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NOTES ON ABSOLUTELY CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES

Abstract

Let $\Omega \subset \mathbb{R}^n$ be a domain. The result of J. Kauhanen, P. Koskela and J. Malý [4] states that a function $f : \Omega \to \mathbb{R}$ with a derivative in the Lorentz space $L^{n,1}(\Omega, \mathbb{R}^n)$ is n-absolutely continuous in the sense of [5]. We give an example of an absolutely continuous function of two variables, whose derivative is not in $L^{2,1}$. The boundary behavior of n-absolutely continuous functions is also studied.

1 Introduction.

Absolutely continuous functions of one variable are admissible transformations for the change of variables in the Lebesgue integral. Recently, J. Malý [5] introduced a class of n-absolutely continuous functions giving an n-dimensional analogue of the notion of absolute continuity from this point of view. For the recent development in the theory of n-absolutely continuous functions also see [2] and [3].

Suppose that $\Omega \subset \mathbb{R}^n$ is a domain. A function $f : \Omega \to \mathbb{R}^m$ is said to be n-absolutely continuous if for each $\varepsilon > 0$ there is $\delta > 0$ such that for each disjoint finite family $\{B_i\}$ of open balls in Ω we have

$$\sum_i \mathcal{L}_n(B_i) < \delta \Longrightarrow \sum_i (\operatorname{osc}_{B_i} f)^n < \varepsilon.$$

It was shown in [5] that n-absolute continuity implies weak differentiability with gradient in L^n, differentiability a.e., area and coarea formula.

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It was proved by J. Kauhanen, P. Koskela and J. Malý [4] that a function $f : \Omega \to \mathbb{R}$ has an n-absolutely continuous representative if $\nabla f \in L^{n,1}(\Omega, \mathbb{R}^n)$. This result gains in interest if we realize that $L^{n,1}(\Omega)$ is the largest rearrangement invariant Banach space of functions on $\mathbb{R}^n$ with such a property, (see [1]). In the third section we give an example of 2-absolutely continuous function, whose derivative is not in the Lorentz space $L^{2,1}$.

Sections 4 and 5 are devoted to the study of the boundary behavior of n-absolutely continuous functions. The aim of these sections is to find conditions on the domain Ω which guarantee that every n-absolutely continuous function on Ω can be continuously extended to $\partial\Omega$. Let $0 < \alpha < 1$. Example 4.3 demonstrates that the existence of a continuous extension is not generally guaranteed by the condition that a domain Ω has $C^{1,\alpha}$ boundary. On the other hand, in Section 5 it is shown that a continuous extension exists if Ω has a $C^{1,1}$ boundary. (See Preliminaries for the definition of $C^{1,\alpha}$ boundary.)

2 Preliminaries.

We will denote by $\mathcal{L}_n$ the n-dimensional Lebesgue measure. We will use the symbol α_n to denote the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

We will denote by $B(x, r)$ the n-dimensional Euclidean open ball with the center x and diameter r and by $\overline{B(x, r)}$ the corresponding closed ball. Throughout the paper, we will use the letter B only for open balls.

For a mapping $f : \Omega \to \mathbb{R}$, we denote by $f'(x)$ the vector of all partial derivatives of f at x. We write ∇f for the weak (distributional) derivative.

The convex hull of a set $A \subset \mathbb{R}^n$ will be denoted by $\operatorname{conv}(A)$. The closure of a set A is denoted by $\overline{A}$ and its boundary is denoted by ∂A. We denote by $|x|$ the Euclidean norm of a point $x \in \mathbb{R}^d$.

Let $A \subset \mathbb{R}^d$ be an open set and $0 < \alpha \leq 1$. A function $F : A \to \mathbb{R}^d$ is said to be α-Hölder continuous if there is a constant $K > 0$ such that

$$|F(x) - F(y)| \leq K|x - y|^\alpha \text{ for every } x, y \in A. \tag{2.1}$$

As usual, F is called Lipschitz if it is 1-Hölder continuous. We will denote by $C^{1,\alpha}(A)$ the family of functions from A to $\mathbb{R}$ whose derivative, as a function from A to $\mathbb{R}^d$, is α-Hölder continuous. Let us denote by $C^1(A)$ the family of functions whose derivative is continuous.

We will use the letter Ω to denote a domain; i.e., a connected open set in $\mathbb{R}^n$, $n \geq 2$. Let $0 < \alpha \leq 1$. A domain Ω is said to have $C^{1,\alpha}$ boundary (or C^1 boundary) $\partial\Omega$ if for every $x_0 \in \partial\Omega$ there is a ball $B(x_0, r_0) \subset \mathbb{R}^n$, $i \in \{1, \ldots, n\}$, an open set $D \subset \mathbb{R}^{n-1}$ and $h \in C^{1,\alpha}(\mathbb{R}^{n-1})$ (or $h \in C^1(\mathbb{R}^{n-1})$)

such that

$$\partial\Omega \cap B(x_0, r_0) = \{x \in \mathbb{R}^n : [x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n] \in D \text{ and } h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = x_i\} \tag{2.2}$$

and that either $G^+ \subset \Omega$ and $G^- \cap \Omega = \emptyset$ or $G^- \subset \Omega$ and $G^+ \cap \Omega = \emptyset$ where

$$\begin{aligned} G^+ &= \{x \in B(x_0, r_0) : h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) < x_i\} \\ \text{and } G^- &= \{x \in B(x_0, r_0) : h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) > x_i\}. \end{aligned} \tag{2.3}$$

We will need the following version of the Taylor theorem which holds for $C^{1,1}(\mathbb{R}^d)$ mappings.

Proposition 2.1. *Let $h : \mathbb{R}^d \to \mathbb{R}$ be a $C^{1,1}$ mapping. Let K denotes the Lipschitz constant of h' (i.e., $|h'(x) - h'(y)| \le K|x - y|$ for every $x, y \in \mathbb{R}^d$). Then*

$$\big|h(\tilde{x}_0 + \tilde{x}) - h(\tilde{x}_0) - h'(\tilde{x}_0)\tilde{x}\big| \le \frac{K}{2}|\tilde{x}|^2 \tag{2.4}$$

for every $\tilde{x}_0, \tilde{x} \in \mathbb{R}^d$.

If $f : \Omega \to \mathbb{R}$ is a mapping and $x \in \Omega$, we write $\operatorname{mlip}(f, x)$ for the "maximal function" version of Lipschitz constant

$$\operatorname{mlip}(f, x) = \sup\Big\{\Big|\frac{f(x) - f(y)}{x - y}\Big| : y \in \Omega \setminus \{x\} \text{ and } x, y \in B \text{ for some ball } B \subset \Omega\Big\}.$$

We write $\operatorname{osc}_{B(x,r)} f$ for the oscillation of f over the ball $B(x, r)$, which is the diameter of the image $\overline{f(B(x, r))}$. The support of a function $f : \Omega \to \mathbb{R}$ is denoted by $\operatorname{spt}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$.

Throughout this paper, we use the letter γ for a continuous mapping $\gamma : [0, 1] \to \Omega$. Set $\langle\gamma\rangle = \{\gamma(t) : t \in [0, 1]\}$. The length of the curve γ is denoted by $\ell(\gamma)$. For $x, y \in \Omega$, we will denote by $\rho_\Omega(x, y)$ the distance of x and y in Ω; i.e.,

$$\rho_\Omega(x, y) = \inf\{\ell(\gamma);\ \gamma : [0, 1] \to \Omega,\ \gamma(0) = x \text{ and } \gamma(1) = y\}.$$

We use the convention that C denotes some positive constant. The value of this constant may differ from occurrence to occurrence but for a fixed n (the dimension of the underlying space $\mathbb{R}^n$) it is always an absolute constant.

Given a function $f : \Omega \to \mathbb{R}$, the n-variation of f on Ω is defined by

$$V_n(f, \Omega) = \sup\Big\{\sum_i (\operatorname{osc}_{B_i} f)^n : \{B_i\} \text{ is a disjoint finite family of balls in } \Omega\Big\}.$$

We define the space $AC^n(\Omega)$ to be the family of all n-absolutely continuous functions $f : \Omega \to \mathbb{R}$ such that $V_n(f, \Omega) < \infty$.

A function $f : \Omega \to \mathbb{R}$ is said to satisfy the RR-condition (written $f \in RR(\Omega)$) if there is a function $g \in L^1(\Omega)$, called the weight, such that

$$\Big(\operatorname{osc}_{B(x,r)} f\Big)^n \leq \int_{B(x,r)} g$$

for every ball $B(x, r) \subset \Omega$. A condition similar to RR was used by Rado and Reichelderfer [6] as a sufficient condition for the area formula and for the differentiability a.e. It was shown in [5] that the RR-condition easily implies n-absolute continuity.

Theorem 2.2 (RR-condition). *Suppose that a function $f : \Omega \to \mathbb{R}$ satisfies the RR-condition. Then $f \in AC^n(\Omega)$.*

Moreover the results of M. Csörnyei [2] give $RR(\Omega) = AC^n(\Omega)$, but we will not need this fact in this paper.

3 Lorentz Space $L^{n,1}$.

If $f : \Omega \to \mathbb{R}^m$ is a measurable function, we define its distributional function $m(\cdot, f)$ by

$$m(\sigma, f) = \mathcal{L}_n(\{x : |f(x)| > \sigma\}), \quad \sigma > 0,$$

and the nonincreasing rearrangement $f^\star$ of f by

$$f^\star(t) = \inf\{\sigma : m(\sigma, f) \leq t\}.$$

The Lorentz space $L^{n,1}(\Omega, \mathbb{R}^m)$ is defined to be the class of all measurable functions $f : \Omega \to \mathbb{R}^m$ such that

$$\int_0^\infty t^{\frac{1}{n}} f^\star(t) \frac{dt}{t} < \infty.$$

For abbreviation, we write $L^{n,1}(\Omega)$ instead of $L^{n,1}(\Omega, \mathbb{R})$. For an introduction to Lorentz spaces see for instance [7].

The following theorem of J. Kauhanen, P. Koskela and J. Malý [4] states that functions with the distributional derivative in the Lorentz space $L^{n,1}$ are n-absolutely continuous.

Theorem 3.1. *Suppose that $\nabla f \in L^{n,1}(\Omega, \mathbb{R}^n)$. Then there is a representative of f such that $f \in AC^n(\Omega)$.*

This result is quite sharp, because A. Cianchi and L. Pick [1] proved that $L^{n,1}$ is the largest rearrangement invariant Banach space of functions on $\mathbb{R}^n$ with the property $\nabla f \in L^{n,1}(\Omega, \mathbb{R}^n) \Rightarrow f \in C(\Omega)$ (see also [4, Theorem F]).

The rest of this section is devoted to the proof that there are $\varepsilon > 0$ and $f \in AC^2(B([0,0],\varepsilon))$ such that $\nabla f \notin L^{2,1}(B([0,0],\varepsilon), \mathbb{R}^2)$. It follows that these two classes of functions do not coincide.

Lemma 3.2. *Let $B(0,R) \subset \mathbb{R}^n$ and let $f : B(0,R)\backslash\{0\} \to \mathbb{R}^+$ be a continuous function. Suppose that there is a decreasing function $g : (0,R) \to \mathbb{R}^+$ such that $f(x) = g(|x|)$. Then $f \in L^{n,1}(B(0,R))$ if and only if $\int_0^R g < \infty$.*

PROOF. Since $m(\sigma, f) = \mathcal{L}_n(\{x : |f(x)| > \sigma\}) = \alpha_n(g^{-1}(\sigma))^n$, it follows that

$$f^\star(t) = \inf\{\sigma : m(\sigma, f) \le t\} = \inf\{\sigma : \alpha_n(g^{-1}(\sigma))^n \le t\} = g\Big(\frac{\sqrt[n]{t}}{\sqrt[n]{\alpha_n}}\Big).$$

From this we have

$$\begin{aligned}\int_0^\infty t^{\frac{1}{n}} f^\star(t)\frac{dt}{t} &= \int_0^{\alpha_n R^n} t^{\frac{1}{n}} f^\star(t)\frac{dt}{t} \\ &= \int_0^{\alpha_n R^n} t^{\frac{1}{n}} g\Big(\frac{\sqrt[n]{t}}{\sqrt[n]{\alpha_n}}\Big)\frac{dt}{t} = C\int_0^R g(s)ds.\end{aligned}$$ □

Lemma 3.3. *Let $B(0,R) \subset \mathbb{R}^n$ and let $G : [0,R] \to \mathbb{R}^+$ be an increasing continuous function which is differentiable on $(0,R)$. Assume further that G' is a continuous decreasing function on $(0,R)$. Then a function $F(x) = G(|x|)$ satisfies $F' \in L^{n,1}(B(0,R), \mathbb{R}^n)$.*

PROOF. Set $f = |F'|$ and $g = G'$. Clearly, f and g satisfy the assumptions of Lemma 3.2 and $\int_0^R g = \int_0^R G' = G(R) - G(0) < \infty$. □

Remark 3.4. From Lemma 3.3 and Theorem 3.1 we have that $AC^n(\Omega)$ functions can have arbitrarily "bad" modulus of continuity even on compact subsets of Ω. Note that functions from $AC^n(\Omega)$ are not necessarily uniformly continuous on Ω if $\partial\Omega$ is not "nice" (see Section 4 for details).

The following lemma provides a criterion for absolute continuity.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $f : \Omega \to \mathbb{R}$. If $g(x) = \mathrm{mlip}^n(f,x) \in L_1(\Omega)$ then f satisfies the RR-condition with weight Cg, and hence $f \in AC^n(\Omega)$.*

PROOF. Fix $B = B(z,r) \subset \Omega$ and $x \in B$. There exist $a, b \in B$ such that

$$\frac{\operatorname{osc}_B f}{2} \le |f(a) - f(b)|.$$

Since $|a - x| \le 2r$ and $|b - x| \le 2r$, we have

$$\begin{aligned}\frac{\operatorname{osc}_B^n f}{r^n} &\le C\frac{|f(a)-f(b)|^n}{r^n} \le C\left(\frac{|f(a)-f(x)|^n}{(2r)^n} + \frac{|f(b)-f(x)|^n}{(2r)^n}\right)\\ &\le C\left(\frac{|f(a)-f(x)|^n}{|a-x|^n} + \frac{|f(b)-f(x)|^n}{|b-x|^n}\right) \le C \operatorname{mlip}^n(f,x) = Cg(x)\end{aligned}$$

It follows that

$$\operatorname{osc}_{B(z,r)}^n f = C\int_{B(z,r)} \frac{\operatorname{osc}_{B(z,r)}^n f}{r^n} dx \le C\int_{B(z,r)} g(x)dx.$$

Hence f satisfies the RR-condition with weight Cg, and the desired conclusion follows from Theorem 2.2. □

Example 3.6. *There is a function* $f : B([0,0],1/2) \to \mathbb{R}$ *such that* $f \in AC^2(B([0,0],1/2))$*, but* $\operatorname{mlip}^2(f,x) \notin L^1(B([0,0],1/2))$.

PROOF. Set

$$f(x) = \begin{cases} \frac{1}{|\log|x||^{1/2}} & \text{for } x \in B([0,0],1/2),\\ 0 & \text{for } x = [0,0].\end{cases}$$

Clearly, Lemma 3.3 and Theorem 3.1 give that $f \in AC^2(B([0,0],1/2))$. An easy computation shows that

$$\begin{aligned}\int_B \operatorname{mlip}^2(f,x)\ dx &= \int_B \left|\frac{f(x)-f(0)}{x-0}\right|^2 dx = \int_B \frac{1}{|x|^2\big|\log|x|\big|}\ dx\\ &= C\int_0^{\frac{1}{2}} \frac{1}{r^2|\log r|} r\ dr = C\int_{-\infty}^{\log\frac{1}{2}} \frac{1}{|a|}\ da = \infty.\end{aligned}$$ □

Theorem 3.7. *There exist* $0 < \varepsilon_0 < 1/2$ *and* $F : B([0,0],\varepsilon_0) \to \mathbb{R}$ *such that* $F \in AC^2(B([0,0],\varepsilon_0))$ *and* $\nabla F \notin L^{2,1}(B([0,0],\varepsilon_0))$.

PROOF. Set

$$g(r) = \begin{cases} \frac{1}{\ln r} r \sin\frac{1}{r} & \text{for } r \in (0,1/2),\\ 0 & \text{for } r = 0.\end{cases}$$

We claim that the function $F(x) = g(|x|)$ satisfies desired conditions if ε_0 is small enough. Plainly, $F' \in C(B([0,0],1/2) \setminus \{0\})$ and $\nabla F = F'$ a.e.

Let us first prove that $F' \notin L^{2,1}(B([0,0],\varepsilon_0))$. We compute

$$|F'(x)| = |g'(|x|)| = \Big|\frac{1}{\ln|x|}|x|\frac{-1}{|x|^2}\cos\frac{1}{|x|} + \frac{1}{\ln|x|}\sin\frac{1}{|x|} + \frac{1}{|x|}\frac{-1}{\ln^2|x|}|x|\sin\frac{1}{|x|}\Big|.$$

Let

$$M = \left\{r \in \left(0,\frac{1}{2}\right) : \left|\cos\frac{1}{r}\right| \geq \frac{1}{2}\right\} = \bigcup_{k\in\mathbb{N}} \left[\frac{1}{\frac{\pi}{3}+k\pi}, \frac{1}{-\frac{\pi}{3}+k\pi}\right]. \tag{3.1}$$

We have

$$|g'(r)| \geq \left|\frac{1}{r\ln r}\cos\frac{1}{r}\right| - \left|\frac{1}{\ln r}\sin\frac{1}{r}\right| - \left|\frac{1}{\ln^2 r}\sin\frac{1}{r}\right| \geq \frac{-1}{2r\ln r} - \left|\frac{1}{\ln r}\right| - \frac{1}{\ln^2 r}$$

for every $r \in M$. Clearly, there is $k_0 \in \mathbb{N}\setminus\{1\}$ such that for $\varepsilon_0 = \dfrac{1}{-\frac{\pi}{3}+k_0\pi}$ we have

$$|g'(r)| \geq \frac{-1}{4r\ln r} \text{ for every } r \in M\cap(0,\varepsilon_0). \tag{3.2}$$

Set

$$f(x) = \frac{-1}{4|x|\ln|x|},\ x \in B([0,0],\varepsilon_0).$$

We claim that the nonincreasing rearrangements of F' and f satisfy

$$(F')^\star(t) \geq f^\star(4t). \tag{3.3}$$

From (3.2) we have

$$|F'(x)| \geq |f(x)| \text{ for } |x| \in M\cap(0,\varepsilon_0). \tag{3.4}$$

An elementary computation gives

$$\begin{aligned} &3\mathcal{L}_2\left(\left\{x:\ |x| \in \left[\frac{1}{\frac{\pi}{3}+k\pi}, \frac{1}{-\frac{\pi}{3}+k\pi}\right]\right\}\right)\\ &> \mathcal{L}_2\left(\left\{x:\ |x| \in \left[\frac{1}{-\frac{\pi}{3}+k\pi}, \frac{1}{\frac{\pi}{3}+(k-1)\pi}\right]\right\}\right)\end{aligned} \tag{3.5}$$

for every $k \in \mathbb{N}\setminus\{1\}$. From (3.4), (3.5) and

$$[0,\varepsilon_0]\cap M = \bigcup_{k\in\mathbb{N},\ k\geq k_0} \left[\frac{1}{\frac{\pi}{3}+k\pi}, \frac{1}{-\frac{\pi}{3}+k\pi}\right]$$

we obtain $4m(\sigma,F') \geq m(\sigma,f)$. The inequality (3.3) easily follows.

Since $\int_0^{\varepsilon_0} \frac{-1}{4r\ln r}\,dr = \infty$, we have $f \notin L^{2,1}(B([0,0],\varepsilon_0))$ by Lemma 3.2. Thus (3.3) implies

$$F' \notin L^{2,1}(B([0,0],\varepsilon_0)).$$

Using Lemma 3.5 we will prove that $F \in AC^2(B([0,0],\varepsilon_0))$. Clearly,

$$\operatorname{mlip}^2(F,x) = \operatorname{mlip}^2(g,|x|).$$

For every r such that $0 < r < \varepsilon_0 < 1/e$ we have

$$\begin{aligned} |g'(r)| &= \Big|\frac{1}{\ln r} r \frac{-1}{r^2}\cos\frac{1}{r} + \frac{1}{\ln r}\sin\frac{1}{r} + \frac{1}{r}\frac{-1}{\ln^2 r} r \sin\frac{1}{r}\Big| \\ &\leq \frac{-1}{r\ln r} + \frac{-1}{\ln r} + \frac{1}{\ln^2 r} \leq \frac{-3}{r\ln r}. \end{aligned} \tag{3.6}$$

Fix r such that $r < \varepsilon_0 < 1/e$ and t such that $1/r + 2\pi \leq 1/t \leq 1/r + 4\pi$ and define $\tilde{t} = t/(1-2\pi t)$ (i.e., $1/\tilde{t} = 1/t - 2\pi$). Since the function $t/\ln t$ is decreasing on the interval $(0, 1/e)$, we obtain $|g(\tilde{t})| \geq |g(t)|$ and therefore

$$\sup_{t,\ \frac{1}{t}\in\left[\frac{1}{r}+2\pi,\frac{1}{r}+4\pi\right]} |g(r) - g(t)| \leq \sup_{\tilde{t},\ \frac{1}{\tilde{t}}\in\left[\frac{1}{r},\frac{1}{r}+2\pi\right]} |g(r) - g(\tilde{t})|.$$

Analogously, we conclude that

$$\sup_{t,\ \frac{1}{t}>\frac{1}{r}+2\pi} |g(r) - g(t)| \leq \sup_{\tilde{t},\ \frac{1}{\tilde{t}}\in\left[\frac{1}{r},\frac{1}{r}+2\pi\right]} |g(r) - g(\tilde{t})|.$$

This and $5/r > 1/r + 2\pi$ for $r < \varepsilon_0 < 1/e$ give

$$\sup_{0\leq t\leq\varepsilon_0}\left|\frac{g(r)-g(t)}{r-t}\right| = \sup_{\frac{r}{5}\leq t\leq\varepsilon_0}\left|\frac{g(r)-g(t)}{r-t}\right|. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\begin{aligned} \operatorname{mlip}(g,r) &= \sup_{0\leq t\leq\varepsilon_0}\Big|\frac{g(r)-g(t)}{r-t}\Big| \\ &= \sup_{\frac{r}{5}\leq t\leq\varepsilon_0}\Big|\frac{g(r)-g(t)}{r-t}\Big| \leq \sup_{\frac{r}{5}\leq\xi\leq\varepsilon_0} |g'(\xi)| \leq \frac{-3}{\frac{r}{5}\ln\frac{r}{5}}. \end{aligned}$$

An easy computation yields

$$\begin{aligned} \int_{B([0,0],\varepsilon_0)} \operatorname{mlip}^2(F,x) &\leq \int_{B([0,0],\varepsilon_0)} \Big(\frac{-3}{\frac{|x|}{5}\ln\frac{|x|}{5}}\Big)^2\,dx \\ &\leq C\int_0^{\varepsilon_0}\Big(\frac{1}{r\ln\frac{r}{5}}\Big)^2 r\,dr = C\int_{-\infty}^{\ln\frac{\varepsilon_0}{5}} \frac{1}{a^2}\,da < \infty. \end{aligned}$$

Therefore $F \in AC^2(B([0,0],\varepsilon_0))$ by Lemma 3.5. □

4 Boundary Behavior—Negative Results.

In this section we give examples of domains $\Omega \subset \mathbb{R}^n$ for which there is a function $f \in AC^n(\Omega)$ that fails to have a continuous extension to $\partial\Omega$ (i.e., there is no $\tilde{f} \in C(\overline{\Omega})$ such that $f = \tilde{f}$ on Ω). When Ω is bounded, this is equivalent to the fact that there is $f \in AC^n(\Omega)$ which is not uniformly continuous on Ω.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and suppose that there is $x \in \partial\Omega$ such that for all balls $B \subset \Omega$ we have $x \notin \partial B$. Then there is $f \in AC^n(\Omega)$ such that there is no continuous extension of f to $\partial\Omega$.*

Proof. This theorem is an easy consequence of Theorem 4.2. □

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $0 < R < 1$. Suppose that there is $x \in \partial\Omega$ such that $x \notin \partial B$ for every ball $B \subset \Omega$. Then there is $f \in AC^n(\Omega)$ such that $f \geq 0$, $\operatorname{spt}(f) \subset \overline{B(x,R)}$ and $\lim_{\substack{y\to x\\ y\in\Omega}} f(y) = +\infty$. Moreover, there is $g \in L^1(\Omega)$, $\operatorname{spt} g \subset \overline{B(x,R)}$ such that f satisfies the RR-condition with weight g.*

Proof. For every $m \in \mathbb{N}$ we set

$$M_m = \bigcup\Big\{B\Big(z, \frac{1}{m}\Big) : z \in \Omega, \operatorname{dist}(z, \partial\Omega) \geq \frac{1}{m}\Big\}.$$

Since it is not possible to touch $\partial\Omega$ at the point x with a ball of radius $1/m$, we have $r_m = \operatorname{dist}(x, M_m) > 0$.

Set $a_1 = R$. We define a sequence $\{a_m\}_{m=2}^{\infty}$ by induction. Given a_m, we will show that there is a_{m+1} such that $0 < a_{m+1} < a_m$ and for every ball B

$$\begin{aligned}\Big[B \cap B(x, a_{m+1}) \neq \emptyset \text{ and } B \cap \Big(B(x, a_m) \setminus B\Big(x, \frac{a_m}{2}\Big)\Big) \neq \emptyset\Big] \\ \Longrightarrow B \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset.\end{aligned} \tag{4.1}$$

Fix $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{a_m}{6}$. For every $B(z,r)$ we have

$$\begin{aligned}\Big[r \leq \frac{1}{k} \text{ and } B(z,r) \cap \Big(B(x, a_m) \setminus B\Big(x, \frac{a_m}{2}\Big)\Big) \neq \emptyset\Big] \\ \Longrightarrow B(z,r) \cap B\Big(x, \frac{a_m}{6}\Big) = \emptyset.\end{aligned} \tag{4.2}$$

We prove that (4.1) holds for $a_{m+1} = \min(a_m/6, r_k)$ by contradiction. If there were a ball $B(z,r)$ such that (4.1) failed, we would have

$$B(z,r) \cap B(x, r_k) \neq \emptyset \text{ and } B(z,r) \cap (\mathbb{R}^n \setminus \Omega) = \emptyset \Longrightarrow r \leq \frac{1}{k},$$

by the definition of r_k. From (4.2) we obtain $B(z,r)\cap B(x,a_m/6)=\emptyset$ and therefore $B(z,r)\cap B(x,a_{m+1})=\emptyset$, contrary to the assumption in (4.1).

Let f be defined for $y\in\Omega$ by

$$f(y)=\begin{cases}0 & y\in\Omega\setminus B(x,a_1),\\ \sum_{i=1}^{m-1}\frac{1}{i}+\frac{1}{m}\frac{2}{a_m}(a_m-|x-y|) & y\in B(x,a_m)\setminus B(x,\frac{a_m}{2}),\ m\in\mathbb{N},\\ \sum_{i=1}^{m}\frac{1}{i} & y\in B(x,\frac{a_m}{2})\setminus B(x,a_{m+1}),\ m\in\mathbb{N}.\end{cases}$$

Clearly, $\lim_{\substack{y\to x\\ y\in\Omega}} f(y)=+\infty$. Set

$$g(y)=\begin{cases}\left(\frac{2}{a_m m}\right)^n & y\in B(x,a_m)\setminus B(x,\frac{a_m}{2}),\ m\in\mathbb{N},\\ 0 & y\in B(x,\frac{a_m}{2})\setminus B(x,a_{m+1}),\ m\in\mathbb{N}.\end{cases}$$

From (4.1) we have

$$g(y)=\mathrm{mlip}^n(f,y)\ \text{for}\ \ y\in B(x,a_m)\setminus B\Big(x,\frac{a_m}{2}\Big),\ m\in\mathbb{N}.$$

Lemma 3.5 now gives $\mathrm{osc}_B^n f\leq C\int_B g$ for every ball $B\subset B(x,a_m)\setminus B(x,\frac{a_m}{2})$.

From (4.1) and the definition of f it is evident that for every ball $B\subset\Omega$ there is a ball $B'\subset B$ such that $\mathrm{osc}_B f=\mathrm{osc}_{B'} f$ and $B'\subset B(x,a_m)\setminus B(x,\frac{a_m}{2})$ for some $m\in\mathbb{N}$. Thus

$$\mathrm{osc}_B^n f=\mathrm{osc}_{B'}^n f\leq C\int_{B'} g(y)\ dy\leq C\int_B g(y)\ dy.$$

Hence f satisfies the RR-condition with weight Cg. An easy computation gives that

$$\begin{aligned}\int_\Omega g&\leq\sum_{m=1}^{\infty}\mathcal{L}_n\Big(B(x,a_m)\setminus B\Big(x,\frac{a_m}{2}\Big)\Big)\Big(\frac{2}{a_m m}\Big)^n\\ &\leq\sum_{m=1}^{\infty}Ca_m^n\Big(\frac{2}{a_m m}\Big)^n=C2^n\sum_{m=1}^{\infty}\Big(\frac{1}{m}\Big)^n<\infty.\end{aligned}$$ □

Example 4.3. *Let $0<\alpha<1$. There exist a domain $\Omega\subset\mathbb{R}^n$ with $C^{1,\alpha}$ boundary and $f\in AC^n(\Omega)$ such that there is no continuous extension of f to $\partial\Omega$.*

PROOF. Set $\Omega=\{[x_1,\dots,x_n]\in\mathbb{R}^n:\ \ x_1>|[x_2,\dots,x_n]|^{\alpha+1}\}$. It is not difficult to show that Ω has $C^{1,\alpha}$ boundary and that for every ball $B\subset\Omega$ we have $0\notin\partial B$. Thus Theorem 4.2 shows that there is $f\in AC^n(\Omega)$ such that there is no continuous extension of f to the point $0\in\partial\Omega$. □

The following example shows that it is not enough to assume that we can touch every point of a boundary by a ball.

Example 4.4. *There is a bounded, convex domain $\Omega \subset \mathbb{R}^2$ with C^1 boundary such that for all $x \in \partial\Omega$ we have $x \in \partial B$ for some ball $B \subset \Omega$. Moreover, there is $f \in AC^2(\Omega)$ such that there is no continuous extension of f to $\partial\Omega$.*

Proof. For $i \in \mathbb{N}_0$ set $x_i = \left[\frac{1}{2^i}, \frac{1}{2^{2i}}\right]$ and

$$A_i = \left\{[x, y] \in \mathbb{R}^2 : x \in \left[\frac{1}{2^{i+1}}, \frac{1}{2^i}\right], y = \frac{3}{2^{i+1}}x - \frac{1}{2^{2i+1}}\right\}.$$

Define $\Omega_1 = \operatorname{conv}(S)$, where we have set

$$S = \bigcup_{i=0}^{\infty} A_i \cup \{[x, y] : \ x^2 + (y-1)^2 = 1, \ y \geq 1\} \\ \cup \{[x, y] : \ x^2 + (y-1)^2 = 1, \ x \leq 0\}.$$

Clearly, there is a continuous function $h : [-1, 1] \to \mathbb{R}$ such that

$$\Omega_1 = \{[x, y] : \ x \in (-1, 1), \ h(x) < y < 1 + \sqrt{1 - x^2}\}.$$

For every $j \in \mathbb{N} \setminus \{1, 2, 3\}$ and $\frac{1}{2^j} \leq x \leq \frac{1}{2^{j-1}}$ we have

$$h(x) \leq h\Big(\frac{1}{2^{j-1}}\Big) = \frac{1}{2^{2(j-1)}} = 4\frac{1}{2^{2j}} \leq 4x^2 \leq \frac{1}{8} - \sqrt{\frac{1}{8^2} - x^2}.$$

Thus $B([0, 1/8], 1/8) \subset \Omega_1$.

Applying Theorem 4.2 to Ω_1, x_i and $r_i = \frac{1}{2^{i+3}}$ we obtain functions f_i, g_i such that $\operatorname{spt}(g_i)$, $i \in \mathbb{N}$, are pairwise disjoint. Consider $\{a_i\}_{i=0}^{\infty}$, $a_i \in \mathbb{R}$, $a_i > 0$ such that $\sum_{i=0}^{\infty} a_i \int_{\Omega_1} g_i < \infty$. Set $f = \sum_{i=0}^{\infty} a_i f_i$. Clearly, f satisfies the RR-condition with weight $g = \sum_{i=0}^{\infty} a_i g_i$ and hence $f \in AC^2(\Omega_1)$.

There are $y_i \in \Omega_1$ such that

$$\operatorname{dist}(y_i, A_i) = \operatorname{dist}(y_i, A_{i-1}) \text{ and } a_i f_i(y_i) = 1$$

and there is $y_0 \in \Omega_1$ such that

$$\operatorname{dist}(y_0, A_0) = \operatorname{dist}(y_0, \partial B([0, 1], 1)) \text{ and } a_0 f_0(y_0) = 1.$$

Let $B_i = B(y_i, \operatorname{dist}(y_i, \partial\Omega_1))$. Fix $z_i \in A_i \cap \partial B_i$ and $z \in \partial B([0, 1], 1) \cap \partial B_0$. Set

$$\Omega_1^i = \big(\Omega_1 \setminus B(x_i, |x_i - z_i|)\big) \cup B_i, \ i \in \mathbb{N},$$
$$\Omega_1^0 = \left(\Omega_1 \setminus B\left(\frac{z + z_0}{2}, \frac{|z - z_0|}{2}\right)\right) \cup B_0.$$

Let $\Omega = \bigcap_{i=0}^{\infty} \Omega_1^i$. Now Ω obviously satisfies all assumptions. Further, $f \in AC^2(\Omega)$ and there is no continuous extension of f to the point $[0,0]$ since

$$[0,y] \overset{y\to 0+}{\longrightarrow} [0,0] \text{ and } f([0,y]) \overset{y\to 0+}{\longrightarrow} 0 \text{ but } y_i \to [0,0] \text{ and } f(y_i) \to 1. \qquad \square$$

Remark 4.5. In much the same way we can prove that there is a domain $\Omega \subset \mathbb{R}^n$ with the same properties as in Example 4.4.

5 Boundary Behavior—Positive Results.

Definition 5.1. A domain $\Omega \subset \mathbb{R}^n$ is said to have the property (P) if the following holds. There are $k \in \mathbb{N}$, $\eta > 0$ and a function $h : [0,\eta) \to [0,\infty)$ such that $h(0) = 0$, h is continuous at 0, and for every $x, y \in \Omega$ satisfying $|x - y| < \eta$ we have:

$$\begin{aligned} &\text{There are balls } B_i = B(s_i, r_i) \subset \Omega, i \in \{1,\dots,k\} \text{ such that} \\ &x \in B_1,\ B_i \cap B_{i+1} \neq \emptyset \text{ for all } i \in \{1,\dots,k-1\}, \\ &y \in B_k \text{ and } \ r_i \le h(|x-y|) \text{ for all } i \in \{1,\dots,k\}. \end{aligned} \tag{5.1}$$

For abbreviation of (5.1), we say that the points x and y are joined in Ω by k balls.

Lemma 5.2. *Suppose that a domain Ω has the property (P) and let $f : \Omega \to \mathbb{R}$. Suppose that for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\big[B(c,r) \subset \Omega,\ r < \delta\big] \Rightarrow \operatorname{osc}_{B(c,r)} f < \varepsilon. \tag{5.2}$$

Then there is $\tilde{f} \in C(\overline{\Omega})$ such that $f = \tilde{f}$ on Ω.

PROOF. To obtain a contradiction, suppose that there are Ω and $f : \Omega \to \mathbb{R}$ satisfying (5.2) such that there is no continuous extension of f to the point $x \in \overline{\Omega} \setminus \Omega$. Then we can find sequences $\{a_j\}_{j=1}^{\infty} \subset \Omega$, $\{b_j\}_{j=1}^{\infty} \subset \Omega$ and $\tilde{\varepsilon} > 0$ such that

$$a_j \to x,\ b_j \to x,\ |a_j - b_j| < \eta \text{ and } |f(a_j) - f(b_j)| \ge \tilde{\varepsilon}$$

where η is occurring in the definition of the property (P). Applying (P) to points a_j, b_j we obtain balls $B_1^j, B_2^j, \dots, B_k^j$ such that

$$a_j \in B_1^j, B_i^j \cap B_{i+1}^j \neq \emptyset \text{ for } i \in \{1,\dots,k-1\} \text{ and } b_j \in B_k^j.$$

By the triangle inequality, we have

$$\tilde{\varepsilon} \le |f(a_j) - f(b_j)| \le \sum_{i=1}^{k} \operatorname{osc}_{B_i^j}(f).$$

Therefore there is $d(j) \in \{1,2,\dots,k\}$ such that $\operatorname{osc}_{B^j_{d(j)}}(f) \geq \tilde{\varepsilon}/k$. Let us denote by r_j the radius of $B^j_{d(j)}$. From $|a_j - b_j| \to 0,\ r_j \leq h(|a_j - b_j|),\ h(0) = 0$ and the continuity of h at 0 we obtain $r_j \to 0$. Hence $\operatorname{osc}_{B^j_{d(j)}}(f) \geq \tilde{\varepsilon}/k$ contradicts (5.2). □

Lemma 5.3. *Let $R > 0$ and let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose that we have a continuous curve $\gamma : [0,1] \to \Omega$ such that $\operatorname{diam}(\langle\gamma\rangle) < R$ and that for every $z \in \langle\gamma\rangle$ there is a ball $B_z = B(c_z, R) \subset \Omega$ such that $z \in B_z$. Then there are $z_1, \dots, z_{2\cdot 3^n} \in \langle\gamma\rangle$ such that $x = \gamma(0)$ and $y = \gamma(1)$ are joined by $B_{z_1}, B_{z_2}, \dots, B_{z_{2\cdot 3^n}}$ in Ω.*

Proof. Find $z_1, z_2, \dots, z_k \in \langle\gamma\rangle$ such that x and y are joined by $B_{z_1} \dots B_{z_k}$ and k is minimal in the sense

$$\left[z_1^{'},\ z_2^{'}, \dots, z_l^{'} \in \langle\gamma\rangle,\ B_{z_1^{'}}, \dots, B_{z_l^{'}} \subset \Omega \text{ join } x \text{ and } y\right] \Longrightarrow k \leq l. \tag{5.3}$$

If there were $a, b, c \in \{1, \dots, k\},\ a \neq b \neq c \neq a$ such that $B_{z_a} \cap B_{z_b} \cap B_{z_c} \neq \emptyset$, then one of the balls $B_{z_a}, B_{z_b}, B_{z_c}$ would be redundant in joining x and y which contradicts the minimality of k in the sense of (5.3). From this and $B_{z_i} \subset B(x, 3R)$ we have

$$\mathcal{L}_n\Big(\bigcup_{i=1}^{k} B_i\Big) \leq 2\mathcal{L}_n(B(x,3R)) \Rightarrow k \leq \frac{2\mathcal{L}_n(B(x,3R))}{\mathcal{L}_n(B(0,R))} = 2\cdot 3^n. \qquad \square$$

Lemma 5.4. *Given $r > 0$ and $A \subset \mathbb{R}^n$ suppose that $\Omega = \bigcup_{a\in A} B(a,r)$ is a bounded domain. Suppose that for every $z \in \partial\Omega$ and for every sequences $\{x_i\}_{i=1}^{\infty},\ \{y_i\}_{i=1}^{\infty} \subset \Omega$ we have*

$$x_i \to z,\ y_i \to z \Longrightarrow \rho_\Omega(x_i, y_i) \to 0. \tag{5.4}$$

Then Ω has the property (P).

Proof. Set

$$g(t) = \sup\{\rho_\Omega(x,y) :\ x, y \in \Omega,\ |x-y| \leq t\} \text{ for } t \geq 0.$$

We claim that the function g is continuous at 0. Conversely, suppose that there are $\delta > 0$ and $\{x_i\}_{i\in\mathbb{N}}, \{y_i\}_{i\in\mathbb{N}} \subset \Omega$ such that $|x_i - y_i| \to 0$ and $\rho_\Omega(x_i, y_i) > \delta$. Since $\overline{\Omega}$ is compact, we may assume that there is $z \in \overline{\Omega}$ such that $x_i \to z$ and $y_i \to z$. Clearly this would not be possible if $z \in \Omega$ and therefore $z \in \partial\Omega$. However this contradicts condition (5.4).

Fix $\eta > 0$ small enough such that for $t < \eta$ we have $2g(t) < r$. Set $h(t) = 2g(t)$ and $k = 2\cdot 3^n$. We claim that Ω satisfies the property (P) with the constants $k,\ \eta$ and the function h.

Fix $x, y \in \Omega$ such that $|x-y| < \eta$. It follows from the choice of η that $h(|x-y|) < r$. By the definition of $\rho_\Omega(x,y)$, there is a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\ell(\gamma) < 2\rho_\Omega(x,y)$. Clearly,

$$\operatorname{diam}(\langle\gamma\rangle) < 2\rho_\Omega(x,y) \leq 2g(|x-y|) = h(|x-y|).$$

For every $z \in \langle\gamma\rangle$ we can find $B(c_z, h(|x-y|)) \subset \Omega$ with $z \in B(c_z, h(|x-y|))$ since $\Omega = \bigcup_{a\in A} B(a,r)$ and $h(|x-y|) < r$. Applying Lemma 5.3 to $R = h(|x-y|)$ we obtain points $z_1, \dots, z_k \in \langle\gamma\rangle$ such that $B(c_{z_1}, R), \dots, B(c_{z_k}, R)$ join x and y in Ω. □

Thanks to Lemma 5.2 we can rephrase Lemma 5.4 as follows.

Theorem 5.5. *Let $A \subset \mathbb{R}^n$ and $r > 0$. Suppose that $\Omega = \bigcup_{a\in A} B(a,r)$ is a bounded domain such that for every $z \in \partial\Omega$ and for every sequences $\{x_i\}_{i=1}^\infty$, $\{y_i\}_{i=1}^\infty \subset \Omega$ we have*

$$x_i \to z,\ y_i \to z \Longrightarrow \rho_\Omega(x_i, y_i) \to 0. \tag{5.5}$$

Let $f : \Omega \to \mathbb{R}$ be a function such that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\big[B(c,r) \subset \Omega,\ r < \delta\big] \Rightarrow \operatorname{osc}_{B(c,r)} f < \varepsilon. \tag{5.6}$$

Then there is $\tilde f \in C(\overline{\Omega})$ such that $f = \tilde f$ on Ω.

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a domain with $C^{1,1}$ boundary. Then for every n-absolutely continuous function $f : \Omega \to \mathbb{R}$ there is $\tilde f \in C(\overline{\Omega})$ such that $f = \tilde f$ on Ω.*

PROOF. We only give the main ideas of the proof. We can assume that Ω is bounded, for the existence of the extension is a local property. Clearly, every n-absolutely continuous function $f : \Omega \to \mathbb{R}$ satisfies (5.6) and hence it remains to verify the assumptions of Theorem 5.5 about the domain Ω.

Let $x_0 \in \partial\Omega$ and find $r_0 > 0$, $D \subset \mathbb{R}^{n-1}$ and a function $h \in C^{1,1}(\mathbb{R}^{n-1})$ occurring in (2.2). Without loss of generality we may assume that $i = 1$, $x_0 = 0$,

$$\partial\Omega \cap B(0, r_0) = \{x \in \mathbb{R}^n :\ [x_2, \dots, x_n] \in D \text{ and } h(x_2, \dots, x_n) = x_1\},$$

$G^+ \subset \Omega$ and $G^- \cap \Omega = \emptyset$ (where G^+ and G^- are defined in (2.3)). It is clear from this description that (5.5) holds for $z = x_0$. Now it remains to show that $\Omega = \bigcup_{a\in A} B(a,r)$ for some $A \subset \mathbb{R}^n$ and $r > 0$.

Let us denote by $V \in \mathbb{R}^{n-1}$ the vector of partial derivatives of h at 0. Choose a constant $K > 0$ large enough such that K is greater than the Lipschitz constant of h' (i.e., $|h'(x) - h'(y)| \leq K|x - y|$ for every $x, y \in \mathbb{R}^{n-1}$) and moreover

$$B\Big(0, \frac{\sqrt{1+|V|^2}}{K}\Big) \subset D \text{ and } B\Big(0, \frac{\sqrt{1+|V|^2}}{K}\Big) \subset B(x_0, r_0). \tag{5.7}$$

We claim that

$$\tilde{B} := B\Big(\Big[\frac{1}{2K}, \frac{-V_1}{2K}, \dots, \frac{-V_{n-1}}{2K}\Big], \frac{1}{2K}\sqrt{1+|V|^2}\Big) \subset \Omega. \tag{5.8}$$

Let $x \in \partial\tilde{B}\setminus\{0\}$. Set $\tilde{x} = [x_2, \dots, x_n]$ and notice that $\tilde{x} \in D$ and $x \in B(x_0, r_0)$ by (5.7). From (5.8) we have $|x|^2 = \frac{1}{K}x_1 - \frac{1}{K}V\tilde{x}$. Proposition 2.1 now gives

$$h(\tilde{x}) \leq V\tilde{x} + \frac{K}{2}|\tilde{x}|^2 < V\tilde{x} + K|x|^2 = x_1$$

which implies $x \in \Omega$ since $G^+ \subset \Omega$. Clearly $\partial\tilde{B} \subset \Omega \cup \{0\}$, implies $\tilde{B} \subset \Omega$. Note that the radius of $\tilde{B}$ depends only on h, r_0 and D, and not on a particular point x_0. Therefore it is possible to find $\tilde{r}_0 > 0$ and $r_1 > 0$ such that for every $x \in \partial\Omega \cap B(x_0, \tilde{r}_0)$ there exists a ball $B(c_x, r_1) \subset \Omega$ such that $x \in \partial B(c_x, r_1)$.

Since $\partial\Omega$ is compact, this implies that there is $r_2 > 0$ such that for every $x \in \partial\Omega$ there is a ball $B(c_x, r_2) \subset \Omega$ such that $x \in \partial B(c_x, r_2)$. From this and the definition of $C^{1,1}$ boundary it is not difficult to deduce that $\Omega = \bigcup_{a\in A} B(a, r)$ for some $A \subset \mathbb{R}^n$ and $r > 0$. □

The following example shows that the assumptions of Lemma 5.4 are not equivalent to the property (P).

Example 5.7. *There is a bounded domain $\Omega \subset \mathbb{R}^2$ which has the property (P) and does not satisfy the assumptions of Lemma 5.4.*

Proof. Set

$$A = \big\{[x, y] : \; x^2 + (y-1)^2 = 1 \text{ and } \big((x \leq 0) \text{ or } (y \geq 1)\big)\big\}$$

and

$$B_i = B\left(\left[\frac{1}{2^i}, \frac{1}{2^i} + \frac{1}{8\;2^{2i}}\right], \frac{1}{2^i}\right).$$

We claim that the domain $\Omega = \operatorname{conv}\Big(A \cup \bigcup_{i=1}^{\infty} B_i\Big)$ has the desired properties. Since $\partial B_i \cap \partial\Omega \neq \emptyset$ and $\operatorname{diam} B_i \to 0$, we have $\Omega \neq \bigcup_{a\in A} B(a, r)$ for any $r > 0$ and $A \subset \mathbb{R}^2$. Thus Ω does not satisfy the assumptions of Lemma 5.4. The proof of the property (P) for Ω is straightforward and not difficult and hence we omit it. □

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