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CONVERGENCE OF A SERIES WHOSE TERMS ARE ITERATES OF QUADRATIC MAPS †

Abstract

The functional equation $k(p(x)) + k(x) = x,\ p(x) = x^2 + c$, was used to find quadratic invariant curves of a planar mapping. The continuity of its solutions k on an interval is tied to its series representation through $\sum_{i=0}^{\infty}(p^{(2i)}(x) - p^{(2i+1)}(x))$, where the terms contain iterates of p. The intervals of convergence of the series deserve much attention. Because of the presence of iteration, such maximal intervals are sometimes difficult to determine. In this paper we show how numerical computations using Maple V5.1 and the use of discriminants and resultants may assist such development.

1 Introduction.

The planar mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$,

$$G(x, y) = (y, 2y - x - \frac{1}{2}(g(y) + g(x))), \tag{1.1}$$

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is induced from a differential equation with piecewise constant arguments ([6, 7]). The finding of quadratic invariant curves $\Gamma : y = f(x) = \alpha x^2 + \beta x + \gamma$ of G led us to the functional equation

$$k(p(x)) + k(x) = x, \quad p(x) = x^2 + c. \tag{1.2}$$

In [6] and [7] solutions of (1.2) and their continuity are discussed. Its continuity is tied to the convergence of the series

$$S(x) := \sum_{i=0}^{\infty} (p^{(2i)}(x) - p^{(2i+1)}(x)), \tag{1.3}$$

where $p^{(i)}$ denotes the i-th iterate of p; i.e., $p^{(k)}(x) = p(p^{(k-1)}(x))$ and $p^{(0)}(x) \equiv x$. The intervals of convergence of the series $S(x)$ deserve much attention. Some interesting work on an equation very close to (1.2) can be found in [5]. Because of the presence of iteration, such maximal intervals are sometimes difficult to determine. This also requires one to determine whether or not the series representation of k on a maximal open interval has a continuous extension to include a boundary point.

The problem of convergence of $S(x)$ is more difficult when the parameter c is in $[-3/4, 0]$. The quadratic map p has two fixed points

$$x_1 = (1 - \sqrt{1-4c})/2, \quad x_2 = (1 + \sqrt{1-4c})/2.$$

When $c \in (-3/4, 0)$ it is established through the ratio test that this series is convergent on the open interval $(-x_2, x_2)$, which contains x_1. Here we ask whether the function $S(x)$ can be continuously extended to the boundary point x_2. We give numerical evidence suggesting that the answer is no. When $c = -3/4$ the convergence cannot be determined by using the ratio test. Its divergence at each point in the interval $[-3/4, x_1) \cup (x_1, 0]$ is nonetheless proven using other means ([7]). Here we are able to give an alternate proof demonstrating the use of discriminants and resultants ([8, 9]) in proving inequalities which infer divergence. Maple V5.1 is used in such computations.

When $c = 0$, $S(x)$ is known to converge on $x \in (-1, 1)$ ([6]). Here we ask whether it has a continuous extension to the boundary point 1. Numerical rendering of the function does not give strong evidence pointing to an answer. We are grateful to Professor W. Rudin who suggested that the Hadamard's gap condition can be checked, leading to a negative answer. This in turn posts challenge of whether or not a different proof can be obtained with numerical methods.

2 The Case $c = -3/4$.

In this case, the following result is given in [7]:

Theorem 1. *When $c = -3/4$ the series $S(x)$ diverges for all $x \in [-3/4, -1/2) \cup (-1/2, 0]$.*

It shows that no solution of (1.2) is continuous at $x = -1/2$. Now we want to give a different proof demonstrating the use of discriminant and resultant calculations under Maple V5.1.

By taking $u_i(x) := p^{(2i)}(x) - p^{(2i+1)}(x)$, the series $S(x)$ can be written as

$$S(x) = \sum_{i=0}^{\infty} u_i(x). \tag{2.4}$$

Let us consider the ratio

$$\Delta_n(x) := u_{n+1}(x)/u_n(x). \tag{2.5}$$

For simplicity we let $x^{(n)} := p^{(n)}(x)$. Then,

$$\begin{aligned} u_{n+1}(x) &= p^{(2n+2)}(x) - p^{(2n+3)}(x) = p(x^{(2n+1)}) - p(x^{(2n+2)}) \\ &= (x^{(2n+1)})^2 - (x^{(2n+2)})^2 = (x^{(2n+1)} - x^{(2n+2)})(x^{(2n+1)} + x^{(2n+2)}) \\ &= (x^{(2n)} - x^{(2n+1)})(x^{(2n)} + x^{(2n+1)})(x^{(2n+1)} + x^{(2n+2)}) \\ &= u_n(x)(x^{(2n)} + x^{(2n+1)})(x^{(2n+1)} + x^{(2n+2)}). \end{aligned}$$

Thus for $x \in [-3/4, -1/2) \cup (-1/2, 0]$, noting $u_n(x) \neq 0$,

$$\Delta_n(x) = \frac{u_{n+1}(x)}{u_n(x)} = (x^{(2n)} + x^{(2n+1)})(x^{(2n+1)} + x^{(2n+2)}). \tag{2.6}$$

It implies that $\Delta_{n+1}(x) = \Delta_n(p^{(2)}(x))$ and by induction that

$$\Delta_n(x) = \Delta_0(p^{(2n)}(x)), \tag{2.7}$$

where

$$\Delta_0(x) = (x + p(x))(p(x) + p^{(2)}(x))$$

has meaning extended to include $x = -1/2$. The convergence of (2.4) cannot be determined by the ratio-test since $x_1 = -1/2$ is an attractive fixed point of p; i.e., $p(x_1) = x_1$ and $p^n(x) \to x_1$ as $n \to +\infty$ for each $x \in [-3/4, -1/2) \cup (-1/2, 0]$, and $\lim_{n \to +\infty} \Delta_n(x) = \Delta_0(-1/2) = 1$. The divergence of (2.4) was given in [7] not by using $\Delta_n(x)$. The following lemma and tighter controls over the ratios $\Delta_n(x)$ lead to a new proof of Theorem 1.

Lemma 1. *Let $w_n > 0$, $n = 1, 2, \ldots$, and w_n tends to 0 decreasingly. If*

$$\frac{n}{n+1} \leq \frac{w_{n+1}}{w_n} \leq 1, \tag{2.8}$$

then the series $\sum_{i=0}^{\infty} w_i$ diverges.

PROOF. Under condition (2.8), $w_{n+1} \geq \frac{n}{n+1} w_n$. By induction we have $w_{n+j} \geq \frac{n}{n+j} w_n$. Thus

$$\begin{aligned}\sum_{i=n}^{n+j} w_i &= w_n + w_{n+1} + ... + w_{n+j} \\ &\geq (1 + \frac{n}{n+1} + ... + \frac{n}{n+j}) w_n \\ &\geq \frac{jn}{n+j} w_n.\end{aligned}$$

In particular, $\sum_{i=n}^{2n} w_i \geq \frac{n^2}{2n} w_n = \frac{n}{2} w_n$. From (2.8), $nw_n \geq (n-1)w_{n-1}$. It follows by induction that $nw_n \geq w_1$, where $w_1 > 0$ is definite. Therefore $\sum_{i=n}^{2n} w_i \geq \frac{1}{2} w_1 > 0$ and the sequence $h_n := \sum_{i=0}^{n} w_i$ cannot be a Cauchy sequence. The divergence is implied by Cauchy's criterion of convergence. This completes the proof. □

3 Basic Properties of $\Delta_0(x)$.

Lemma 1 gives the divergence of $S(x)$ as long as we can show that the ratios $\Delta_n(x)$ satisfies (2.8); i.e., $\frac{n}{n+1} \leq \Delta_n(x) \leq 1$ for all $x \in [-3/4, 0] \backslash \{x_1\}$. For this reason we shall examine the properties of $\Delta_0(x)$ in more detail. Using Maple V5.1 we get

$$\Delta_0(x) = \frac{1}{64}(2x-1)(2x+3)(4x^2-5)(4x^2+3), \tag{3.9}$$

which has no zeros on the interval $[-3/4, 0]$.

Lemma 2. *On* $[-3/4, 0]$ *the function* $\Delta_0(x)$ *is positive and has a unique maximal value* 1 *at* $x_1 = -1/2$. *It is strictly increasing on* $[-3/4, -1/2)$ *and strictly decreasing on* $(-1/2, 0]$. *We also have* $\Delta_0(0) < \Delta_0(-3/4)$.

PROOF. Using Maple V5.1 we get the derivative

$$\Delta_0'(x) = \frac{1}{2}(x + \frac{1}{2})(48x^4 + 16x^3 - 48x^2 + 12x - 15). \tag{3.10}$$

Observe the factor

$$\begin{aligned}\beta(x) &:= 48x^4 + 16x^3 - 48x^2 + 12x - 15 \\ &= x\{x[x(48x+16) - 48] + 12\} - 15 < 0 \end{aligned} \tag{3.11}$$

for $x \in [-3/4, 0]$. In fact, $x[x(48x+16) - 48] \geq 0$ and $x\{x[x(48x+16) - 48] + 12\} \leq 0$ since $-20 \leq 48x + 16 \leq 16$ and $-60 \leq x(48x+16) - 48 \leq -33$. This implies the monotonicity of $\Delta_0(x)$ on $[-3/4,-1/2)$ and $(-1/2,0]$. Furthermore, it is easy to calculate that $\Delta_0(-1/2)=1$, $\Delta_0'(-1/2)=0$, $\Delta_0''(-1/2)=-4<0$. Direct calculation yields $\Delta_0(0)=\frac{45}{64}\approx 0.703$ and $\Delta_0(-3/4)=\frac{3465}{4096}\approx 0.8459$, confirming $\Delta_0(0)<\Delta_0(-3/4)$. This completes the proof. $\square$

Next, we consider the function

$$\gamma(y) := \Delta_0\left(-\frac{3}{16}-\frac{3}{2}y^2+y^4\right). \tag{3.12}$$

Lemma 3. *For $n\geq 1$ in $\mathbb{Z}_+$ and $y<0$, $\Delta_0(y)=\frac{n}{n+1}$ implies $\gamma(y)>\frac{n+1}{n+2}$.*

PROOF. Let

$$\delta(\kappa,y) := \Delta_0(y)-\frac{\kappa}{\kappa+1}, \qquad \eta(\kappa,y) := \gamma(y)-\frac{\kappa+1}{\kappa+2},$$

where $\kappa\geq 1$ is a real constant. Note that $\delta(\kappa,y)$ is a polynomial in y of degree 6. Let $y_j(\kappa)$, $j=1,2,...,6$, denote the six branches of the algebraic function (p.300, [1]) of κ defined by $\delta(\kappa,y)=0$. They are clearly continuous in κ. We first claim that for each j the continuous function $\eta(\kappa,y_j(\kappa))$ has no zero in $[1,8]\cup[9,+\infty)$.

By the Product Formula in Chapter 12 of [3], the discriminant $\mathrm{Dis}(\delta(\kappa,y))$ satisfies

$$\begin{aligned}\mathrm{Dis}(\delta(\kappa,y)) &:= (-1)^{n(n-1)/2}a_0^{2n-2}\prod_{1\leq i<j\leq 6}(y_i(\kappa)-y_j(\kappa))^2\\ &= \frac{1}{a_0}\mathrm{res}\left(\delta(\kappa,y),\frac{d}{dy}\delta(\kappa,y),y\right)\\ &= \frac{-64(344\kappa^4+47\kappa^3+1080\kappa^2+783\kappa+135)}{(\kappa+1)^5},\end{aligned} \tag{3.13}$$

where $a_0=1$ is the leading coefficient of $\delta(\kappa,y)$ in y and $\mathrm{res}(f,g,y)$ denotes the *resultant* of f and g in y. From (3.13), the discriminant $\mathrm{Dis}(\delta(\kappa,y))$ has no zero when $\kappa\geq 1$, so the number of real branches defined by $\delta(\kappa,y)=0$ is a constant integer and is independent of $\kappa\geq 1$. On the other hand, when $\kappa=1$,

$$\delta(1,y)=y^6+y^5-\frac{5}{4}y^4-\frac{1}{2}y^3-\frac{9}{16}y^2-\frac{15}{16}y+\frac{13}{64}. \tag{3.14}$$

The Maple command *readlib(realroot)*, designed by Descartes' rule of signs as explained in [2], returns a list of isolating intervals for all real roots of a univariate polynomial with integer coefficients, although multiplicity information is not included. It is obviously applicable to $\delta(1,y)$ with rational coefficients. Using this command we get

$$\text{realroot}(\delta(1,y),1/16)=[[\frac{3}{16},\frac{1}{4}],[\frac{9}{8},\frac{19}{16}],[\frac{-15}{16},\frac{-7}{8}],[\frac{-13}{8},\frac{-25}{16}]], \tag{3.15}$$

implying that there are exactly four intervals of diameter 1/16 each covering only one real root of $\delta(1,y)$. This indicates that there are exactly four real branches, denoted by $y_j(\kappa), j=1,2,3,4$. For our purpose it is adequate to consider only the real branches. Furthermore, using Maple V5.1 we get

$$\prod_{j=1}^{6}\eta(\kappa,y_j(\kappa))=\text{res}(\delta(\kappa,y),\eta(\kappa,y),y)=\frac{F(\kappa)}{(\kappa+2)^6(\kappa+1)^{24}}, \tag{3.16}$$

where

$$\begin{aligned} F(\kappa):=&-1024\kappa^{26}-16896\kappa^{25}-92864\kappa^{24}+178944\kappa^{23}+6211776\kappa^{22}\\ &+49682112\kappa^{21}+250195264\kappa^{20}+923952256\kappa^{19}\\ &+2654250048\kappa^{18}+6107481792\kappa^{17}+11431687088\kappa^{16}\\ &+17534730304\kappa^{15}+22084987424\kappa^{14}+22797180128\kappa^{13}\\ &+19202459056\kappa^{12}+13145665248\kappa^{11}+7343342128\kappa^{10}\\ &+3454612320\kappa^{9}+1496148668\kappa^{8}+670443280\kappa^{7}\\ &+308826380\kappa^{6}+126545612\kappa^{5}+40393016\kappa^{4}+9231296\kappa^{3}\\ &+1399372\kappa^{2}+124848\kappa^{1}+4913. \end{aligned}$$

Maple V5.1 also helps us to know that on $(0,\infty)$, $F(\kappa)$ has a unique zero which lies in the open interval $(8,9)$. Therefore, every $\eta(\kappa,y_j(\kappa))$, $j=1,2,3,4$, has no zeros in $[1,8]\cup[9,+\infty)$. This proves what we claimed.

In order to determine whether each $\eta(\kappa,y_j(\kappa))$, $j=1,2,3,4$, is positive, we consider zeros $y_j(1)$, $j=1,2,3,4$, of $\delta(1,y)$ again. Choosing diameter $1/2^{10}$, much smaller than that in (3.15), we get

$$\begin{gathered}\text{realroot}(\delta(1,y),1/2^{10})=\\ [[\frac{97}{512},\frac{195}{1024}],[\frac{601}{512},\frac{1203}{1024}],[\frac{-239}{256},\frac{-955}{1024}],[\frac{-801}{512},\frac{-1601}{1024}]];\end{gathered} \tag{3.17}$$

i.e., four intervals are obtained and each of them covers exactly one of the real roots $y_j(1)$, $j=1,2,3,4$. Similarly, for the function $\eta(1,y)$ we obtain that

$$\begin{gathered}\text{realroot}(\eta(1,y),1/2^{10})=\\ [[\frac{655}{512},\frac{1311}{1024}],[\frac{749}{512},\frac{1499}{1024}],[\frac{-1311}{1024},\frac{-655}{512}],[\frac{-1499}{1024},\frac{-749}{512}]];\end{gathered} \tag{3.18}$$

i.e., another four intervals of diameter $1/2^{10}$ are obtained and all real roots of $\eta(1,y)$ are covered by them. The eight intervals given in (3.17) and (3.18) do not intersect each other. So $\delta(1,y)$ and $\eta(1,y)$ have no common real zeros. Evaluating $\eta(1,y)$ at an end-point of each interval given in (3.17), for example,

$$\eta(1,97/512) = \frac{7306101197017380431595603177628561664374454063731382253441440435 5}{3159368750056715600937540830510112969566852862016473337629329326 08} > 0,$$

we assert that $\eta(1,y) > 0$ for y in $[97/512, 195/1024]$, the first interval of (3.17), and similarly $\eta(1,y) > 0$ for y in the other three intervals. Hence all $\eta(1, y_j(1)) > 0,\ j = 1,2,3,4$. This further implies that

$$\eta(\kappa, y_j(\kappa)) > 0, \quad j = 1,2,3,4, \quad \forall \kappa \in [1,8] \tag{3.19}$$

since the sign of each $\eta(\kappa, y_j(\kappa))$, a continuous function with no zeros in $[1,8]$ as claimed in the beginning of the proof, cannot change for all $\kappa \in [1,8]$.

For $\kappa \geq 9$, computation gives

$$\delta(9,y) = y^6 + y^5 - \frac{5}{4}y^4 - \frac{1}{2}y^3 - \frac{9}{16}y^2 - \frac{15}{16}y - \frac{63}{320}. \tag{3.20}$$

Using the same method as for $\delta(1,y)$ in (3.15), we know that the polynomial $\delta(9,y)$ has exactly four real zeros, three of which are negative and one is positive. Since $y < 0$ is required, we need only to consider the negative three, denoted by $y_j(9),\ j = 1,2,3$. It is also easy to check that $\eta(9, y_j(9)) > 0,\ j = 1,2,3$, as done above for $\kappa = 1$. Note that $y_j(\kappa) < 0,\ j = 1,2,3,\ y_4(\kappa) > 0,\ \forall \kappa \geq 9$, because

$$\delta(\kappa, 0) = \frac{45}{64} - \frac{\kappa}{\kappa+1} > 0, \quad \forall \kappa > \frac{45}{19}.$$

Thus

$$\eta(\kappa, y_j(\kappa)) > 0, j = 1,2,3, \quad \forall \kappa \in [9, +\infty) \tag{3.21}$$

since the sign of each $\eta(\kappa, y_j(\kappa))$ cannot change for $\kappa \geq 9$.

Summarizing (3.19) and (3.21) we see that $\eta(n, y(n)) > 0$ for any integer $n \geq 1$ if $y(n)$ is a negative branch of the algebraic function of n defined by $\delta(n,y) = 0$. This completes the proof. $\square$

4 Proof of Divergence.

We are now ready to give a proof for Theorem 1 using the ratios $\Delta_n(x)$ and the comparison test. As explained in the beginning of section 3, it suffices to prove that

$$\frac{n}{n+1} \le \Delta_n(x) \le 1 \tag{4.22}$$

for all $x \in [-3/4, 0]\backslash\{x_1\}$ and integers $n > 0$. By Lemma 2, $\Delta_n(x) = \Delta_0(p^{(2n)}(x)) \le 1$ for all $x \in [-3/4, 0]\backslash\{x_1\}$ since $p^{(2n)}(x) \in [-3/4, 0]$ for all $x \in [-3/4, 0]$. Hence the inequality on the right hand side of (4.22) holds.

In order to prove the inequality on the left hand side of (4.22), observe that $\Delta_n(x) = \Delta_0(p^{(2n)}(x)) \ge \Delta_n(0)$ for $x \in (x_1, 0]$ and $\Delta_n(x) = \Delta_0(p^{(2n)}(x)) \ge \Delta_n(-3/4)$ for $x \in [-3/4, x_1)$ since $x_1 < p^{(2n)}(x) \le p^{(2n)}(0)$ and $p^{(2n)}(-3/4) \le p^{(2n)}(x) < x_1$ respectively for x in the two intervals. Thus, proving the inequality on the left-hand side of (4.22) is reduced to the following lemma.

Lemma 4. *$\Delta_n(0) \ge \frac{n}{n+1}$ and $\Delta_n(-\frac{3}{4}) \ge \frac{n}{n+1}$ for all integers $n > 0$.*

Proof. Clearly $\Delta_1(0) \ge \frac{1}{2}$, i.e., the first inequality in Lemma 4 holds for $n = 1$. Assume that the first inequality holds for the natural number n. Then $\Delta_0(0^{(2n)}) = \Delta_n(0) \ge \frac{n}{n+1}$ by (2.7). Noticing that Δ_0 is strictly decreasing on $(-1/2, 0]$, shown in Lemma 2, and that $0^{(2n)} \in (-1/2, 0]$, we get that the inverse Δ_0^{-1} of Δ_0 exists on the interval $[45/64, 1)$ with the range $(-1/2, 0]$ and that $0^{(2n)} \le \Delta_0^{-1}(\frac{n}{n+1})$. Since $p^{(2)}(x) = -\frac{3}{16} - \frac{3}{2}x^2 + x^4$ is strictly increasing on $(-1/2, 0]$, we have

$$\begin{aligned} 0^{(2n+2)} &= p^{(2)}(0^{(2n)}) \le p^{(2)}(\Delta_0^{-1}(\frac{n}{n+1})) \\ &= -\frac{3}{16} - \frac{3}{2}(\Delta_0^{-1}(\frac{n}{n+1}))^2 + (\Delta_0^{-1}(\frac{n}{n+1}))^4. \end{aligned}$$

By (2.7) we get $\Delta_{n+1}(0) = \Delta_0(0^{(2n+2)}) \ge \gamma(\Delta_0^{-1}(\frac{n}{n+1})) \ge \frac{n+1}{n+2}$, where $\gamma(y)$ is defined in (3.12)and Lemma 3 is applied. This proves the first inequality in Lemma 4 by induction.

The proof to the second inequality in Lemma 4 is similar. It is also easy to verify that it holds at $n = 1$. Assume that it holds for the natural number n. By (2.7), $\Delta_0(p^{(2n)}(-3/4)) = \Delta_n(-3/4) \ge \frac{n}{n+1}$. Noticing that Δ_0 is strictly increasing on $[-3/4, -1/2)$, as shown in Lemma 2, and that $p^{(2n)}(-\frac{3}{4}) \in [-3/4, -1/2)$, we get that the inverse Δ_0^{-1} also exists on the interval $[3465/4096, 1)$ with the range $[-3/4, -1/2)$ and that $p^{(2n)}(-3/4) \ge \Delta_0^{-1}(\frac{n}{n+1})$. Since $p^{(2)}(x) = -\frac{3}{16} - \frac{3}{2}x^2 + x^4$ is also strictly increasing on

$[-3/4, -1/2)$, we get

$$p^{(2n+2)}(-\frac{3}{4}) = p^{(2)}(p^{(2n)}(-\frac{3}{4})) \geq p^{(2)}(\Delta_0^{-1}(\frac{n}{n+1}))$$
$$= -\frac{3}{16} - \frac{3}{2}(\Delta_0^{-1}(\frac{n}{n+1}))^2 + (\Delta_0^{-1}(\frac{n}{n+1}))^4.$$

By (2.7) and Lemma 3, $\Delta_{n+1}(-\frac{3}{4}) = \Delta_0(p^{(2n+2)}(-\frac{3}{4})) \geq \gamma(\Delta_0^{-1}(\frac{n}{n+1})) \geq \frac{n+1}{n+2}$. This proves the second inequality in Lemma 4 by induction and the proof of Lemma 4 is completed. □

The result of Lemma 4 completes the proof of Theorem 1.

5 Case $c = 0$ and Case $-3/4 < c < 0$.

For $c = 0$, the map p has two fixed points at $x_1 = 0$ and $x_2 = 1$. It was shown in [6] that equation (1.2) on the interval $[0, 1)$ has a unique continuous solution given by

$$k(x) = \sum_{i=0}^{\infty}(p^{(2i)}(x) - p^{(2i+1)}(x)) = \sum_{i=0}^{\infty}(-1)^i x^{2^i}. \tag{5.23}$$

Its extendability to a continuous solution k on $[0, 1]$ is equivalent to

$$\lim_{x\to 1-} S(x) = 1/2.$$

Numerical experiments with Maple V5.1 to compute the sum

$$S(x, n) = \sum_{i=0}^{n}(-1)^i x^{2^i}$$

of the first n terms give many crude data, for example,

$$S(0.999, 12) = .500399872685635403709706656021,$$
$$S(0.999^2, 12) = .498875854474142353725438028783$$

in accuracy of 30 digits and

$$S(0.999, 12) = .5003998726856354037097066560212515168593,$$
$$S(0.999^2, 12) = .4988758544741423537254380287830162041506$$

in accuracy of 40 digits, which do not allow us to dismiss this possibility. Through personal communications, Professor Walter Rudin offered a negative answer to the existence of $\lim_{x\to 1-} S(x)$. Thus we conclude that when $c = 0$ equation (1.2) does not have a continuous solution on $[0, 1]$.

Theorem 2. *The one-sided limit* $\lim_{x\to 1-}\sum_{i=0}^{\infty}(-1)^i x^{2^i}$ *does not exist.*

PROOF. We first state the "high indices theorem" reported in [4]: Suppose $\{\lambda_n\}$ is an increasing sequence of positive numbers which satisfies Hadamard's gap condition

$$\liminf_{n\to\infty}(\lambda_{n+1}/\lambda_n) > 1, \tag{5.24}$$

and let a_n be real numbers such that the series

$$\xi(x) = \sum_{n=0}^{\infty} a_n x^{\lambda_n} \tag{5.25}$$

converges for $0 \le x < 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if $\xi(x)$ tends to a finite limit as $x \to 1$ from the left.

Clearly, $\{2^i : i = 0, 1, 2, ...\}$ satisfies (5.24) and $\sum_{i=0}^{\infty}(-1)^i x^{2^i}$ converges for $x \in [0,1)$. Since $\sum_{i=0}^{\infty}(-1)^i$ diverges, $\lim_{x\to 1-}\sum_{i=0}^{\infty}(-1)^i x^{2^i}$ does not exist. This completes the proof. □

We may ask whether numerical computations can provide such a conclusion —that the one-sided limit does not exist. If so, can it reveal the values of $\limsup_{x\to 1-}\sum_{i=0}^{\infty}(-1)^i x^{2^i}$ and $\liminf_{x\to 1-}\sum_{i=0}^{\infty}(-1)^i x^{2^i}$?

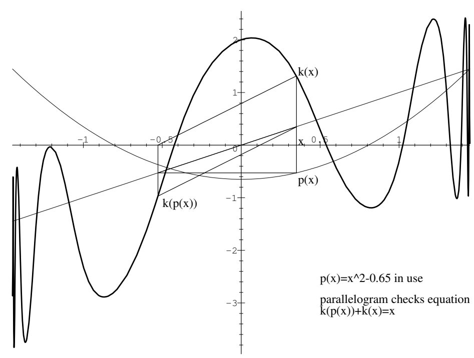
When $c = 0$ equation (1.2) is a special case of a more general equation studied in [5]. There, readers may find many interesting results on the solutions and their representations by series.

When $-3/4 < c < 0$, it is shown in Theorem 4.2 of [7] that equation (1.2) on $(-x_2, x_2)$ has a unique continuous solution k given by

$$k(x) = \frac{x_1}{2} + \sum_{i=0}^{\infty}(p^{(2i)}(x) - p^{(2i+1)}(x)). \tag{5.26}$$

Similar to the case of $c = 0$, whether it can be extended continuously to the boundary x_2 is tied to the existence of the limit $\lim_{x\to x_2-} S(x)$. The following graph of k is obtained from using the Maple V5.1 plotting facility. The large oscillation near the end points $-x_2$ and x_2 suggests that it has no continuous extension to the closed interval $[-x_2, x_2]$. Is there a simple proof?

Figure3. Graphs of y=x , p (thin) and k (thick)



References

[1] L. V. Ahlfors, *Complex Analysis* (Third Edition), McGraw-Hill, New York, 1979.

[2] B. Buchberger, G. E. Collins and R. Loos, *Computer Algebra—Symbolic and Algebraic Computation*, Second Edition, Springer, New York, 1983.

[3] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhauser, Boston, 1994.

[4] G. H. Hardy and J. E. Littlewood, *A further note on the converse of Abel's theorem*, Proc. London Math. Soc., **25** (1926), 219–236.

[5] M. Kuczma and W. Smajdor, *On the radius of convergence of series solutions of a functional equation*, Ann. Polon. Math., **24** (1971), 233–240.

[6] C. T. Ng and W. Zhang, *Invariant curves for planar mappings*, J. Difference Eqns. Appl., **3** (1997), 147–168.

[7] C. T. Ng and W. Zhang, *Quadratic invariant curves for a planar mapping*, J. Difference Eqns. Appl., **6** (2000), 147–163.

[8] R. Peretz, *Applications of induced resultants to polynomial maps*, Israel J. Math., **109** (1999), 181–187.

[9] I. M. Syam, *The resultant method for approximating real fixed points of polynomials*, Comput. Math. Appl., **41** (2001), 7–8, 879–891.