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SEPARATION BY AMBIVALENT SETS

Abstract

A characterization of when two sets in $\mathbb{R}$ can be separated by ambivalent sets is given. Two applications of the characterization are also presented.

A set is said to be ambivalent if it is G_δ and F_σ simultaneously. Ambivalent sets form an algebra of sets [3, p. 65]. The following characterization of separation of sets in $\mathbb{R}$ by ambivalent sets has turned out to be a useful tool in proving various facts about Baire class one functions. It would be of interest to find a proof of the proposition not resting on the use of transfinite induction.

Proprosition 1. *Let A and B be disjoint subsets of* $[0, 1]$. *Then the following statements are equivalent:*

(i) A and B can be separated by ambivalent sets[1].

(ii) A and B can be separated by a Baire class one function[2].

(iii) There is no perfect set K such that both A and B are dense in K.

Proof. (i) $\Rightarrow$ (ii). Let U be an ambivalent set that contains A and that is disjoint from B. Then the characteristic function of the complement of U is of Baire class one and separates A and B.

(ii) $\Rightarrow$ (iii). If (iii) were false, then the function f separating A and B would have no continuity point when restricted to K. This is impossible for f is of Baire class one.

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[1] It means that there are disjoint ambivalent sets U and V such that $A \subset U$ and $B \subset V$.

[2] It means that there is a Baire class one function $f: [0, 1] \to [0, 1]$ such that $f\big|_A \equiv 0$ and $f\big|_B \equiv 1$.

(iii) $\Rightarrow$ (i). Let A and B be disjoint sets that are not simultaneously dense in any perfect set K. Set $F_0 = \overline{A} \cup \overline{B}$ and define a transfinite sequence $(F_\alpha)_{\alpha<\omega_1}$ of subsets of [0, 1] as follows. If an ordinal $\alpha \geq 1$ has a predecessor then we set

$$F_\alpha = \overline{A \cap F_{\alpha-1}} \cap \overline{B \cap F_{\alpha-1}} ,$$

and if $\alpha \geq 1$ is a limit ordinal, then we set

$$F_\alpha = \bigcap_{\gamma<\alpha} F_\gamma .$$

Then $(F_\alpha)_{\alpha<\omega_1}$ is a nonincreasing sequence of closed sets and by [1, Thm 3.10] there is the smallest $\alpha_0 < \omega_1$ such that $F_\alpha = F_{\alpha_0}$ for all $\alpha > \alpha_0$.

Suppose F_{α_0} is nonempty. Then the equality $F_{\alpha_0+1} = F_{\alpha_0}$ implies that

$$F_{\alpha_0} = \overline{A \cap F_{\alpha_0}} \cap \overline{B \cap F_{\alpha_0}} ,$$

and hence both A and B are dense in F_{α_0}. Since the sets A and B are disjoint, F_{α_0} must be perfect which contradicts (iii). Therefore $F_{\alpha_0} = \emptyset$.

Now for $\alpha \leq \alpha_0$ let us define sets

$$U_\alpha = F_{\alpha-1} \setminus \overline{B \cap F_{\alpha-1}} \qquad \text{and} \qquad V_\alpha = \overline{B \cap F_{\alpha-1}} \setminus F_\alpha$$

if α has a predecessor, and define $U_\alpha = V_\alpha = \emptyset$ otherwise.

Observe that for every α such that $1 \leq \alpha \leq \alpha_0$ we get

$$U_\alpha \sqcup F_\alpha \sqcup V_\alpha = \bigcap_{\lambda<\alpha} F_\lambda$$

(here the symbol $\sqcup$ denotes a union of pairwise disjoint sets) and

$$F_0 = \bigsqcup_{\lambda\leq\alpha} U_\lambda \sqcup F_\alpha \sqcup \bigsqcup_{\lambda\leq\alpha} V_\lambda .$$

All sets in the sequences $(U_\alpha)_{\alpha\leq\alpha_0}$ and $(V_\alpha)_{\alpha\leq\alpha_0}$ are F_σ and so are the unions

$$U \overset{\text{df}}{=} \bigsqcup_{\alpha\leq\alpha_0} U_\alpha \qquad \text{and} \qquad V \overset{\text{df}}{=} \bigsqcup_{\alpha\leq\alpha_0} V_\alpha .$$

Furhter we get $A \subset U$ and $B \subset V$ since the inclusions

$$A \setminus F_\alpha \subset \bigsqcup_{\lambda\leq\alpha} U_\lambda \qquad \text{and} \qquad B \setminus F_\alpha \subset \bigsqcup_{\lambda\leq\alpha} V_\lambda$$

hold for all α. Clearly $U \cap V = \emptyset$ and $U \cup V = F_0$. Since F_0 is closed, $U = F_0 \setminus V$ is a G_δ in addition to being F_σ. Thus both U and V are ambivalent sets which completes the proof. □

The following corollary is a special case ($\alpha = 1$) of Sierpiński theorem on separation by ambivalent sets [5].

Corollary 1. *Any two disjoint G_δ sets in $\mathbb{R}$ can be separated by ambivalent sets.*

PROOF. If two disjoint G_δ sets were both dense in the same perfect set K, then K would be a union of two disjoint residual sets which is impossible. □

In [2] S. Kempisty gave a proof of an approximation theorem on Baire class one functions (see also [3, Proposition 3.37 and the following Remark there]). At the end of his note Kempisty refined the result by proving that given an $\epsilon > 0$, for every function f of Baire class one there is a function of Baire class one that differs from f by less than 2ϵ and that takes values only in the set of integer multiples of ϵ. Actually, Kempisty claimed that the function g differs from f by less than ϵ, but the claim is not supported by his proof. However, a simple application of the above separation property yields a proof of the original statement of refined approximation theorem.

Proprosition 2. *Let $f : [0, 1] \to \mathbb{R}$ be a Baire class one function. Given $\epsilon > 0$, there is a Baire class one function g such that $|f(x) - g(x)| < \epsilon$ on $[0, 1]$ and values of g are integer multiples of ϵ .*

PROOF. Given an integer i, let $h_i : \mathbb{R} \to [0, 1]$ be a continuous function defined by
$$h_i(x) \; = \; \min\left\{1, \max\left\{0, \frac{x - i\epsilon}{\epsilon}\right\}\right\} .$$
Then $h_i \circ f$ is a Baire class one function that separates sets $\{x : \; f(x) \le i\epsilon\}$ and $\{x : \; f(x) \ge (i + 1)\epsilon \}$. Hence by Proposition 1 for every integer i there is an ambivalent set A_i such that
$$\{x : \; f(x) \le i\epsilon\} \; \subset \; A_i \; \subset \; \{x : \; f(x) < (i + 1)\epsilon \} .$$

Setting $B_i = A_i \setminus A_{i-1}$ for $i \in \mathbb{Z}$, we get a partition of $[0, 1]$ into disjoint ambivalent sets and hence the function $g = \sum_{i\in\mathbb{Z}} i\epsilon\chi_{B_i}$ is the required Baire class one function. □

The second application of our separation property consists of a short proof of a characterization of Baire class one functions found by D. Preiss [4]. Incidentally, the new proof yields easily a slightly strengthened condition (see (iii) below).

Proprosition 3 ([4]). *Let $f : [a, b] \to \bar{\mathbb{R}}$. The following assertions are equivalent:*

(i) *f is of Baire class one.*

(ii) *For each closed subset P of $[a, b]$ and for any real numbers $\alpha < \beta$ at most one of the sets $\{ x \in P : f(x) \geq \beta \}$ and $\{ x \in P : f(x) \leq \alpha \}$ is dense in P.*

(iii) *For each closed subset P of $[a, b]$ and for any rational numbers $\alpha < \beta$ at most one of the sets $\{ x \in P : f(x) \geq \beta \}$ and $\{ x \in P : f(x) \leq \alpha \}$ is dense in P.*

PROOF. (i) $\Rightarrow$ (ii). Since the sets $\{ x : f(x) \geq \beta \}$ and $\{ x : \underline{f}(x) \leq \alpha \}$ are disjoint, it suffices to prove (ii) for perfect sets only. Let $h : \overline{\mathbb{R}} \to \mathbb{R}$ be a continuous function such that $h(y) = 0$ for $y \leq \alpha$ and $h(y) = 1$ for $y \geq \beta$. Then $h \circ f$ is a Baire class one function that separates the sets $\{ x : f(x) \geq \beta \}$ and $\{ x : f(x) \leq \alpha \}$. Hence by Proposition 1 (ii) holds.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (i). Given rationals $\alpha < \beta$, there is by Proposition 1 an ambivalent set $A_{\alpha, \beta}$ such that

$$\{ x : f(x) \leq \alpha \} \subset A_{\alpha, \beta} \subset \{ x : f(x) < \beta \} .$$

Thus, given $a \in \mathbb{R}$, we get

$$\{ x : f(x) < a \} = \bigcup_{\substack{\alpha < \beta < a \\ \alpha, \beta \in \mathbb{Q}}} A_{\alpha, \beta}$$

and

$$\{ x : f(x) > a \} = \bigcup_{\substack{a < \alpha < \beta \\ \alpha, \beta \in \mathbb{Q}}} CA_{\alpha, \beta}$$

(where the symbol CE denotes the complement of a set E), and since both unions are taken over countable families of indices, the sets $\{ x : f(x) < a \}$ and $\{ x : f(x) > a \}$ are F_σ which completes the proof that f is Baire class one. $\square$

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