kp+1} + h^{kp+1} \\ &\lesssim \log^{2kp+2}(1/h) h^{kp+1} = o(h^{kq+1}). \end{aligned}$$

If $\ell \geq m$, the estimate is even simpler; we get

$$|I \cap C^c| \leq \sum_{s=m}^{\infty} a(s) c_{kp} 2^{-(kp+1)s} \lesssim h^{kp+1} = o(h^{kq+1}). \quad \square$$

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