Real Analysis Exchange
Vol. 30(1), 2004/2005, pp. 207–216

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A NEW NECESSARY CONDITION FOR EXISTENCE OF THE STIELTJES INTEGRAL

Abstract

The paper is devoted to establishing a necessary condition for the existence of the Riemann-Stieltjes integral of continuous functions and simultaneously to finding a limit form of its relationship with the classical Riemann integral depending on a parameter. The mathematical technique for this study is based on changing the weighted first difference of a given function into the simple difference of a function determined by a first-order difference equation.

1 Functional Properties of the Solution to a Difference Equation.

The main mathematical idea we use below is to change the weighted first difference $[f(t+h)-f(t)]\,g(t)$ into the simple first difference of a function F of variable t. Undoubtedly, such a change, aside from applying here, is of certain independent interest. To state the problem correctly, consider two continuous functions $g(t)$ and $f(t)$, respectively, on intervals $[0,a]$ and $D = (-H, a+H)$ for sufficiently small $H > 0$. The desired function F is to be a solution of the difference equation

$$x(t+h) - x(t) = \varphi(t,h) \;\forall\; t \in [0,a] \tag{1.1}$$

with the positive step $h \in (0,H)$ and the right-hand side

$$\varphi(t,h) = [f(t+h)-f(t)]\,g(t) \;\forall\; t \in [0,a]. \tag{1.2}$$

Key Words: Riemann-Stieltjes integral, difference equation, weighted first difference
Mathematical Reviews subject classification: Primary 26A42; Secondary 28A75, 39A10
Received by the editors April 1, 2004
Communicated by: Peter Bullen

Clearly, for the unique solvability of this equation it is necessary to complete the definition of the function $\varphi(t,h)$ on the initial interval $[-h,0)$. Namely, let

$$\varphi(t,h) = \varphi(0,h)\left(1+\frac{t}{h}\right)\ \forall\ t\in[-h,0). \tag{1.3}$$

It follows from definitions (1.2) and (1.3) that $\varphi(t,h)$ is continuous with respect to (t,h) jointly in the trapezium $G = \{(t,h) : h\in(0,H)\,, t\in[-h,a]\}$ and possesses the evident properties:

$$\varphi(-h,h) = 0,\ \lim_{h\to 0^+}\varphi(y(h),h) = 0\ \forall\ y(h)\in[-h,a]\,. \tag{1.4}$$

Unfortunately, the above excellent idea to change the weighted difference into the simple one can not be fully realized. The problem is that the solution of difference equation (1.1) is to be a function not only of the variable t but also of the step h, because the right-hand side of this equation depends on h too. Thus, the sought for solution is a function $F(t,h)$ and the desired change of the weighted difference into the simple one is represented as

$$[f(t+h)-f(t)]\,g(t) = F(t+h,h)-F(t,h)\ \forall\ t\in[0,a]. \tag{1.5}$$

It is seen from here, that the introduced function $F(t,h)$ should be defined in the trapezium $Q = \{(t,h) : h\in(0,H)\,, t\in[0,a+h]\}$. It should be noted that for $h=0$ the solution $F(t,h)$ of difference equation (1.1) is not defined at all, because in this case the equation makes no sense, but one may talk about the limit of $F(t,h)$ as $h\to 0^+$.

On the basis of a trivial modification of the method for solution of first-order difference equations from [1, p. 248] we define the sought for solution $F(t,h)$ on each interval of length h of the t axis by

$$F(t,h) = \sum_{k=-1}^{n-1}\varphi(t-(n-k)h,h)\ \forall\ nh\le t<(n+1)h,\ n=0,1,2,\dots,N,N+1, \tag{1.6}$$

where $N = \left[\frac{a}{h}\right]$ which means the integral part of a number. It is easy to show that the function $F(t,h)$ defined by (1.6) is a solution of difference equation (1.1). Indeed, according to (1.6)

$$\begin{aligned} F(t+h,h) &= \sum_{k=-1}^{n}\varphi(t+h-(n+1-k)h,h) \\ &= \sum_{k=-1}^{n}\varphi(t-(n-k)h,h) = F(t,h)+\varphi(t,h); \end{aligned}$$

i.e., $F(t,h)$ satisfies equation (1.1) for all $t \in [0,a]$.

We now establish continuity of the sought for function $F(t,h)$ with respect to t in the domain Q. On each of intervals $[nh,(n+1)h)$ separately this property is evident because the sum from the right-hand side of equality (1.6) is finite, whereas the function $\varphi(t,h)$ is continuous in t. Thus the t-continuity of the function $F(t,h)$ is in doubt only on the boundaries $t = mh$ $(0 \le m < n)$ of the above intervals. To remove such a suspicion, compare the value of the function $F(t,h)$ at $t = mh$ with its left-hand limit at this point. To that end, using the definition (1.6), the continuity of the function $\varphi(t,h)$ in t, and its boundary value from (1.4), we easily find

$$F\,(mh,h) = \sum_{k=-1}^{m-1} \varphi(mh-(m-k)h,h) = \sum_{k=-1}^{m-1} \varphi(kh,h) = \sum_{k=0}^{m-1} \varphi(kh,h),$$

$$\begin{aligned} F\left(mh^-,h\right) &= \sum_{k=-1}^{m-2} \varphi(mh^- - (m-1-k)h,h) = \sum_{k=-1}^{m-2} \varphi((k+1)h,h) \\ &= \sum_{i=0}^{m-1} \varphi(ih,h). \end{aligned}$$

Hence $F\,(mh,h) = F\,(mh^-,h)$, and therefore the function $F(t,h)$ is continuous in t at any point $t = mh$.

Next we investigate the existence of the double limit of the function $F(t,h)$ in the domain Q at its boundary point $t = a$ and $h = 0$. To this end, we analyze the values $F(a+\tau,h)$ of the function in question for sufficiently small increments τ and $h > 0$ to be admissible in Q. By (1.6)

$$F(a+\tau,h) = \sum_{k=-1}^{m-1} \varphi(a+\tau-(m-k)h,h), \;\; m = \left[\frac{a+\tau}{h}\right]. \tag{1.7}$$

Theorem 1. *If the functions $f(t)$ and $g(t)$ are continuous, respectively, in D and $[0,a]$ and the finite Riemann-Stieltjes integral $\int_0^a g(t)\,df(t)$ exists, then the conditional double limit*

$$\lim_{\substack{h\to 0^+ \\ |\tau|<h}} F(a+\tau,h) = \int_0^a g(t)\,df(t). \tag{1.8}$$

Proof. We transform (1.7) by introducing the number $p = m - N$ and the values $\eta = a - Nh$ and $\mu = \tau + \eta - ph$ satisfying the relations

$$p = \left[\frac{a+\tau}{h}\right] - N = \left[\frac{Nh+\eta+\tau}{h}\right] - N = \left[\frac{\eta+\tau}{h}\right], \quad 0 \le \eta < h, \quad 0 \le \mu < h.$$

It is seen from here, that for various combinations of values η and τ, as soon as $|\tau| < h$, the number p can take the only three values: $-1, 0,$ and 1. This allows one to represent expression (1.7) in the form

$$F(a+\tau, h) = \sum_{k=-1}^{N+p-1} \varphi(a+\tau-(N+p-k)h, h) = \sum_{k=-1}^{N-1} \varphi(kh+\mu, h)+S(p), \quad (1.9)$$

where

$$S(p) = \begin{cases} -\varphi((N-1)h+\mu, h) & \text{for} \quad p=-1, \\ 0 & \text{for} \quad p=0, \\ \varphi(Nh+\mu, h) & \text{for} \quad p=1, \end{cases}$$

so that by virtue of the limit property from (1.4)

$$\lim_{h\to 0^+} S(p) = 0 \qquad (1.10)$$

for all values τ such that $|\tau| < h$.

Represent the first sum from the right-hand side of the last equality in (1.9) as

$$\begin{aligned} \sum_{k=-1}^{N-1} \varphi(kh+\mu, h) = &\varphi(-h+\mu, h) - \varphi(0,\mu) + \varphi((N-1)h+\mu, h) \\ &- \varphi((N-1)h+\mu, \varkappa) + \sigma, \end{aligned} \qquad (1.11)$$

where $0 < \varkappa = h - \mu + \eta < 2h$ and the sum

$$\begin{aligned} \sigma = &\varphi(0,\mu) + \sum_{k=0}^{N-2} \varphi(kh+\mu, h) + \varphi((N-1)h+\mu, \varkappa) \\ &= [f(\mu) - f(0)]\, g(0) + \sum_{k=0}^{N-2} [f((k+1)h+\mu) - f(kh+\mu)]\, g(kh+\mu) \\ &\quad + [f(a) - f((N-1)h+\mu)]\, g((N-1)h+\mu). \end{aligned} \qquad (1.12)$$

Obviously, σ is a Stieltjes integral sum for $\int_0^a g(t)\, df(t)$. Then, if $h \to 0^+$, the sum $\sigma \to \int_0^a g(t)\, df(t)$, whereas all the rest of the terms from the right-hand side of equality (1.11) vanish according to the limit property from (1.4). That means we can pass to the limit in equality (1.11) as $h \to 0^+$ and state

$$\lim_{h\to 0^+} \sum_{k=-1}^{N-1} \varphi(kh+\mu, h) = \int_0^a g(t)\, df(t). \qquad (1.13)$$

Moreover, this limit property holds regardless of values τ. Because the reduction of expression (1.7) to the form of (1.9) is possible only under $|\tau| < h$, then the compilation of relations (1.9), (1.10), and (1.13) results in (1.8). □

Corollary 1. *If the functions $f(t)$ and $g(t)$ are continuous, respectively, in D and $[0,a]$, then the conditional double limit*

$$\lim_{\substack{h\to 0^+\\ |\tau|<h}} F(\tau,h) = 0. \tag{1.14}$$

2 Representation of the Stieltjes Integral.

It turns out, that the above established properties of the function $F(t,h)$ defined by (1.6) allow us to formulate a necessary condition for the existence of the Stieltjes integral $\int_0^a g(t)df(t)$ by means of finding its new representation in the form of the limit of a Riemann integral depending on a parameter. First note that the continuity of the function $F(t,h)$ with respect to t in Q ensures the existence of its primitive $\Phi(t,h)$ in t. Moreover, the function $\Phi(t+h,h)$ is directly verified to be a primitive for $F(t+h,h)$ with respect to t. Then, using the Newton-Leibniz formula, we easily obtain

$$\frac{1}{h}\int_0^a \left[F\left(t+h,h\right) - F(t,h)\right]\, dt = \frac{\Phi(a+h,h)-\Phi(a,h)}{h} - \frac{\Phi(h,h)-\Phi(0,h)}{h}. \tag{2.1}$$

And now, find the limits of the two last fractions as $h \to 0^+$. To this end, use the mean value theorem to transform the difference

$$\Phi(a+h,h) - \Phi(a,h) = \int_a^{a+h} F(t,h)\, dt = hF(a+\xi,h), \tag{2.2}$$

where the value ξ, being a solution of the last integral equation, depends on h, but $|\xi| < h$. Then, in the equality

$$\frac{\Phi(a+h,h)-\Phi(a,h)}{h} = F(a+\xi,h), \tag{2.3}$$

arising from (2.2), we can pass to the limit as $h \to 0^+$ provided that the conditions of Theorem 1 hold. As a result

$$\lim_{h\to 0^+} \frac{\Phi(a+h,h)-\Phi(a,h)}{h} = \lim_{\substack{h\to 0^+\\ |\xi|<h}} F(a+\xi,h) = \int_0^a g(t)\, df(t). \tag{2.4}$$

In a similar way, but by using Corollary 1, we also obtain

$$\lim_{h\to 0^+} \frac{\Phi(h,h)-\Phi(0,h)}{h} = \lim_{\substack{h\to 0^+\\ |\xi|<h}} F(\xi,h) = 0. \tag{2.5}$$

Relations (2.3)-(2.5) allow us to pass to the limit in equality (2.1) as $h \to 0^+$ and to state the existence of the limit

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^a \left[F\left(t+h,h\right)-F(t,h)\right]\,dt = \int_0^a g(t)\,df(t).$$

Hence, taking (1.5) into account, we come to the following final statement.

Theorem 2. *If the functions $f(t)$ and $g(t)$ are continuous, respectively, in D and $[0,a]$ and the finite Riemann-Stieltjes integral $\int_0^a g(t)\,df(t)$ exists, then it is represented in the form of the following limit of a Riemann integral depending on a parameter:*

$$\int_0^a g(t)\,df(t) = \lim_{h\to 0^+} \frac{1}{h}\int_0^a \left[f(t+h)-f(t)\right]g(t)\,dt. \tag{2.6}$$

The above proof shows that the requirement of continuity of the functions $f(t)$ and $g(t)$ is essential for validity of this representation. Notably, the formula (2.6) itself can be used both for approximate and for exact calculations of the Stieltjes integral $\int_0^a g(t)\,df(t)$ if one succeeds in explicitly finding the Riemann integral in its right-hand side.

Corollary 2. *If the functions $f(t)$ and $g(t)$ are continuous in D and the finite Riemann-Stieltjes integral $\int_0^a g(t)\,df(t)$ exists, then the two following relations are valid:*

$$\begin{aligned}&\lim_{h\to 0^+} \frac{1}{h}\int_0^a \left[f(t+h)-f(t)\right]g(t)\,dt\\ &+\lim_{h\to 0^+} \frac{1}{h}\int_0^a \left[g(t+h)-g(t)\right]f(t)\,dt = f(t)g(t)\mid_0^a,\end{aligned} \tag{2.7}$$

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^a \left[f(t+h)-f(t)\right]\cdot\left[g(t+h)-g(t)\right]\,dt = 0. \tag{2.8}$$

Proof. The first follows easily from the formula for integration by parts [2,c. 203]

$$\int_0^a f(t)dg(t) = f(t)g(t)\mid_0^a - \int_0^a g(t)\,df(t)$$

and from successively using Theorem 2 for the two Stieltjes integrals here. Interestingly, the formula (2.7) can be interpreted as an extension of the classical formula $\int_0^a [f(t)g(t)]'\,dt = f(t)g(t)\,|_0^a$ to the case of non-differentiable functions $f(t)$ and $g(t)$.

The second one, i.e. (2.8), generalizing the trivial statement $\int_0^a f'(t)\,[g(t+h)-g(t)]\,dt \to 0$ as $h \to 0^+$, can be obtained from (2.6) and (2.7) by means of the following identical transforms:

$$\begin{aligned}
&\lim_{h\to 0^+} \frac{1}{h}\int_0^a [f(t+h)-f(t)]\cdot[g(t+h)-g(t)]\,dt \\
&= \lim_{h\to +0} \frac{1}{h}\int_0^a [f(t+h)g(t+h)-f(t)g(t)]\,dt \\
&\quad - \lim_{h\to +0} \frac{1}{h}\int_0^a \{f(t+h)-f(t)]\,g(t) + [g(t+h)-g(t)]\,f(t)\}\,dt \\
&= \int_0^a d\,[f(t)g(t)] - f(t)g(t)|_0^a = 0.
\end{aligned}$$

□

3 On Existence Conditions for the Stieltjes Integral.

In Sections 1 and 2 the Stieltjes integral $\int_0^a g(t)\,df(t)$ is assumed to exist. Here we are not concerned with sufficient conditions for existence of the integral and only point out that along with the conventional sufficient conditions [2, p. 204] based on boundedness of variation of $f(t)$, there are the little-known ones [3] in which the functions $f(t)$ and $g(t)$ belong to some Hőlder classes. Nevertheless, the new condition necessary for the existence of the Stieltjes integral $\int_0^a g(t)\,df(t)$ of continuous functions will be discussed here in more detail. Namely, this condition directly follows from Theorem 2 and consists in existence of the finite limit

$$\lim_{h\to 0^+} \frac{1}{h}\int_0^a [f(t+h)-f(t)]\,g(t)\,dt = A. \tag{3.1}$$

It is reasonable to question, in what sense the condition (3.1) is sufficient for the existence of the Stieltjes integral $\int_0^a g(t)\,df(t)$. The answer is in the following statement.

Theorem 3. *If the functions $f(t)$ and $g(t)$ are continuous, respectively, in D and $[0,a]$ and the finite limit (3.1) exists, then there exists at least one Stieltjes integral sum $\overline{\sigma}$ corresponding to $\int_0^a g(t)\,df(t)$ and converging to the number A as all related subintervals dividing the integration interval $[0,a]$ into parts vanish.*

PROOF. Indeed, suppose that the limit (3.1) exists. Then, according to (1.5) and (2.1),

$$\lim_{h\to 0^+}\left[\frac{\Phi(a+h,h)-\Phi(a,h)}{h}-\frac{\Phi(h,h)-\Phi(0,h)}{h}\right]=A. \tag{3.2}$$

Because relation (2.5) holds without assuming the existence of the Stieltjes integral $\int_0^a g(t)\,df(t)$, we easily obtain from (2.3) and (3.2)

$$\lim_{\substack{h\to 0^+\\ |\xi|<h}} F(a+\xi,h)=A. \tag{3.3}$$

Using (1.4) and (1.9)-(1.11), we can represent $F(a+\xi,h)$ in the form

$$F(a+\xi,h)=\overline{\sigma}+\sigma_0 \text{ for } |\xi|<h, \tag{3.4}$$

where $\overline{\sigma}$ is the Stieltjes integral sum σ from (1.12) for $\tau=\xi$ and σ_0 is the finite sum of terms vanishing as $h\to 0^+$. That means, taking into account the limit property (3.3), we can pass to the limit in equality (3.4) as $h\to 0^+$ and state

$$\lim_{h\to 0^+}\overline{\sigma}=A, \tag{3.5}$$

where $\overline{\sigma}$ is just the desired Stieltjes integral sum in question. □

Conversely, suppose that the limit (3.5) exists for the Stieltjes integral sum $\overline{\sigma}$ given by (1.12) in which the value $\tau=\xi$ is a solution of the integral equation from (2.2). Then, by virtue of (3.4), the limit (3.3) exists. Moreover, by compiling relations (1.5), (2.1), and (2.3) we easily find

$$\frac{1}{h}\int_0^a [f(t+h)-f(t)]\,g(t)\,dt=F(a+\xi,h)-F(\xi_0,h), \tag{3.6}$$

where ξ_0 is defined from the above integral equation for $a=0$. On the basis of limit properties (1.14) and (3.3) we can pass to the limit in equality (3.6) as $h\to 0^+$ that results in (3.1). Combine this result with that is in Theorem 3 as follows.

Corollary 3. *If $f(t)$ and $g(t)$ are continuous, respectively, in D and $[0,a]$, then the existence of one of two limits $\lim\limits_{h\to 0^+}\frac{1}{h}\int_0^a[f(t+h)-f(t)]\,g(t)\,dt$ and $\lim\limits_{h\to 0^+}\overline{\sigma}$ implies the existence of another. In this case the two limits are equal.*

In other words, the condition (3.1) is necessary and sufficient for convergence of the Stieltjes integral sum $\overline{\sigma}$ as $h\to 0^+$. Notably, in this case each vanishing number sequence $\{h_i\}_{i=1}^{\infty}$ on interval $(0, H)$ generates the appropriate sequence of Stieltjes integral sums $\{\overline{\sigma}_i\}_{i=1}^{\infty}$ to be convergent to the number A. Thus, under condition (3.1) there exist infinitely many such integral-sum sequences to be convergent to the same number. However, each of sums $\overline{\sigma}_i$ defined by (1.12) for $h = h_i$ and $\tau = \xi$ corresponds to dividing the integration interval $[0, a]$ into equal h_i -long parts except for its first and last ones.

Conclusion. The obtained results can be extended to the case of negative h. Therefore, all statements of theorems and corollaries remain valid if in all their appearing passages to the limit we change the tendency $h \to 0^+$ for $h \to 0$. The proof of this fact is based on investigation of functional properties of the solution to difference equation (1.1) with a negative step h and is not given here because of its bulky volume and full analogy to the above.

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