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# MORE ABOUT SIERPIŃSKI-ZYGMUND UNIFORM LIMITS OF EXTENDABLE FUNCTIONS

**Abstract**

Let $SZ, D, Ext$, and $\overline{Ext}$ denote respectively the spaces of Sierpiński-Zygmund functions, Darboux functions, extendable connectivity functions, and uniform limits of sequences of extendable connectivity functions, with the metric of uniform convergence on them. We show that the subspaces $SZ \cap D$ and $SZ \cap \overline{Ext}$ are each porous in the space $SZ$, but $SZ \cap \overline{Ext}$ is not porous in the space $\overline{Ext}$. We also show that every real function can be expressed as a sum of two Sierpiński-Zygmund functions one of which belongs to $\overline{Ext}$. Ciesielski and Natkaniec show in [4] that if $\mathbb{R}$ is not the union of less than $\mathfrak{c}$-many nowhere dense subsets, then there exist Sierpiński-Zygmund bijections $f, g : \mathbb{R} \to \mathbb{R}$ such that $f^{-1} \notin SZ$ and $g^{-1} \in SZ$, but here we can additionally have $f$ and $g$ belonging to $\overline{Ext}$.

A *Sierpiński-Zygmund* ($SZ$) function $f : \mathbb{R} \to \mathbb{R}$ has the property that all restrictions $f \restriction_B$ to subsets $B$ of cardinality $\mathfrak{c}$ are discontinuous. This is equivalent to having $\operatorname{card}(f \cap g) < c$ for all continuous functions $g$ defined on $G_\delta$ subsets of $\mathbb{R}$ [12].

A function $h : \mathbb{R} \to \mathbb{R}$ is called *extendable connectivity* if there is a function $F : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $F(x,0) = h(x)$ for all $x \in \mathbb{R}$ and $F \restriction_C$ is connected for each connected set $C \subset \mathbb{R} \times [0,1]$, and such a function $h$ must be *Darboux*, which means $h(K)$ is connected for each connected subset $K$ of $\mathbb{R}$.

According to [7], if an extendable connectivity function $h$ has a dense graph in $\mathbb{R}^2$, then there exists a decomposition of $\mathbb{R}$ into a sequence $\{A_n\}_{n=0}^{\infty}$ of the following "special sets": $A_0$ is a dense $G_\delta$ subset of $\mathbb{R}$ that is *h-negligible* with respect to $Ext$. This means that if $h$ is arbitrarily redefined just on $A_0$, the

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resulting function still belongs to $Ext$. Moreover, $\mathbb{R} \setminus A_0 = \cup_{n=1}^{\infty} A_n$, where the sets $A_n$, $n \geq 1$, are pairwise disjoint and nowhere dense in $\mathbb{R}$ and therefore $h$-negligible.

In [10], we show that under MA, $SZ \cap \overline{Ext}$ cannot be characterized by pre-images of sets. We obtain more results starting with the porosity of the function space $SZ \cap \overline{Ext}$.

## 1 Porosity.

The porosity of a subspace $M$ in a metric space $X$ is a measurement of how thin $M$ is in $X$. In a metric space $(X, d)$, $B(x, r)$ denotes the open ball centered at $x$ with radius $r > 0$. For $x \in X$, let

$$\gamma(x, r, M) = \sup\Big\{ s > 0 : \exists z \in X \text{ such that } B(z, s) \subset B(x, r) \setminus M \Big\}.$$

$M$ is *porous at* $x$ if

$$p(x) = \limsup_{r \to 0^+} \frac{\gamma(x, r, M)}{r}$$

is a positive real number. M is *porous in* $X$ if $M$ is porous at each $x \in X$. A set $M$ porous in $X$ is a boundary set in $X$, which means $\overline{X \setminus M} = X$.

Each function space has on it the metric $d$ of uniform convergence defined by

$$d(f, g) = \min\Big\{ 1, \sup\big\{ |f(x) - g(x)| : x \in \mathbb{R} \big\} \Big\}.$$

See [9] for some results on the porosity of Darboux-like function spaces.

According to [11], $SZ \cap Ext = \emptyset$ but in [10, Theorem 1], it is shown there exists a function $f \in SZ \cap \overline{Ext}$ whose graph is dense in $\mathbb{R}^2$. Balcerzak, Ciesielski and Natkaniec show in [1] that in ZFC an extra hypothesis is needed in order to have $SZ \cap D \neq \emptyset$.

**Theorem 1.** *$SZ \cap \overline{Ext}$ is not porous in $\overline{Ext}$ but is a boundary set in $\overline{Ext}$.*

PROOF. Let $g \in SZ \cap \overline{Ext}$ have a dense graph in $\mathbb{R}^2$ and let $0 < r \leq 1$. Pick an arbitrary $\varphi \in B(g, r) \subset \overline{Ext}$ and an arbitrary positive number $s < r$ such that $B(\varphi, s) \subset B(g, r)$. Then there exists $h \in Ext$ such that $d(\varphi, h) < \frac{s}{3}$ on $\mathbb{R}$. Notice the graphs of $\varphi$ and $h$ are dense in $\mathbb{R}^2$ just like $g$. Let $\{A_n\}_{n=0}^{\infty}$ be a decomposition of $\mathbb{R}$ into those $h$-negligible special sets described above. Let $\mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\}$ and henceforth let $C_{G_\delta} = \{g_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions defined on $G_\delta$ subsets of $\mathbb{R}$. Define a function $f : \mathbb{R} \to \mathbb{R}$ in $SZ$ so that $f(x_\alpha) \in \mathbb{R} \setminus \big\{ g_\xi(x_\alpha) : \xi \leq \alpha \big\}$ but we require

$$\big| f(x_\alpha) - h(x_\alpha) \big| < \frac{s}{2n + 2} \text{ whenever } x_\alpha \in A_n \text{ for some } n \geq 0.$$

Then

$$h_0 = \begin{cases} f & \text{on } A_0 \\ h & \text{on } \mathbb{R} \setminus A_0 \end{cases}$$

is in $Ext$, and for $n \geq 1$,

$$h_n = \begin{cases} f & \text{on } A_n \\ h_{n-1} & \text{on } \mathbb{R} \setminus A_n \end{cases}$$

is in $Ext$. Since $f$ is the uniform limit of $h_n$, we have $f \in \overline{Ext}$. But $f \in B(\varphi, s) \cap SZ \cap \overline{Ext}$ since

$$d(\varphi, f) \leq d(\varphi, h) + d(h, f) < \frac{s}{3} + \frac{s}{2} < s \text{ on } \mathbb{R}.$$

So

$$\gamma(g, r, SZ \cap \overline{Ext}) = \sup \emptyset = -\infty \text{ and } p(g) = -\infty.$$

This shows $SZ \cap \overline{Ext}$ cannot be porous in $\overline{Ext}$ at any function $g$ in $SZ \cap \overline{Ext}$ with graph dense in $\mathbb{R}^2$.

To see $SZ \cap \overline{Ext}$ is a boundary set in $\overline{Ext}$, let $f \in SZ \cap \overline{Ext}$, which implies $f$ is a uniform limit of a sequence $h_n$ in $Ext \subset \overline{Ext} \setminus SZ$. That is, every open neighborhood of $f$ in $\overline{Ext}$ contains all but finitely many $h_n$ and therefore meets $\overline{Ext} \setminus SZ$. □

**Theorem 2.** *$SZ \cap D$ and $SZ \cap \overline{Ext}$ are each porous in $SZ$.*

PROOF. If $SZ \cap D = \emptyset$, the first result is true because the porosity of $\emptyset$ is 1 everywhere in $SZ$. Therefore suppose $f \in \overline{SZ \cap D} \subset SZ \cap \overline{D}$, where closure is taken in $SZ$. For sufficiently small $r$ with $0 < r \leq 1$, there exist numbers $a < b$ such that $\frac{r}{4} < \left|f(a) - f(b)\right| < \frac{r}{2}$ and we may suppose $f(a) < f(b)$. Let

$$B = (a, b) \cap f^{-1}\left(\left(f(a) + \frac{r}{16}\,,\, f(b) - \frac{r}{16}\right)\right) = \{x_\alpha : \alpha < c\}.$$

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x)$ if $x \in \mathbb{R} \setminus B$, and for every $\alpha < \mathfrak{c}$, pick

$$g(x_\alpha) \in \Big(f(a)\,,\, f(b)\Big) \setminus \left(\left(f(a) + \frac{r}{16}\,,\, f(b) - \frac{r}{16}\right) \cup \Big\{g_\xi(x_\alpha) : \xi \leq \alpha\Big\}\right).$$

Then $g \notin \overline{D}$ and $g \in B\big(f\,,\, \frac{r}{2}\big)$ because $\left|f(a) - f(b)\right| < \frac{r}{2}$, and $B\big(g\,,\, \frac{r}{16}\big) \subset B(f, r) \setminus D$ because $\left|f(a) - f(b)\right| > \frac{r}{4}$.

To see $g \in SZ$, suppose $X \subset \mathbb{R}$ and $\operatorname{card} X = \mathfrak{c}$. Either (1) $\operatorname{card}(X \setminus B) = \mathfrak{c}$ or (2) $\operatorname{card}(X \cap B) = \mathfrak{c}$. If (1) holds, then $g\restriction_{(X\setminus B)} = f\restriction_{(X\setminus B)}$ is discontinuous. If (2) holds, then

$$\Big\{x \in X \cap B : g(x) = g_\xi(x)\Big\} \subset \Big\{x \in B : g(x) = g_\xi(x)\Big\} \subset \big\{x_\alpha : \alpha < \xi\big\},$$

which has cardinality $< \mathfrak{c}$, and therefore $g\restriction_{(X\cap B)}$ is discontinuous. This shows $g \in SZ \setminus \overline{D}$. Since $\gamma(f, r, SZ \cap D) \geq \frac{r}{16}$,

$$p(f) = \limsup_{r\to 0^+} \frac{\gamma(f, r, SZ \cap D)}{r} \geq \frac{1}{16} > 0$$

and so $SZ \cap D$ is porous at $f$. $SZ \cap \overline{Ext}$ is porous in $SZ$ because it is a subspace of $SZ \cap \overline{D}$, which is porous in $SZ$ according to the above argument with $\overline{D}$ in place of $D$. □

It is left as an open problem whether or not $SZ \cap D$, if nonempty, is porous in $D$.

**Theorem 3.** *$SZ \cap D$ is a boundary set in $D$.*

PROOF. Assume $f \in \overline{SZ \cap D} \subset \overline{SZ} \cap D$, where closure is taken in $D$. Given $0 < r \leq 1$, there exist $a, b \in \mathbb{R}$ such that $0 < s \equiv \frac{f(b)-f(a)}{2} < r$. Define the continuous function

$$g(x) = \begin{cases} x + s & \text{if } x < f(a) \\ \frac{f(a)+f(b)}{2} & \text{if } f(a) \leq x \leq f(b) \\ x - s & \text{if } x > f(b). \end{cases}$$

Since $d(g, \mathit{identity}) = s$, $d(g \circ f, f) = s < r$. The Darboux function $g \circ f$ is constant (hence continuous) on $f^{-1}\big((f(a), f(b))\big)$ which has cardinality $\mathfrak{c}$. Therefore $g \circ f \in D \setminus SZ$. This shows $SZ \cap D$ is a boundary set of $D$. □

## 2 Sums of Functions.

In [3], Ciesielski and Natkaniec show that every real function can be expressed as the sum of two $SZ$ functions. In [6], Płotka shows that under $CH$, every function $f : \mathbb{R} \to \mathbb{R}$ can be represented as a sum of an almost continuous ($AC$) function and an $SZ$ function. (Each open neighborhood of the graph of an *almost continuous* function $f : \mathbb{R} \to \mathbb{R}$ in $\mathbb{R}^2$ contains the graph of a continuous function defined on $\mathbb{R}$.) As a corollary to a theorem in [1] about the existence of a model with no Darboux $SZ$ function, Płotka obtains the equality $\mathbb{R}^\mathbb{R} = AC + SZ$ is independent of $ZFC$. According to the following first corollary, $\mathbb{R}^\mathbb{R} = \big(SZ \cap \overline{Ext}\big) + SZ$.

**Theorem 4.** *For each family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ with $\operatorname{card} \mathcal{F} \leq \mathfrak{c}$, there exists $g \in \overline{Ext}$ such that $g + \mathcal{F} \subset SZ$.*

Proof. Let $C_{G_\delta} = \{g_\alpha : \alpha < \mathfrak{c}\}$, $\mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\}$, and $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{c}\}$. There exists $h \in Ext$ with graph dense in $\mathbb{R}^2$ [5], [8], and so there is a decomposition of $\mathbb{R}$ into a sequence $\{A_n\}_{n=0}^{\infty}$ of $h$-negligible special sets. For every $\alpha < c$, pick $g(x_\alpha) \in \mathbb{R} \setminus \{g_\gamma(x_\alpha) - f_\beta(x_\alpha) : \beta, \gamma \leq \alpha\}$ as done in [6], but here we require

$$\Big|g(x_\alpha) - h(x_\alpha)\Big| < \frac{1}{n+1} \text{ whenever } x_\alpha \in A_n \text{ for some } n \geq 0.$$

Then $g \in \overline{Ext}$ and for every $\beta, \gamma < c$,

$$\Big\{x : g(x) + f_\beta(x) = g_\gamma(x)\Big\} \subset \Big\{x_\alpha : \alpha < \max\{\beta, \gamma\}\Big\},$$

which has cardinality $< \mathfrak{c}$. Since $\operatorname{card}\big((g + f_\beta) \cap g_\gamma\big) < \mathfrak{c}$ for all $\beta, \gamma < \mathfrak{c}$, $g + f_\beta \in SZ$ for every $\beta < \mathfrak{c}$. □

Letting $\mathcal{F} = \{0, f\}$ in Theorem 4 gives the next result.

**Corollary 1.** *Each function $f \in \mathbb{R}^{\mathbb{R}}$ is the sum of a function in $SZ \cap \overline{Ext}$ and a function in $SZ$.*

As in [6], for $\mathcal{F}_1$ and $\mathcal{F}_2 \subset \mathbb{R}^{\mathbb{R}}$, define $\operatorname{Add}(\mathcal{F}_1, \mathcal{F}_2) = \min\big(\{\operatorname{card} \mathcal{F} : \mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ and there exists no $g \in \mathcal{F}_1$ such that $g + \mathcal{F} \subset \mathcal{F}_2\} \cup \{(2^{\mathfrak{c}})^+\}\big)$. This is a generalization for $\mathcal{F} \subset \mathbb{R}^X$ of

$$A(\mathcal{F}) = \min\{\operatorname{card} F : F \subset \mathbb{R}^X \text{ and there is no } g \in \mathbb{R}^X \text{ such that } g + F \subset \mathcal{F}\}.$$

It turns out $\mathfrak{c}^+ \leq A(\overline{Ext}) \leq 2^{\mathfrak{c}}$. There is no $g \in \overline{Ext}$ such that $g + \overline{Ext} \subset SZ$ because $g - g = 0 \notin SZ$, and also $\operatorname{card}(\overline{Ext}) = 2^{\mathfrak{c}}$. Therefore by Theorem 4, we have the following.

**Corollary 2.** $\mathfrak{c}^+ \leq \operatorname{Add}\big(\overline{Ext}, SZ\big) \leq 2^{\mathfrak{c}}$.

## 3 Inverses of Uniform Limits.

In [4], Ciesielski and Natkaniec verify these two facts:

Fact 1: There exists a one-to-one $SZ$ function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{-1} : f(\mathbb{R}) \to \mathbb{R}$ does not belong to $SZ$.

Fact 2: Assume $\mathbb{R}$ cannot be covered by less than $\mathfrak{c}$-many meager sets. Then

(a) there exists an $SZ$ bijection $f:\mathbb{R}\to\mathbb{R}$ such that $f^{-1}\notin SZ$;

(b) there exists an $SZ$ bijection $f:\mathbb{R}\to\mathbb{R}$ such that $f^{-1}=f$.

**Theorem 5.** *In each of the above two facts, $f$ can be constructed to belong to $\overline{Ext}$.*

PROOF. We show how to modify Ciesielski and Natkaniec's proof to make $f\in\overline{Ext}$ in Fact 1 and leave how to in Fact 2 to the reader. Let $C^*_{G_\delta}=\{g_\alpha:\alpha<\mathfrak{c}\}$ be the collection of all nowhere constant continuous functions defined on $G_\delta$ subsets of $\mathbb{R}=\{x_\alpha:\alpha<\mathfrak{c}\}$, and let $\varphi:[0,1]\to[0,1]$ be a nowhere constant continuous function like that given in [2, p. 222] such that $\varphi(0)=0$, $\varphi(1)=1$, and $\operatorname{card}\bigl(\varphi^{-1}(y)\bigr)=\mathfrak{c}$ for every $y\in[0,1]$. Extend $\varphi$ to a continuous function $g:\mathbb{R}\to\mathbb{R}$ by defining $g(x)=\varphi(x-n)+n$ on $[n,n+1]$ for each integer $n$. Let $h:\mathbb{R}\to\mathbb{R}$ be an extendable connectivity function having a dense graph in $\mathbb{R}^2$, and let $\{A_n\}_{n=0}^{\infty}$ be a decomposition of $\mathbb{R}$ into those special $h$-negligible subsets.

Define $f:\mathbb{R}\to\mathbb{R}$ by

$$y_\alpha=f(x_\alpha)\in\begin{cases} g^{-1}(x_\alpha)\setminus\Bigl(\bigl\{y_\beta:\beta<\alpha\bigr\}\cup\bigl\{g_\beta(x_\alpha):\beta\le\alpha\bigr\}\Bigr) & \text{if } x_\alpha\in A_0\\ \mathbb{R}\setminus\Bigl(\bigl\{y_\beta:\beta<\alpha\bigr\}\cup\bigl\{g_\beta(x_\alpha):\beta\le\alpha\bigr\}\Bigr) & \text{if } x_\alpha\in\mathbb{R}\setminus A_0\end{cases}$$

such that if $x_\alpha\in A_n$ and $n>0$, then $\bigl|f(x_\alpha)-h(x_\alpha)\bigr|<\frac{1}{n+1}$ which implies $f\in\overline{Ext}$. Because $f:\mathbb{R}\to\mathbb{R}$ is one-to-one and $\operatorname{card}(f\cap g_\alpha)<\mathfrak{c}$ for each $g_\alpha\in C^*_{G_\delta}$, $f$ is a Sierpiński-Zygmund function according to [4, Lemma 1 ]. So $f\in SZ\cap\overline{Ext}$, and $f^{-1}\notin SZ$ because $f^{-1}(y_\alpha)=x_\alpha=g(y_\alpha)$ if $x_\alpha\in A_0$.

Note that $f$ preserves nowhere dense sets $C\subset A_0$. For $g^{-1}(\overline{C})$ is closed and nowhere dense in $\mathbb{R}$ because $g\Bigl(\overline{g^{-1}(\overline{C})}\Bigr)\subset\overline{g\bigl(g^{-1}(\overline{C})\bigr)}=\overline{C}$ and the nowhere constant function $g$ maps connected sets to connected sets. Since $C\subset A_0$ and $f(C)\subset f(\overline{C})\subset g^{-1}(\overline{C})$, $f(C)$ is nowhere dense in $\mathbb{R}$. □

## References


[1] M. Balcerzak, K. Ciesielski, T. Natkaniec, *Sierpiński-Zygmund functions that are Darboux, almost continuous, or have a perfect road*, Arch. Math. Logic, **37, no. 1** (1997), 29–35.

[2] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics, **659**, Springer-Verlag, Berlin, 1978.

[3] K. Ciesielski, T. Natkaniec, *Algebraic properties of the class of Sierpiński-Zygmund functions*, Topology Appl., **79** (1997), 75–99.

[4] K. Ciesielski, T. Natkaniec, *On Sierpiński-Zygmund bijections and their inverses*, Topology Proc., **22** (1997), 155–164.

[5] K. Ciesielski, I. Recław, *Cardinal invariants concerning extendable and peripherally continuous functions*, Real Analysis Exchange, **21** (1995/96), 459–472.

[6] K. Płotka, *Sum of Sierpiński-Zygmund and Darboux like functions*, Topology Appl., **122** (2002), 547–564.

[7] H. Rosen, *Limits and sums of extendable connectivity functions*, Real Analysis Exchange, **20** (1994/95), 183–191.

[8] H. Rosen, *Every real function is the sum of two extendable connectivity functions*, Real Analysis Exchange, **21** (1995/96), 299–303.

[9] H. Rosen, *Porous and boundary sets in Darboux-like function spaces*, Real Analysis Exchange, **27** (2001/02), 27–40.

[10] H. Rosen, *Sierpiński-Zygmund uniform limits of extendable connectivity functions*, Real Analysis Exchange, **28** (2002/03), 105–110.

[11] H. Rosen, R. G. Gibson, F. Roush, *Extendable functions and almost continuous functions with a perfect road*, Real Analysis Exchange, **17** (1991/92), 248–257.

[12] W. Sierpiński, A. Zygmund, *Sur une fonction qui est discontinue sur tout ensemble de puissance du continu*, Fund. Math., **4** (1923), 316–318.