A p-ADIC CONSTRUCTION OF ATR POINTS ON $\mathbb{Q}$-CURVES

Xavier Guitart and Marc Masdeu

Abstract: In this note we consider certain elliptic curves defined over real quadratic fields isogenous to their Galois conjugate. We give a construction of algebraic points on these curves defined over almost totally real number fields. The main ingredient is the system of Heegner points arising from Shimura curve uniformizations. In addition, we provide an explicit p-adic analytic formula which allows for the effective, algorithmic calculation of such points.

1. Introduction

At the beginning of the 2000's Darmon introduced two constructions of local points on modular elliptic curves over number fields: the Stark–Heegner points [**D1**] and the ATR points [**D2**, Chapter 8]. Both types of points are expected to be algebraic and to behave in many aspects as the more classical Heegner points. Although the two constructions bear some formal resemblances, a crucial difference lies in the nature of the local field involved: while the former is p-adic, the later is archimedean. In order to explain the importance of this distinction, let us briefly recall the constructions and some of the features that are currently known about them.

Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N, and let K be a real quadratic field such that the sign of the functional equation of $L(E/K, s)$ is -1. Let p be a prime that exactly divides N and which is inert in K. Under an additional Heegner-type hypothesis, Stark–Heegner points in $E(K_p)$ are constructed in [**D1**] by means of certain p-adic line integrals, and they are conjectured to be global and defined over narrow ring class fields of K. This construction was generalized by Greenberg [**G2**] to the much broader setting in which E is defined over a totally real number field F of narrow class number 1, the extension K/F

is any non-CM quadratic extension in which some prime divisor of the conductor of E is inert, and $L(E/K, s)$ has sign -1.

There is extensive numerical evidence in support of the rationality of such p-adic points (cf. [**DG**], [**DP**], [**GM1**], [**GM2**]), but the actual proof in general seems to be still far out of reach. In spite of this, in some very special cases Stark–Heegner points are known to be global. In these particular settings they coexist with Heegner points, and they can actually be seen to be related to them (cf. [**BD2**], [**LV**], [**GSS**]). The p-adic nature of the points seems to play a key role in these arguments, by means of the connection between the formal logarithm of the Stark–Heegner points and the special values of suitable p-adic L-functions (see also [**BDP**], [**DR1**], [**DR2**], and [**BDR**]).

The archimedean counterparts to these points, as introduced in [**D2**, Chapter 8] and later generalized by Gärtner [**Gär**], seem to be even more mysterious. The simplest and original setting is that of an elliptic curve E defined over a totally real number field F, and M/F a quadratic *Almost Totally Real* (ATR) extension (i.e., M has exactly one complex place). In this case the points are constructed by means of complex integrals and thus they lie in $E(\mathbb{C})$. They are also expected to be global, and there is some numerical evidence of it [**DL**], [**GM1**].

However, in the archimedean constructions it is not (the logarithm of) the points which is expected to be related to special values of complex L-functions, but their heights, very much in the spirit of Gross–Zagier formulas. It is this crucial difference with the p-adic case what seems to prevent the success of any attempt of showing their rationality, even in the very particular instances in which they coexist with Heegner points. It could be arguably said that complex ATR points are much more difficult to handle than their p-adic counterparts. Thus, even in the simplest situations in which one wants to compare them with Heegner points in order to show their rationality, it is desirable to have p-adic constructions of such points at one's disposals.

In light of the above discussion, the goal of the present paper is to present a p-adic construction of algebraic points defined over ATR fields. To be more precise, we consider a real quadratic field F and a non-CM elliptic curve E/F that is F-isogenous to its Galois conjugate (this is sometimes referred to as a $\mathbb{Q}$-curve in the literature). Suppose that M/F is a quadratic ATR extension such that the sign of the functional equation of $L(E/M, s)$ is -1. We describe a p-adic construction of algebraic points in $E(M)$, which are manufactured by means of suitable Heegner points in a certain Shimura curve parametrizing E.

The points that we construct are algebraic (for they essentially come from Heegner points in certain modular abelian varieties) and given in terms of *p*-adic line integrals. Observe that in this set up one can also consider *p*-adic Stark–Heegner points, e.g., the ones constructed by Greenberg [**G2**], and it would be very interesting to investigate the possible relationship between these two types of points. Of course, it would also be of interest to compare them with the ATR points constructed by Darmon and Gärtner, although as explained before in this case only information about the heights of the points seems to be directly available from the involved *L*-functions.

The fact that our construction is given in terms of *p*-adic line integrals also has another consequence, which constitutes in fact one of the remarkable features of the construction: it gives rise to a completely explicit and efficient algorithm for computing the points.

Our construction is inspired by the work of Darmon, Rotger, and Zhao [**DRZ**]. Since it builds on this work, the next section is devoted to recalling the points introduced in [**DRZ**], as well as to giving an overview of the rest of the paper.

2. Background and outline of the construction

Our construction can be seen as a generalization of that of [**DRZ**]. In order to put it in context, it is illustrative to examine first the case of elliptic curves over $\mathbb{Q}$. So let us (temporarily) denote by E an elliptic curve over $\mathbb{Q}$ of conductor N. The Modularity Theorem [**Wil**], [**TW**], [**BCDT**] provides a non-constant map

$$(2.1) \qquad\qquad \pi_E \colon X_0(N) \longrightarrow E,$$

where $X_0(N)$ denotes the modular curve parametrizing cyclic isogenies $C \to C'$ of degree N. This moduli interpretation endows $X_0(N)$ with a canonical set of algebraic points known as CM or Heegner points which give rise, when projected under π_E, to a systematic construction of algebraic points on E.

To be more precise, suppose that K is a quadratic imaginary field and $\mathcal{O} \subset K$ is an order of discriminant coprime with N. In addition, suppose that K satisfies the *Heegner condition*:

(H) All the primes dividing N are split in K.

Under this assumption, there exist elliptic curves C and C' with complex multiplication by $\mathcal{O}$, together with a cyclic isogeny $C \rightarrow C'$ of degree N. The theory of complex multiplication implies that the point in $X_0(N)$ corresponding to $C \rightarrow C'$ is, in fact, algebraic and defined over the ring class field of $\mathcal{O}$.

Moreover, the corresponding Heegner point on E can be computed by means of the complex uniformization derived from (2.1) which, in view of the identifications $X_0(N)(\mathbb{C}) \simeq \Gamma_0(N)\backslash(\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))$ and $E(\mathbb{C}) \simeq \mathbb{C}/\Lambda_E$, is of the form

$$(2.2) \qquad \pi_E \colon \Gamma_0(N)\backslash(\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})) \longrightarrow \mathbb{C}/\Lambda_E.$$

The formula for computing the Heegner point corresponding to $C \rightarrow C'$ is then

$$(2.3) \qquad \Phi_{\mathrm{W}}\left(\int_\tau^{i\infty} 2\pi i f_E(z)\, dz\right),$$

where $f_E(z) = \sum_{n \geq 1} a_n e^{2\pi i n z}$ denotes the weight two newform for $\Gamma_0(N)$ whose L-function equals that of E, the map $\Phi_W \colon \mathbb{C}/\Lambda_E \rightarrow E(\mathbb{C})$ is the Weierstrass uniformization, and $\tau \in \mathcal{H} \cap K$ is such that $C \simeq \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and $C' \simeq \mathbb{C}/\mathbb{Z} + N\tau\mathbb{Z}$.

This type of Heegner points are one of the main ingredients intervening in the proof of the Birch and Swinnerton-Dyer conjecture for curves over $\mathbb{Q}$ of analytic rank at most 1 [**GZ**], [**Kol**]. Moreover, and perhaps more relevant to the purpose of the present note, the formula (2.3) is completely explicit and computable, as the Fourier coefficients a_n can be obtained by counting points on the several reductions of E (mod p). In other words, (2.3) provides with an effective algorithm for computing points on E over abelian extensions of K, which turn out to be of infinite order whenever the analytic rank is 1. See, e.g., [**E1**] for a discussion of this method and examples of computations.

Suppose now that K does not satisfy the Heegner condition, and factor N as $N = N^+N^-$, where N^+ contains the primes that split in K and N^- those that are inert (we assume that the discriminant of K is coprime to N). In this case there is a generalization of the above Heegner point construction, which works under the less restrictive *Heegner–Shimura condition*:

(H') N^- is squarefree and the product of an even number of primes.

In this set up, when $N^- > 1$ the map (2.1) is replaced by a uniformization of the form

$$(2.4) \qquad\qquad \pi_E^{N^-} : X_0(N^+, N^-) \longrightarrow E,$$

where $X_0(N^+, N^-)$ is the Shimura curve of level N^+ associated to the indefinite quaternion algebra $B/\mathbb{Q}$ of discriminant N^-. The moduli interpretation of $X_0(N^+, N^-)$, combined with the theory of complex multiplication, can also be used to construct Heegner points on E that are defined over ring class fields of orders $\mathcal{O} \subset K$ of conductor coprime to N.

There is also an analogue of formula (2.3), but in this case it seems to be much more difficult to compute in practice. Indeed, one needs to integrate modular forms associated to B and, since B is division, the Shimura curve $X_0(N^+, N^-)$ has no cusps. Therefore the corresponding modular forms do not admit Fourier expansions, which are the crucial tool that allow for the explicit calculation of (2.3). Elkies developed methods for performing such computations under some additional hypothesis [**E2**]. More recently Voight–Willis [**VW**] using Taylor expansions and Nelson [**Nel**] using the Shimizu lift have been able to compute some of these CM points.

In a different direction, there is an alternative method that allows for the numerical calculation of Heegner points associated to quaternion division algebras. The key idea is to fix a prime $p \mid N^-$ and to use the rigid analytic p-adic uniformization derived from (2.4), instead of the complex one. The Čerednik–Drinfel'd theorem provides a model for $X_0(N^+, N^-)$ as the quotient of the p-adic upper half plane $\mathcal{H}_p$ by Γ, a certain subgroup in a definite quaternion algebra. Bertolini and Darmon [**BD1**], building on previous work of Gross [**Gro**], give an explicit formula for the uniformization map

$$\Gamma \backslash \mathcal{H}_p \longrightarrow E(\mathbb{C}_p)$$

in terms of the so-called multiplicative p-adic line integrals of rigid analytic modular forms for Γ (see also [**Mok**] for a generalization to curves over totally real fields). Such integrals can be very efficiently computed, thanks to the methods of M. Greenberg [**G1**] (which adapt Pollack–Stevens' overconvergent modular symbols technique [**PS**]) and to the explicit algorithms provided by Franc–Masdeu [**FM**].

Let us now return to the setting that we consider in the present note. Namely, F is a real quadratic field and E/F is an elliptic curve without complex multiplication that is F-isogenous to its Galois conjugate. As

a consequence of Serre's modularity conjecture [**KW1**], [**KW2**] and re-
sults of Ribet [**R3**], E can be parametrized by a modular curve of the
form $X_1(N)$, where the integer N is related to the conductor $\mathfrak{N}_E$ of E/F
(cf. §3.1 for more details on the precise relation). This property was ex-
ploited by Darmon–Rotger–Zhao in [**DRZ**] in order to construct certain
algebraic ATR points on E by means of Heegner points on $X_1(N)$. Let
us briefly explain the structure of the construction.

Consider the uniformization mentioned above

$$(2.5) \qquad\qquad \pi_E \colon X_1(N) \longrightarrow E.$$

We assume, for simplicity, that N is squarefree. We remark that π_E is
defined over F. Let M/F be a quadratic extension that has one com-
plex and two real places (this is what is known as an Almost Totally
Real (ATR) extension, because it has exactly one complex place). There
is a natural quadratic imaginary field K associated to M as follows: if
$M = F(\sqrt{\alpha})$ for some $\alpha \in F$, then $K = \mathbb{Q}(\sqrt{\mathrm{Nm}_{F/\mathbb{Q}}(\alpha)})$. Suppose that
K satisfies the following Heegner-type condition, which might be called
the Heegner–Darmon–Rotger–Zhao condition:

(**DRZ**) All the primes dividing N are split in K.

Under this assumption, the method presented in [**DRZ**] uses Heegner
points on $X_1(N)$ associated to orders in K to construct points in $E(M)$,
which are shown to be of infinite order in situations of analytic rank
one. One of the salient features of this construction is that it is explicitly
computable. In fact, there is a formula analogous to (2.3), giving the
points as integrals of certain classical modular forms for $\Gamma_1(N)$.

In the first part of the paper, which consists of Sections 3 to 5, we
extend the construction of [**DRZ**] to the situation in which K, whose
discriminant is assumed to be coprime to N, satisfies the following, less
restrictive, Heegner–Shimura-type condition (where, as before, we write
$N = N^+N^-$ with N^+ containing the primes that split in K and N^-
containing those that remain inert):

(**DRZ'**) N^- is squarefree and the product of an even number of primes.

As we will see, this condition is satisfied whenever $L(E/M, s)$ has
sign -1 (see Proposition 3.6 below). In particular, it is satisfied when
the analytic rank of E/M is 1.

The idea of our construction, inspired by the case of curves over $\mathbb{Q}$
reviewed above, consists in replacing (2.5) by a uniformization of the
form

$$(2.6) \qquad\qquad \pi_E^{N^-} \colon X_1(N^+, N^-) \longrightarrow E,$$

where $X_1(N^+, N^-)$ is a suitable Shimura curve attached to an indefinite quaternion algebra $B/\mathbb{Q}$ of discriminant N^- and level structure "of Γ_1-type". This main construction of ATR points in $E(M)$ is presented in §5, after developing some preliminary results. Namely, in §3 we briefly review $\mathbb{Q}$-curves and we prove some results in Galois theory that relate certain ring class fields of K with M, and in §4 we define the CM points on the Shimura curves that will play a role in our construction and determine their field of definition.

Just as in the classical case of curves over $\mathbb{Q}$, the CM points in $X_1(N^+, N^-)$ (and hence the points that we construct in $E(M)$) are difficult to compute using the complex uniformization. Once again, the absence of cusps in $X_1(N^+, N^-)$ and thus the lack of Fourier coefficients makes it difficult to compute the integrals that appear in the explicit formula (cf. (5.6) below).

The second part of the article gives a p-adic version of the construction. As has been mentioned in the introduction, this might be useful in order to relate it to p-adic Stark–Heegner points. Another advantage of this p-adic construction is that it is explicitly computable. Concretely, in §6 we exploit the p-adic uniformization of $X_1(N^+, N^-)$ given by the Čerednik–Drinfel'd theorem and the explicit uniformization of Bertolini–Darmon in terms of multiplicative p-adic integrals which, combined with a slight generalization of the algorithms of Franc–Masdeu [**FM**], provides an efficient algorithm for computing algebraic ATR points in $\mathbb{Q}$-curves. We conclude with an explicit example of such computation in §7.

3. $\mathbb{Q}$-curves and ATR extensions

In this section we recall some basic facts on $\mathbb{Q}$-curves and their relation with classical modular forms for $\Gamma_1(N)$. We also give some preliminary results on certain Galois extensions associated to ATR fields that will be needed in the subsequent sections, as they will be related to the field of definition of the Heegner points under consideration.

3.1. $\mathbb{Q}$-curves and modular forms. An elliptic curve E over a number field is said to be a $\mathbb{Q}$-*curve* if it is isogenous to all of its Galois conjugates. One of the motivations for studying $\mathbb{Q}$-curves is that they arise as the 1-dimensional factors over $\overline{\mathbb{Q}}$ of the modular abelian varieties attached to classical modular forms. More precisely, let $f = \sum_{n \geq 1} a_n q^n$ be a normalized newform of weight two for $\Gamma_1(N)$, and denote by $K_f = \mathbb{Q}(\{a_n\})$ the number field generated by its Fourier coefficients. Let A_f be the abelian variety over $\mathbb{Q}$ associated to f by Shimura in [**Shi**, Theorem 7.14]. The dimension of A_f is equal to $[K_f : \mathbb{Q}]$ and $\mathrm{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q}$,

its algebra of endomorphisms defined over $\mathbb{Q}$, is isomorphic to K_f. In particular, A_f is simple over $\mathbb{Q}$. However, it is not necessarily absolutely simple and, in general, it decomposes up to $\overline{\mathbb{Q}}$-isogeny as $A_f \sim_{\overline{\mathbb{Q}}} C^n$, for some $n \geq 1$ and some absolutely simple abelian variety $C/\overline{\mathbb{Q}}$ which is isogenous to all of its Galois conjugates. Therefore, if C turns out to be of dimension 1, then C is a $\mathbb{Q}$-curve. Conversely, as a consequence of Serre's modularity conjecture and results of Ribet, any $\mathbb{Q}$-curve can be obtained, up to isogeny, by this construction.

We will be interested in certain $\mathbb{Q}$-curves defined over quadratic fields. In the next proposition we characterize them in terms of the modular construction.

Proposition 3.1. *Let $f \in S_2(\Gamma_1(N))$ be a non-CM newform of Nebentypus ψ, and let F be the field associated to ψ (by identifying ψ with a character of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via class field theory). Suppose that ψ has order 2 and that $[K_f : \mathbb{Q}] = 2$. Then:*

(1) *F is real quadratic;*
(2) *K_f is imaginary;*
(3) *A_f is F-isogenous to E^2, where E/F is a $\mathbb{Q}$-curve.*

In addition, if N is odd or squarefree then the conductor $\mathfrak{N}_E$ of E/F is generated by a rational integer, say $\mathfrak{N}_E = N_0 \mathcal{O}_F$ for some $N_0 \in \mathbb{Z}_{\geq 0}$, and $N = N_0 \cdot N_\psi$ where N_ψ stands for the conductor of ψ.

Proof: The field F is real because ψ, being the Nebentypus of a modular form of even weight, is an even character.

The Nebentypus induces the complex conjugation on the Fourier coefficients:

$$(3.1) \qquad a_p = \overline{a}_p \psi(p), \text{ for almost all primes } p \text{ (cf., e.g., } [\mathbf{R1}, \S 1]).$$

Since ψ has order 2 it is non-trivial, which implies that K_f is imaginary.

Property (3.1) also implies that ψ is an inner twist of f, in the sense of $[\mathbf{R2}, \S 3]$. In fact, it is the only non-trivial inner twist because $[K_f : \mathbb{Q}] = 2$. By $[\mathbf{GL}, \text{Proposition 2.1}]$ F is the smallest number field where all the endomorphisms of A_f are defined. If A_f were absolutely simple, then $\mathrm{End}_F(A_f) \otimes \mathbb{Q}$ would be isomorphic to a quaternion division algebra over $\mathbb{Q}$ by $[\mathbf{Pyl}, \text{Proposition 1.3}]$. But in that case the minimal field of definition of all the endomorphisms of A_f would be quadratic imaginary by $[\mathbf{Rot}, \text{Lemma 2.3 (i)}]$, so this case is not possible. We conclude that A_f is not absolutely simple, so that A_f is F-isogenous to E^2, where E/F is a $\mathbb{Q}$-curve.

Suppose now that N is odd or squarefree. Then, by the main theorem in $[\mathbf{GG}, \text{p. 2}]$ one has that the conductor $\mathfrak{N}_E$ of E/F is of the

form $N_0\mathcal{O}_F$ for some $N_0 \in \mathbb{Z}_{\geq 0}$, and the relation $N_0 N_F = N$ is satisfied, where N_F stands for the conductor of F. But $N_F = N_\psi$ by the conductor-discriminant formula [**Was**, Theorem 3.11], and this finishes the proof. □

From now on we will assume that E/F is an elliptic curve obtained as in the above proposition. That is to say, it is the absolutely simple factor of A_f for some non-CM newform f whose level N is odd or squarefree, its Nebentypus is of order 2, and its field of Fourier coefficients is quadratic. In addition, in the case where N is not squarefree we will also assume that $(N_0, N_\psi) = 1$, since this condition will be needed in Proposition 3.6. One can easily find many examples of modular forms satisfying these conditions, for instance by consulting the table [**Que**, §4.1].

Remark 3.2. Observe that the above assumptions imply that N_ψ is squarefree. This is clear if N is squarefree. If N is odd, it follows from the fact that the conductor of a quadratic character is squarefree away from 2.

3.2. ATR extensions. Let M/F be a quadratic almost totally real (ATR) extension of discriminant prime to $\mathfrak{N}_E$, the conductor of E, and such that the L-function $L(E/M, s)$ has sign -1. This condition is equivalent (see, e.g., the discussion of [**D2**, §3.6]) to the set

$$(3.2) \qquad \{\mathfrak{p} \mid \mathfrak{N}_E : \mathfrak{p} \text{ is inert in } M\}$$

having even cardinality.

We have that $M = F(\sqrt{\alpha})$ for some $\alpha \in F$. We set $M' = F(\sqrt{\alpha'})$, where α' stands for the Galois conjugate of α. Then $\mathcal{M} = MM'$ is the Galois closure of M and its Galois group $\mathrm{Gal}(\mathcal{M}/\mathbb{Q})$ is isomorphic to $D_{2\cdot4}$, the dihedral group of 8 elements. The diagram of subfields of $\mathcal{M}$ is of the form

(3.3)

where $K = \mathbb{Q}(\sqrt{\alpha\alpha'})$. Observe that K is a quadratic imaginary field, for M is ATR and necessarily $\alpha\alpha' = \mathrm{Nm}_{F/\mathbb{Q}}(\alpha) < 0$. From now on, we will assume that the discriminant of K is relatively prime to N.

We will see that all the primes dividing N_ψ are split in K (see Lemma 3.5). We consider a decomposition of N of the form $N = N^+ N^-$, where

- $N^+ = N_\psi N_0^+$, and N_0^+ contains the primes $\ell \mid N_0$ such that ℓ splits in K, and
- N^- is squarefree and contains the primes $\ell \mid N_0$ such that ℓ is inert in K.

As we already mentioned in the introduction, one of the central ideas of [**DRZ**] is that Heegner points on A_f can be used to manufacture points on $E(M)$. Indeed, an explicit such construction is provided in [**DRZ**, §4], under the assumption that $\mathfrak{N}_E = (1)$. Such construction, in fact, is easily seen to be valid under the following slightly more general Heegner-type condition:

(**DRZ**) $N^- = 1$ (i.e., all the primes dividing N are split in K).

Let us briefly review the structure of the construction in this case (we refer to [**DRZ**] for the details). Let us (temporarily) denote by $\Gamma_0(N)$ the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of upper triangular matrices modulo N, and by $\Gamma_\psi(N)$ the congruence subgroup

$$\Gamma_\psi(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : N \mid c,\ \psi(a) = 1 \right\} \subset \Gamma_0(N).$$

Let $X_0(N)$ (resp. $X_\psi(N)$) denote the modular curve associated to $\Gamma_0(N)$ (resp. to $\Gamma_\psi(N)$), and let $J_0(N)$ (resp. $J_\psi(N)$) denote its Jacobian. The variety $A_f/\mathbb{Q}$ turns out to be a quotient of $J_\psi(N)/\mathbb{Q}$. Since A_f is isogenous over F to E^2, it follows that E admits a morphism (defined over F) from $J_\psi(N)$. Therefore we obtain a uniformization

$$(3.4) \qquad\qquad\qquad J_\psi(N) \longrightarrow E$$

which is defined over F.

On the other hand, the inclusion $\Gamma_\psi(N) \subset \Gamma_0(N)$ induces a degree 2 map $X_\psi(N) \to X_0(N)$, and the Heegner points in $X_\psi(N)$ are the preimages of the Heegner points in $X_0(N)$. Denote by $M_0(N) \subset \mathrm{M}_2(\mathbb{Z})$ the set of matrices which are upper triangular modulo N. An embedding $\varphi \colon K \hookrightarrow M_2(\mathbb{Q})$ is said to be of conductor c and level N if $\varphi^{-1}(M_0(N))$ is equal to $\mathcal{O}_c$, the order of conductor c. The Heegner points in $X_0(N)$ associated to $\mathcal{O}_c$ are in one to one correspondence with the optimal embeddings of level N and conductor c. They are defined over its ring class field H_c, so that their preimages in $X_\psi(N)$ are defined over a certain

quadratic extension L_c of H_c (see §4.2 for the details). This gives rise to Heegner points in $J_\psi(N)$ defined over L_c.

One of the results proved in [**DRZ**] is that, for suitable choices of c, the extension L_c contains L (cf. §3.4 for a generalization of this result). By taking the trace from L_c down to L one obtains a point in $J_\psi(N)(L)$. Summing it with its conjugate by an appropriate element in $\mathrm{Gal}(\mathcal{M}/\mathbb{Q})$ produces a point on $J_\psi(N)(M)$. Finally, projecting to E via (3.4) yields the point on $E(M)$.

The reason why the construction outlined above only works under the hypothesis that $N^- = 1$ is that, otherwise, there do not exist optimal embeddings $\varphi\colon K \hookrightarrow M_2(\mathbb{Q})$ of conductor c and level N. That is to say, there are no Heegner points in $X_0(N)$ defined over ring class fields of K.

The main goal of the present article is to provide a construction of Heegner points on $E(M)$ in the case $N^- > 1$. For that purpose, and similarly to the classical case of Heegner points on curves over $\mathbb{Q}$, we need to consider Heegner points coming from Shimura curves attached to division quaternion algebras. In the next section we introduce the Shimura curves that will play the role of $X_\psi(N)$ in our construction, and we discuss Heegner points on them.

Before that, we state some Galois properties of the fields in Diagram (3.3) and about certain number fields L_c, attached to orders in K of conductor c that will be the fields of definition of Heegner points. We also introduce some more notation that will be in force for the rest of the article.

3.3. Galois properties and the number of primes dividing N^-.
In this subsection we study those properties of the field Diagram (3.3) that are needed later. Let

$$\chi_M, \chi'_M\colon G_F \longrightarrow \{\pm 1\}$$

denote the quadratic characters of $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ cutting out the extensions M and M', respectively. Observe that we can, and often do, view them as characters on the ideles $\mathbb{A}_F^\times$. Similarly we define the characters

$$\chi_L, \chi'_L\colon G_K \longrightarrow \{\pm 1\},$$

and view them as characters of $\mathbb{A}_K^\times$.

Remark 3.3. Observe that $M = F(\sqrt{\alpha})$ by construction, and $M' = F(\sqrt{\alpha'})$, where α' denotes the $\mathrm{Gal}(F/\mathbb{Q})$-conjugate of α. In particular, if p is a prime that splits in F, say as $p\mathcal{O}_F = \mathfrak{p}\mathfrak{p}'$, then $\chi_M(\mathfrak{p}) = \chi'_M(\mathfrak{p}')$, for the splitting behavior of $\mathfrak{p}$ in M/F is the same as that of $\mathfrak{p}'$ in M'/F. A similar observation applies for L.

We also denote by ε_F and ε_K the quadratic characters on $\mathbb{A}_{\mathbb{Q}}^{\times}$ corresponding to F and K, and by

$$\mathrm{Nm}_{F/\mathbb{Q}}\colon \mathbb{A}_F^{\times} \longrightarrow \mathbb{A}_{\mathbb{Q}}^{\times}, \quad \mathrm{Nm}_{K/\mathbb{Q}}\colon \mathbb{A}_K^{\times} \longrightarrow \mathbb{A}_{\mathbb{Q}}^{\times}$$

the norms on the ideles. Observe that, as remarked above, F is the field cut out by ψ. This means that, in fact, $\varepsilon_F = \psi$.

We will make use of the following properties of Diagram (3.3), which are given in Proposition 3.2 of [**DRZ**].

Lemma 3.4.   (1) $\chi_M \cdot \chi_M' = \varepsilon_K \circ \mathrm{Nm}_{F/\mathbb{Q}}$ and $\chi_L \cdot \chi_L' = \varepsilon_F \circ \mathrm{Nm}_{K/\mathbb{Q}}$.
  (2) *The restriction of χ_M and χ_M' to $\mathbb{A}_{\mathbb{Q}}^{\times}$ is ε_K, and the restriction of χ_L and χ_L' to $\mathbb{A}_{\mathbb{Q}}^{\times}$ is ε_F.*
  (3) $\mathrm{Ind}_F^{\mathbb{Q}} \chi_M = \mathrm{Ind}_K^{\mathbb{Q}} \chi_L$.

Let $\mathfrak{d}_{L/K}$ denote the discriminant of the extension L/K, which by the conductor-discriminant formula is the conductor of χ_L.

Lemma 3.5.   (1) *There exists an ideal $\mathfrak{N}_{\psi,L} \subset \mathcal{O}_K$ of norm N_ψ canonically attached to (ψ, L). In particular, all primes dividing N_ψ are split in K.*
  (2) *One may write $\mathfrak{d}_{L/K} = c \cdot \mathfrak{N}_{\psi,L}$, where c belongs to $\mathbb{Z}$ and is coprime to N.*

Proof: From the equality $\mathrm{Ind}_F^{\mathbb{Q}} \chi_M = \mathrm{Ind}_K^{\mathbb{Q}} \chi_L$, using the formula for the conductor of induced representations and the conductor-discriminant formula, we obtain

$$(3.5) \qquad N_\psi \cdot \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{d}_{M/F}) = \mathrm{disc}(K) \cdot \mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{d}_{L/K}).$$

From this it follows that $N_\psi \mid \mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{d}_{L/K})$, since N is coprime to $\mathrm{disc}(K)$. If we write

$$\mathfrak{d}_{L/K} = \prod_{\substack{\mathfrak{p} \mid \mathfrak{d}_{L/K} \\ \mathfrak{p} \nmid N_\psi}} \mathfrak{p}^{e_\mathfrak{p}} \cdot \prod_{\substack{\mathfrak{p} \mid \mathfrak{d}_{L/K} \\ \mathfrak{p} \mid N_\psi}} \mathfrak{p}^{e_\mathfrak{p}}$$

then we have (by setting $p_\mathfrak{p} = \mathfrak{p} \cap \mathbb{Z}$):

$$\mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{d}_{L/K}) = \prod_{\substack{\mathfrak{p} \mid \mathfrak{d}_{L/K} \\ \mathfrak{p} \nmid N_\psi}} p_\mathfrak{p}^{e_\mathfrak{p}'} \cdot \prod_{\substack{\mathfrak{p} \mid \mathfrak{d}_{L/K} \\ \mathfrak{p} \mid N_\psi}} p_\mathfrak{p}^{e_\mathfrak{p}'}, \text{ and } N_\psi = \prod_{\substack{\mathfrak{p} \mid \mathfrak{d}_{L/K} \\ \mathfrak{p} \mid N_\psi}} \mathfrak{p}^{e_\mathfrak{p}'}.$$

Thanks to our running assumption that $(N, \mathrm{disc}(K)) = 1$, for every $\mathfrak{p} \mid N_\psi$ we either have that $\mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{p}) = p_\mathfrak{p}$ when $p_\mathfrak{p}$ is split (in which case $e_\mathfrak{p}' = e_\mathfrak{p}$), or that $\mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{p}) = p_\mathfrak{p}^2$ when $p_\mathfrak{p}$ is inert (in which case

$e'_{\mathfrak{p}} = 2e_{\mathfrak{p}}$). Since N_ψ is squarefree (see Remark 3.2), it must be $e'_{\mathfrak{p}} = 1$ so that $e_{\mathfrak{p}} = 1$ and $p_{\mathfrak{p}}$ is split. Therefore

$$N_\psi = \mathrm{Nm}_{K/\mathbb{Q}}\left(\prod_{\substack{\mathfrak{p}|\mathfrak{d}_{L/K}\\ \mathfrak{p}|N_\psi}} \mathfrak{p}\right),$$

which proves the first assertion by putting $\mathfrak{N}_{\psi,L} = \prod_{\mathfrak{p}|\mathfrak{d}_{L/K},\mathfrak{p}|N_\psi} \mathfrak{p}$.

As for the second part of the lemma, we consider primes p dividing $\mathrm{Nm}_{K/\mathbb{Q}}(\mathfrak{d}_{L/K})$. Suppose first that $p \mid N_\psi$. By the first part p splits in K, say as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. Then $\mathcal{O}_{K,\mathfrak{p}} \simeq \mathbb{Z}_p$, and by part (3.4) of Lemma 3.4 the composition

$$\mathbb{Z}_p^\times \longrightarrow \mathcal{O}_{K,\mathfrak{p}}^\times \times \mathcal{O}_{K,\bar{\mathfrak{p}}}^\times \xrightarrow{\chi_{L,\mathfrak{p}}\cdot\chi_{L,\bar{\mathfrak{p}}}} \{\pm 1\}$$

is equal to ψ_p (the local component of ψ at p), which is non trivial because $p \mid N_\psi$. But since $\chi_{L,\mathfrak{p}}$, $\chi_{L,\bar{\mathfrak{p}}}$ are quadratic characters, then necessarily exactly one them is trivial on $\mathbb{F}_p^\times \simeq \mathcal{O}_{K,\mathfrak{p}}^\times/(1 + \mathfrak{p}) \simeq \mathcal{O}_{K,\bar{\mathfrak{p}}}^\times/(1 + \bar{\mathfrak{p}})$, say $\chi_{L,\bar{\mathfrak{p}}}$. Then $\mathfrak{p}$ divides exactly the conductor of χ_L (which is equal to $\mathfrak{d}_{L/K}$), and $\bar{\mathfrak{p}}$ does not divide it.

Now suppose that $p \nmid N_\psi$. That is to say, ψ_p is trivial on $\mathbb{Z}_p^\times$. Let $\mathfrak{p} \mid p$ be a prime in K such that $\mathfrak{p}^e$ divides exactly the conductor of χ_L. The composition

$$\mathbb{Z}_p^\times \longrightarrow \mathcal{O}_{K,\mathfrak{p}}^\times \xrightarrow{\chi_{L,\mathfrak{p}}} \{\pm 1\}$$

is equal to ψ_p, which is trivial on $\mathbb{Z}_p^\times$. If p was ramified in K, then the above map would restrict to

$$\mathbb{Z}_p^\times/(1 + p^e\mathbb{Z}_p) \xrightarrow{\simeq} \mathcal{O}_{K,\mathfrak{p}}^\times/(1 + \mathfrak{p}^e\mathcal{O}_{K,\mathfrak{p}}) \xrightarrow{\chi_{L,\mathfrak{p}}} \{\pm 1\},$$

contradicting the non-triviality of $\chi_{L,\mathfrak{p}}$ restricted to $\mathcal{O}_{K,\mathfrak{p}}^\times/(1 + \mathfrak{p}^e\mathcal{O}_{K,\mathfrak{p}})$. Hence we see that p cannot ramify. If p is inert in K there is nothing to prove, because $\mathfrak{p}^e = p^e\mathcal{O}_K$ is already a rational ideal. If p splits in K, say as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, then $\chi_{L,\bar{\mathfrak{p}}} \simeq \chi_{L,\mathfrak{p}}^{-1}$ on $\mathbb{Z}_p^\times$, implying that $\bar{\mathfrak{p}}^e$ exactly divides the conductor of χ_L, because $\mathcal{O}_{K,\bar{\mathfrak{p}}} \simeq \mathbb{Z}_p \simeq \mathcal{O}_{K,\mathfrak{p}}$.

We have seen that $\mathfrak{d}_{L/K} = c \cdot \mathfrak{N}_{\psi,L}$ with $c \in \mathbb{Z}$. It remains to prove that c is coprime to N. Recall that $N = N_\psi N_0$, where N_0 is a generator of the conductor $\mathfrak{N}_E$ of E. From what we have seen in the proof so far, it is clear that $(c, N_\psi) = 1$. Recall also our running assumption that $\mathfrak{N}_E$ is coprime to $\mathfrak{d}_{M/K}$ (cf. §3.2), which implies that $(N_0, \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{d}_{M/K})) = 1$. From (3.5) we see that $c \mid \mathrm{Nm}_{F/\mathbb{Q}}(\mathfrak{d}_{M/F})$, and therefore $(c, N_0) = 1$. $\square$

The Heegner points that we will use in our construction arise from a Shimura curve associated to an indefinite algebra of discriminant N^-. Therefore, the following result is key to our purposes.

Proposition 3.6. *The number of primes dividing N^- is even.*

Proof: Recall that $\mathfrak{N}_E = N_0 \mathcal{O}_F$ and that the set

(3.6) $\{\mathfrak{p} \mid \mathfrak{N}_E : \mathfrak{p} \text{ is inert in } M\}$

has even cardinality thanks to our running assumption that $L(E/M, s)$ has sign -1. Every prime in the set (3.6) is above a prime $p \mid N_0$. Thus, in order to prove the proposition it is enough to prove the following claims.

Claim 1. Every prime $p \mid N_0^+$ gives rise to either zero or two primes in (3.6).

Claim 2. Every prime $p \mid N_0^- = N^-$ gives rise to exactly one prime in (3.6).

Proof of Claim 1: Let p be a prime dividing N_0^+. Namely, p is a prime divisor of N_0 that splits in K. Observe that p can not ramify in F because of our assumption that $(N_\psi, N_0) = 1$. If p splits in F, say $p\mathcal{O}_F = \mathfrak{p}\mathfrak{p}'$, by Remark 3.3 and part (1) of Lemma 3.4 we see that

$$\chi_M(\mathfrak{p}) \cdot \chi_M(\mathfrak{p}') = \chi_M(\mathfrak{p}) \cdot \chi'_M(\mathfrak{p}) = \varepsilon_K(\mathrm{Nm}_{\mathbb{Q}}^F(\mathfrak{p})) = \varepsilon_K(p) = 1,$$

so that either both $\mathfrak{p}$ and $\mathfrak{p}'$ are inert in M, or both are split. In other words, either $\mathfrak{p}$ and $\mathfrak{p}'$ belong to (3.6), or none of them does.

If p remains inert in F, by part (2) of Lemma 3.4 we have that

$$\chi_M(p\mathcal{O}_F) = \varepsilon_K(p) = 1,$$

which means that $p\mathcal{O}_F$ is split in M, so that it does not belong to (3.6).

Proof of Claim 2: Let p be a prime dividing $N_0^- = N^-$. Again there are two possibilities.

(1) If p is split in F, say $p\mathcal{O}_F = \mathfrak{p}\mathfrak{p}'$, then by part (1) of Lemma 3.4 we have that

$$\chi_M(\mathfrak{p})\chi_M(\mathfrak{p}') = \chi_M(\mathfrak{p})\chi'_M(\mathfrak{p}) = \varepsilon_K(\mathrm{Nm}_{\mathbb{Q}}^F(\mathfrak{p})) = \varepsilon_K(p) = -1,$$

so exactly one of the primes above p is inert in M and therefore belongs to (3.6).

(2) If p is inert in F, then by part (2) of Lemma 3.4 we see that

$$\chi_M(p\mathcal{O}_F) = \varepsilon_K(p) = -1,$$

and so $p\mathcal{O}_F$ is inert in M.

$\square$

3.4. The field L_c. The aim of this subsection is to define an extension L_c of L, associated to ψ and to the order of conductor c in K. It will turn out to be the field of definition of the Heegner points that we will consider in §4.

Recall that from Lemma 3.5 the discriminant of L/K factors as

$$\mathfrak{d}_{L/K} = c \cdot \mathfrak{N}_{\psi,L},$$

where c is a rational integer with $(c, N) = 1$ and $\mathfrak{N}_{\psi,L}$ is an ideal in K of norm N_ψ. Let $N^+ = N_\psi N_0^+$ and let $\mathfrak{N}^+$ be an ideal of K of norm N^+, such that $\mathfrak{N}_{\psi,L} \mid \mathfrak{N}^+$. We remark that $\mathfrak{N}_{\psi,L}$ is determined by (ψ, L), while there is some freedom in the choice of $\mathfrak{N}^+$. We denote by $\overline{\mathfrak{N}}^+$ its complex conjugate.

Let H_c/K be the ring class field of K of conductor c. Denote by $\mathbb{A}_K$ the adeles of K, and by $\hat{\mathcal{O}}_K = \prod_{\mathfrak{p}} \mathcal{O}_{K,\mathfrak{p}} \subset \mathbb{A}_{K,\mathrm{fin}}$. The reciprocity map of class field theory provides an identification $\mathrm{Gal}(H_c/K) \simeq \mathbb{A}_K^\times/(K^\times U_c)$, where

$$U_c = \hat{\mathbb{Z}}^\times (1 + c\hat{\mathcal{O}}_K)\mathbb{C}^\times \subset \mathbb{A}_K^\times.$$

For an idele $\alpha = \prod_{\mathfrak{p}} \alpha_{\mathfrak{p}}$ and an ideal $\mathfrak{m}$, we denote by $(\alpha)_{\mathfrak{m}}$ the product $\prod_{\mathfrak{p}|\mathfrak{m}} \alpha_{\mathfrak{p}}$. Following [**DRZ**, §4.1] we define

$$U_c^0 = \{\alpha \in U_c : (\alpha)_{\mathfrak{N}^+} \in \ker(\psi) \subset (\mathbb{Z}/N^+\mathbb{Z})^\times\},$$
$$\overline{U}_c^0 = \{\alpha \in U_c : (\alpha)_{\overline{\mathfrak{N}}^+} \in \ker(\psi) \subset (\mathbb{Z}/N^+\mathbb{Z})^\times\}.$$

Here we are using the fact that $\mathfrak{N}^+$ has norm N^+, so that we have isomorphisms

$$\mathcal{O}_{K,\mathfrak{N}^+}^\times/(1 + \mathfrak{N}^+ \mathcal{O}_{K,\mathfrak{N}^+}) \simeq (\mathbb{Z}/N^+\mathbb{Z})^\times,$$
$$\mathcal{O}_{K,\overline{\mathfrak{N}}^+}^\times/(1 + \overline{\mathfrak{N}}^+ \mathcal{O}_{K,\overline{\mathfrak{N}}^+}) \simeq (\mathbb{Z}/N^+\mathbb{Z})^\times,$$

where the notation $\mathcal{O}_{K,\mathfrak{N}^+}$ stands for $\prod_{\mathfrak{p}|\mathfrak{N}^+} \mathcal{O}_{K,\mathfrak{p}}$ and similarly for $\mathcal{O}_{K,\overline{\mathfrak{N}}^+}$.

Let L_c and L'_c be the fields corresponding by class field theory to U_c^0 and $\overline{U}_c^0$ respectively. That is to say

$$(3.7) \qquad \mathrm{Gal}(L_c/K) \simeq \mathbb{A}_K^\times/(K^\times U_c^0), \quad \mathrm{Gal}(L'_c/K) \simeq \mathbb{A}_K^\times/(K^\times \overline{U}_c^0).$$

Both L_c and L'_c are quadratic extensions of H_c, and we denote by $\tilde{H}_c$ the biquadratic extension of K given by $\tilde{H}_c = L_c L'_c$.

Lemma 3.7. *If c is the one given by Lemma 3.5, then L is contained in L_c. Therefore LL' is contained in $\tilde{H}_c$.*

Proof: By class field theory it is enough to show that U_c^0 is contained in $\ker \chi_L$. Recall that the conductor of χ_L is equal to $\mathfrak{d}_{L/K}$ and hence equal to $c\mathfrak{N}_{\psi,L}$, with $\mathfrak{N}_{\psi,L} \mid \mathfrak{N}^+$. This means that $\chi_{L|\widehat{\mathcal{O}}_K^\times}$ factors through a character

$$(3.8) \qquad \chi_L \colon \mathcal{O}_{K,c\mathfrak{N}^+}^\times / (1 + c\mathfrak{N}^+ \mathcal{O}_{K,c\mathfrak{N}^+}) \longrightarrow \{\pm 1\}.$$

Let (α) be a finite idele of K that belongs to U_c^0. We aim to see that $\chi_L(\alpha) = 1$. Since α belongs to U_c, we can write it as $\alpha = a(1 + cx)$ for some $a \in \hat{\mathbb{Z}}^\times$ and some $x \in \hat{\mathcal{O}}_K$. Locally, we can express this as

$$\alpha = a(1 + cx) = a \prod_{\mathfrak{p} \nmid c\mathfrak{N}^+} x_{\mathfrak{p}} \prod_{\mathfrak{p}|c} (1 + \mathfrak{p}^{v_{\mathfrak{p}}(c)} x_{\mathfrak{p}}) \prod_{\mathfrak{p}|\mathfrak{N}^+} x_{\mathfrak{p}}.$$

By (3.8) we see that

$$\chi_L \left(\prod_{\mathfrak{p} \nmid c\mathfrak{N}^+} x_{\mathfrak{p}} \prod_{\mathfrak{p}|c} (1 + \mathfrak{p}^{v_{\mathfrak{p}}(c)} x_{\mathfrak{p}}) \right) = 1.$$

Therefore, we see that

$$\chi_L(\alpha) = \chi_L \left(a \prod_{\mathfrak{p}|\mathfrak{N}^+} x_{\mathfrak{p}} \right) = \chi_L \left(\prod_{\mathfrak{p}|\mathfrak{N}^+} a_{\mathfrak{p}} x_{\mathfrak{p}} \right) \chi_L \left(\prod_{\mathfrak{p}|c} a_{\mathfrak{p}} \right).$$

Since $\prod_{\mathfrak{p}|c} a_{\mathfrak{p}}$ lies in $\mathbb{A}_{\mathbb{Q}}^\times$, we have that

$$\chi_L \left(\prod_{\mathfrak{p}|\mathfrak{N}^+} a_{\mathfrak{p}} x_{\mathfrak{p}} \right) \chi_L \left(\prod_{\mathfrak{p}|c} a_{\mathfrak{p}} \right) = \psi \left(\prod_{\mathfrak{p}|\mathfrak{N}^+} a_{\mathfrak{p}} x_{\mathfrak{p}} \right) \psi \left(\prod_{\mathfrak{p}|c} a_{\mathfrak{p}} \right)$$

$$= \psi((\alpha)_{\mathfrak{N}^+}) \psi \left(\prod_{\mathfrak{p}|c} a_{\mathfrak{p}} \right) = \psi \left(\prod_{\mathfrak{p}|c} a_{\mathfrak{p}} \right).$$

But $\prod_{\mathfrak{p}|c} a_{\mathfrak{p}} \in \prod_{\mathfrak{p}|c} \mathbb{Z}_{\mathfrak{p}_p}^\times$ (where $\mathfrak{p}_p = \mathfrak{p} \cap \mathbb{Z}$). Since the conductor of ψ is N_ψ, which is coprime to c, we have that

$$\psi \left(\prod_{\mathfrak{p}|c} a_{\mathfrak{p}} \right) = 1,$$

as we aimed to show.   □

4. CM points on Shimura curves with quadratic character

In this section we recall some basic facts and well-known properties of Shimura curves. We also introduce the CM points that will play a

key role in our construction of points in $E(M)$ later in §5, and we use Shimura's reciprocity law to deduce their field of definition.

Let $\mathcal{B}/\mathbb{Q}$ be the quaternion algebra of discriminant N^-. Thanks to Proposition 3.6 we see that $\mathcal{B}$ is indefinite so we can, and do, fix an isomorphism

$$\iota_\infty \colon \mathcal{B} \otimes_\mathbb{Q} \mathbb{R} \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{R}).$$

Choose $\mathcal{R}_0 = \mathcal{R}_0(N^+, N^-)$ an Eichler order of level N^+ in $\mathcal{B}$ together with, for every prime $\ell \mid N^+$, an isomorphism

$$\iota_\ell \colon \mathcal{B} \otimes_\mathbb{Q} \mathbb{Q}_\ell \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{Q}_\ell)$$

such that

$$\iota_\ell(\mathcal{R}_0) \simeq \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{M}_2(\mathbb{Z}_\ell) : c \in \ell\mathbb{Z}_\ell \right\}.$$

In this way we also obtain an isomorphism

$$\iota_{N^+} \colon \mathcal{R}_0 \otimes \mathbb{Z}_{N^+} \simeq \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{M}_2(\mathbb{Z}_{N^+}) : c \in N^+\mathbb{Z}_{N^+} \right\},$$

where $\mathbb{Z}_{N^+} = \prod_{p|N^+} \mathbb{Z}_p$. Let $\eta \colon \mathcal{R}_0 \to \mathbb{Z}_{N^+}/N^+\mathbb{Z}_{N^+}$ be the map that sends γ to the upper left entry of $\iota_{N^+}(\gamma)$ taken modulo N^+. The character ψ can be regarded in a natural way as a character $\psi \colon \mathbb{Z}_{N^+}/N^+\mathbb{Z}_{N^+} \to \{\pm 1\}$. Let $\mathcal{U}_0 = \mathcal{R}_0^\times$ be the group of units in $\mathcal{R}_0$, and define

$$(4.1) \qquad \mathcal{U}_\psi = \{\gamma \in \mathcal{U}_0 : \psi \circ \eta(\gamma) = 1\}.$$

Let also Γ_0 (resp. Γ_ψ) denote the subgroup of norm 1 elements in $\mathcal{U}_0$ (resp. $\mathcal{U}_\psi$).

4.1. Shimura curves. Let $X_0 = X_0(N^+, N^-)$ be the Shimura curve associated to Γ_0. Similarly, let $X_\psi = X_\psi(N^+, N^-)$ be the Shimura curve associated to Γ_ψ. See [**BC**, Chapitre III] for the precise moduli description. They are curves over $\mathbb{Q}$, whose complex points can be described as

$$(4.2) \qquad X_0(\mathbb{C}) \simeq \Gamma_0 \backslash \mathcal{H}, \quad X_\psi(\mathbb{C}) \simeq \Gamma_\psi \backslash \mathcal{H},$$

where $\mathcal{H}$ denotes the complex upper half plane, and Γ_0 and Γ_ψ act on $\mathcal{H}$ via ι_∞. The inclusion $\Gamma_\psi \subset \Gamma_0$ induces a degree 2 homomorphism defined over $\mathbb{Q}$

$$\pi_\psi \colon X_\psi \longrightarrow X_0.$$

4.2. CM points. Let c be an integer relatively prime to N and to the discriminant of K, and let $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K$ be the order of conductor c in K. An algebra embedding $\varphi \colon \mathcal{O}_c \hookrightarrow \mathcal{R}_0$ is said to be an *optimal embedding of conductor c* if $\varphi(K) \cap \mathcal{R}_0 = \varphi(\mathcal{O}_c)$. Recall also the ideal $\mathfrak{N}^+ \subset K$ of norm N^+ that we fixed in §3.4, and that we denote by $\overline{\mathfrak{N}}^+$ its complex conjugate.

Definition 4.1. We say that an optimal embedding $\varphi \colon \mathcal{O}_c \hookrightarrow \mathcal{R}_0$ is normalized with respect to $\mathfrak{N}^+$ if it satisfies that

(1) $\iota_\infty(\varphi(a)) \left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right) = a \left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right)$ for all $a \in \mathcal{O}_c$ and all $\tau \in \mathbb{C}$ (here we view $K \subset \mathbb{C}$), and

(2) $\ker(\eta \circ \varphi) = \mathfrak{N}^+$.

We denote by $\mathcal{E}(c, \mathcal{R}_0)$ the set of normalized embeddings with respect to $\mathfrak{N}^+$.

Remark 4.2. Observe that we do not impose any normalization at the primes dividing N^- (in particular, the Galois action that we will introduce below on the set of normalized embeddings will not be transitive). But the involutions at primes dividing the discriminant of the algebra do not play any role for the applications of the present note.

The groups Γ_0 and Γ_ψ act on $\mathcal{E}(c, \mathcal{R}_0)$ by conjugation, and we denote by $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_0$ and $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi$ the corresponding (finite) sets of conjugacy classes. Each $\varphi \in \mathcal{E}(c, \mathcal{R}_0)$ has a unique fixed point τ_φ in $\mathcal{H}$. The image of τ_φ in $\Gamma_0 \backslash \mathcal{H} \simeq X_0(\mathbb{C})$ (resp. in $\Gamma_\psi \backslash \mathcal{H} \simeq X_\psi(\mathbb{C})$) only depends on the class of φ in $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_0$ (resp. $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi$). We will denote the point defined by τ_φ in the Shimura curve by $[\tau_\varphi]$. The points obtained in this way are the so-called *CM points* or *Heegner points*.

We denote by $\mathrm{CM}_0(c)$ the set of *CM points of conductor c* corresponding to optimal embeddings normalized with respect to $\mathfrak{N}^+$. That is to say

$$\mathrm{CM}_0(c) = \{[\tau_\varphi] \in X_0(\mathbb{C}) : \varphi \in \mathcal{E}(c, \mathcal{R}_0)/\Gamma_0\}.$$

Similarly, we denote by $\mathrm{CM}_\psi(c)$ their preimage under π_ψ, which can be described as

$$\mathrm{CM}_\psi(c) = \{[\tau_\varphi] \in X_\psi(\mathbb{C}) : \varphi \in \mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi\}.$$

From now on we identify $\mathrm{CM}_0(c)$ with $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_0$ and $\mathrm{CM}_\psi(c)$ with $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi$ (this is possible because the association $[\varphi] \mapsto [\tau_\varphi]$ is injective). Every element in $\mathrm{CM}_0(c)$ has two preimages in $\mathrm{CM}_\psi(c)$, which are interchanged by the action of any element $W_\psi \in \Gamma_0 \smallsetminus \Gamma_\psi$.

There is an action $\star$ of $\hat{K}^\times$ on $\mathcal{E}(c, \mathcal{R}_0)$, given as follows. For any $x = (x_\mathfrak{p})_\mathfrak{p} \in \hat{K}^\times$ and $\varphi \in \mathcal{E}(c, \mathcal{R}_0)$, the fractional ideal $\hat{\varphi}(x)\hat{\mathcal{R}}_0 \cap \mathcal{B}$ is principal,

say generated by $\gamma_x \in \mathcal{B}^\times$. Let $a_x = \hat{\varphi}(x_{\mathfrak{N}+})^{-1}\gamma_x$. Observe that $a_{x,\mathfrak{p}} \in \mathcal{R}_0^\times$ for every $\mathfrak{p} \mid \mathfrak{N}^+$, and therefore it makes sense to consider $\psi \circ \eta(a_x)$. Modifying each $\gamma_{x,\mathfrak{p}}$ by a unit if necessary and by strong approximation we can assume that γ_x is chosen in such a way that $\psi \circ \eta(a_x) = 1$. That is to say, $\hat{\varphi}(x_{\mathfrak{N}+})^{-1}\gamma_x$ lies in the kernel of $\psi \circ \eta$. Then $x \star \varphi$ is defined as $x \star \varphi := \gamma_x^{-1}\varphi\gamma_x$.

By results of Shimura CM points are defined over K^{ab}, the maximal abelian extension of K. The Galois action on them is given in terms of the reciprocity map of class field theory

$$\mathrm{rec}\colon \hat{K}^\times/K^\times \longrightarrow \mathrm{Gal}(K^{ab}/K)$$

by means of *Shimura's reciprocity law* (cf. [**Shi**, Theorem 9.6]):

$$(4.3) \qquad \mathrm{rec}(x)^{-1}([\tau_\varphi]) = [\tau_{x\star\varphi}].$$

Here the action in the left is the usual Galois action on the $\overline{\mathbb{Q}}$-points of a variety defined over $\mathbb{Q}$. One of its well known consequences is that $\mathrm{CM}_0(c) \subset X_0(H_c)$, i.e. CM points of conductor c on X_0 are defined over the ring class field of conductor c. One can also derive from it the field of definition of $\mathrm{CM}_\psi(c)$, which is precisely the field L_c defined in §3.4.

Proposition 4.3. $\mathrm{CM}_\psi(c) \subset X_\psi(L_c)$.

Proof: It follows directly from (4.3) and the fact that U_c^0 acts trivially on $\mathcal{E}(c, \mathcal{R}_0)/\Gamma_\psi$. $\qquad\square$

5. ATR points on $\mathbb{Q}$-curves

In this section we introduce the main construction of this note, namely an ATR point in E manufactured by means of CM points on X_ψ. To this end, let us briefly recall the setting of §3 and some of the results encountered so far. The initial data is a classical modular form $f = f_E \in S_2(N, \psi)$ such that N is odd or squarefree, ψ is of order 2, and its field of Fourier coefficients K_f is quadratic imaginary. Then the modular abelian variety A_f is F isogenous to the square of a $\mathbb{Q}$-curve E, which is defined over the real quadratic field F corresponding to ψ. In fact, E/F is characterized by the equality of L-functions

$$(5.1) \qquad L(E/F, s) = \prod_{\sigma\colon K_f \hookrightarrow \mathbb{C}} L(f, s).$$

Let M be a quadratic ATR extension of F such that $L(E/M, s)$ has sign -1. This gives rise to a quadratic imaginary extension K, sitting in the field Diagram (3.3). We assume that level N factorizes as $N = N^+N^-$, where N^+ is supported on the primes that split in K and N^- is the squarefree product of an even number of primes that are inert in K.

By Lemma 3.5 the discriminant of L/K factorizes as $c\mathfrak{N}_{\psi,L}$, with $c \in \mathbb{Z}$ and $\mathfrak{N}_{\psi,L} \subset K$ an integral ideal of norm N_ψ. Recall also that we fixed an ideal $\mathfrak{N}^+$ of norm N^+ with $\mathfrak{N}_{\psi,L} \mid \mathfrak{N}^+$. Recall also $\mathrm{CM}_\psi(c)$, the set of Heegner points of conductor c (and normalized with respect to $\mathfrak{N}^+$), which lie in $X_\psi(L_c)$.

5.1. Construction of the ATR point. Recall that, as we have seen before, by Simura's reciprocity law if $\tau \in \mathrm{CM}_\psi(c)$ then $\tau \in X_\psi(L_c)$. Next, we describe how to attach to each such Heegner point τ a point $P_\tau \in E(M)$.

Let $S_2(\Gamma_\psi) = S_2(\Gamma_\psi(N^+, N^-))$ denote the space of weight two newforms with respect to Γ_ψ. Thanks to the Jacquet–Langlands correspondence there exists a newform $g \in S_2(\Gamma_\psi)$ such that $L(g,s) = L(f,s)$. In other words, g has the same system of eigenvalues by the Hecke operators as f. In addition, if we let $J_\psi = \mathrm{Jac}(X_\psi)$ there exists a surjective homomorphism defined over $\mathbb{Q}$ (see [**YZZ**, §1.2.3])

$$(5.2) \qquad\qquad \pi_f \colon J_\psi \longrightarrow A_f.$$

The map π_f is given explicitly in terms of integrals of g and its conjugate (cf. §5.2 below).

The next step is to associate to τ a divisor of degree 0 on X_ψ, hence a point in J_ψ that we can project to A_f. In the case of classical modular curves, i.e. when $N^- = 1$, the usual procedure is to use the embedding $X_\psi \hookrightarrow J_\psi$ given by the choice of the rational cusp ∞ as base point. That is to say, the degree 0 divisor attached to τ would be $(\tau) - (\infty)$. However, when $N^- > 1$ the Shimura curve X_ψ does not have cusps. In this case, Zhang (cf. [**YZZ**, §1.2.2]) uses the map $\phi \colon X_\psi \to J_\psi$ that sends τ to $\tau - \xi$, where $\xi \in \mathrm{Pic}(X_\psi) \otimes \mathbb{Q}$ is the so called Hodge class. Then the point $P'_\tau = \pi_f \circ \phi(\tau)$ belongs to $A_f(L_c)$.

Before continuing with the construction of the point P_τ, it is worth mentioning how P'_τ (or rather a closely related point) can be computed in practice because this will be used in the explicit calculations of §7. For this we follow a remark of [**Mok**, §4.4], based on the fact that ξ is a divisor of degree 1 satisfying that $T_\ell \xi = (\ell+1)\xi$ for all primes ℓ (here T_ℓ stands for the ℓ-th Hecke operator). Thanks to the Hecke-equivariance of π_f, if we let a_ℓ denote the ℓ-th Hecke eigenvalue of f we have that

$$\begin{aligned}
(\ell+1-a_\ell)\pi_f(\phi(\tau)) &= \pi_f((\ell+1-T_\ell)\phi(\tau)) \\
&= \pi_f((\ell+1-T_\ell)(\tau-\xi)) \\
&= \pi_f((\ell+1-T_\ell)\tau).
\end{aligned}$$

This gives an expression for $(\ell + 1 - a_\ell)P'_\tau$, in which we are regarding a_ℓ as an endomorphism of A_f via the identification $K_\ell \simeq \mathrm{End}_{\mathbb{Q}}(A_f) \otimes \mathbb{Q}$.

Returning to the construction of P_τ, recall that by Lemma 3.7 the field L_c contains L. Then we define

$$P_{\tau,L} = \mathrm{Tr}_{L_c/L}(P'_\tau) \in A_f(L).$$

Our running assumption that the sign of the functional equation for E/M is -1 implies that $L(E/M, 1) = 0$. Next, we apply the Gross–Zagier-type formula of [**YZZ**] in order to show that $P_{\tau,L}$ is of infinite order in analytic rank one situations. The natural case to consider is when $L(E/F, 1) \neq 0$, because otherwise the non-torsion point on E would already be defined over F.

Proposition 5.1. *If $L(E/F, 1) \neq 0$ and $L'(E/M, 1) \neq 0$ then $P_{\tau,L}$ is non-torsion.*

Proof: From the basic equality

$$L(E/M, s) = L(E/F, s)L(E/F, \chi_M, s)$$

we see that $L(E/F, \chi_M, 1) = 0$, because of the assumption $L(E/F, 1) \neq 0$. The derivative of the above expression, together with the assumption that $L'(E/M, 1) \neq 0$ also implies that $L'(E/F, \chi_M, 1) \neq 0$. By (5.1), the Artin formalism, and Lemma 3.4 we have that

$$L(E/F, \chi_M, s) = L(f/F \otimes \chi_M, s) = L(f \otimes \mathrm{Ind}_F^{\mathbb{Q}} \chi_M, s)$$
$$= L(f \otimes \mathrm{Ind}_K^{\mathbb{Q}} \chi_L, s) = L(f/K \otimes \chi_L, s),$$

from which we obtain that $L'(f/K \otimes \chi_L, 1) \neq 0$.

The modular form f gives rise to a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ whose central character ω_π is the Nebentypus ψ of f. Since $\omega_\pi \cdot \chi_{L|\mathbb{A}_{\mathbb{Q}}^\times} = 1$ by Lemma 3.4, we are in the position of applying the Gross–Zagier-type formula of [**YZZ**, §1.3.2]. For that, starting with the Heegner point $P'_\tau \in A_f(L_c)$ and regarding χ_L as a character of $\mathrm{Gal}(L_c/K)$ we set

$$(5.3) \qquad P_\tau^{\chi_L} := \sum_{\sigma \in \mathrm{Gal}(L_c/K)} \chi_L(\sigma)^{-1} \cdot \sigma(P'_\tau).$$

Then [**YZZ**, Theorem 1.2] expresses the Neron–Tate heigh of $P_\tau^{\chi_L}$ as a non-zero multiple of $L'(f/K \otimes \chi_L, 1)$. This implies that $P_\tau^{\chi_L}$ is non-torsion. But now, since χ_L is the character corresponding to the quadratic extension L/K, we can write

$$P_\tau^{\chi_L} = \sum_{\sigma \in \mathrm{Gal}(L_c/L)} \sigma(P'_\tau) + \sum_{\sigma \in \mathrm{Gal}(L_c/L)} (-1) \cdot (s\sigma)(P'_\tau),$$

where $s \in \mathrm{Gal}(L_c/K)$ is an element that induces the nontrivial automorphism of L/K. This can be written as

$$(5.4) \qquad P_\tau^{\chi_L} = P_{\tau,L} - s(P_{\tau,L}),$$

and we see that $P_{\tau,L}$ must be non-torsion as well. □

Now let τ_M denote the element in $\mathrm{Gal}(\mathcal{M}/\mathbb{Q})$ whose fixed field is M. We define

$$P_{\tau,M} = P_{\tau,L} + \tau_M(P_{\tau,L}),$$

which belongs to $A_f(M)$. We have the following consequence of Proposition 5.1.

Proposition 5.2. *If $L(E/F,1) \neq 0$ and $L'(E/M,1) \neq 0$ then $P_{\tau,M}$ is non-torsion.*

Proof: The key property is that $P_{\tau,L}$ is defined over L, but it is not defined over K (this follows from (5.4): if it was defined over K, then $P_\tau^{\chi_L}$ would be 0 because s would fix it, but under the hypothesis of the proposition $P_\tau^{\chi_L}$ is non-torsion). The same is true for any multiple $n \cdot P_{\tau,L}$. Since $\tau_M(L) = L'$, we see that $\tau_M(P_{\tau,L})$ is defined over L' (and is not defined over K). But now, if $P_{\tau,L} + \tau_M(P_{\tau,L})$ was torsion, say of order n, we would have that

$$nP_{\tau,L} = -n\tau_M(P_{\tau,L}),$$

which is a contradiction because the point in the left is defined over L, and the point on the right is defined over L'. □

Finally, in order to define P_τ, recall that A_f is F-isogenous to E^2. In particular, $\mathbb{Q} \otimes_{\mathbb{Z}} A_f(M) \simeq \mathbb{Q} \otimes_{\mathbb{Z}} E(M) \times \mathbb{Q} \otimes_{\mathbb{Z}} E(M)$, so that we can choose a projection $\pi_E \colon A_f \to E$ defined over F such that

$$P_\tau = \pi_E(P_{\tau,M}) \in E(M)$$

is of infinite order when $P_{\tau,M}$ is.

Remark 5.3. Observe that the projection π_E is not uniquely determined by the above condition. However, this does not affect the construction in a sensible way because the property for a point being defined over M or being of infinite order is not affected by isogenies defined over F.

One of the main motivations for the construction of the point P_τ is that it extends the construction of [**DRZ**] to the case $N^- > 1$. However, a nice feature of the setting considered in [**DRZ**] is that in that case the points can be effectively computed (cf. the explicit formula of [**DRZ**, Theorem 4.6]) as suitable integrals of the classical modular form f. In our situation, however, the equivalent computation seems to be more

difficult, because the modular forms involved are *quaternionic modular forms*. This is the issue that we address in the next paragraph. As we will see in §6, the effective computation of P_τ can be accomplished by using p-adic methods.

5.2. Complex uniformization and Heegner points. The projection map π_f of (5.2) is given by a generalization of the classical Eichler–Shimura construction (cf. [**D2**, §4]). In this context, the quaternionic modular form g gives rise to a differential form $\omega_g \in H^0(X_\psi, \Omega^1)$. Recall that g is obtained via the Jacquet–Langlands correspondence from an elliptic modular form f. Denote by f' the modular form whose Fourier coefficients are the complex conjugates of those of f, and let g' denote the modular form with respect to Γ_ψ corresponding to f' by Jacquet–Langlands. Observe that ω_g and ω_g' are determined by this construction only up to multiplication by scalars, but they can be normalized so as to satisfy that $\{\omega_g, \omega_g'\}$ is a basis of the space of differential 1-forms defined over F. Let $\Phi = \Phi_{N^+, N^-}$ be the map

$$\Phi \colon \mathrm{Div}^0(\mathcal{H}) \longrightarrow \mathbb{C} \times \mathbb{C}$$
$$z_2 - z_1 \longmapsto \left(\int_{z_1}^{z_2} \omega_g, \int_{z_1}^{z_2} \omega_{\overline{g}} \right).$$

The subgroup generated by the images under Φ of divisors which become trivial in $\Gamma_\psi \backslash \mathcal{H}$ is a lattice $\Lambda_g \subset \mathbb{C} \times \mathbb{C}$, and $\mathbb{C}^2/\Lambda_g$ is isogenous to $A_f(\mathbb{C})$. This gives the following analytic description of π_f:

$$\Phi \colon \mathrm{Div}^0(\mathcal{H}/\Gamma_\psi) \longrightarrow A_f(\mathbb{C})$$
(5.5)
$$z_2 - z_1 \longmapsto \left(\int_{z_1}^{z_2} \omega_g, \int_{z_1}^{z_2} \omega_{\overline{g}} \right).$$

Suppose that $D = \tau_2 - \tau_1 \in \mathrm{Div}^0 \, \mathrm{CM}_\psi(c)$. We see that the point $\pi_f(D) \in A_f(L_c)$ is given, in complex analytic terms, by the formula

(5.6) $$\pi_f(D) = \left(\int_{\tau_1}^{\tau_2} \omega_g, \int_{\tau_1}^{\tau_2} \omega_{\overline{g}} \right) \in \mathbb{C}^2/\Lambda_g \simeq A_f(\mathbb{C}).$$

The effective computation of the above integrals, however, turns out to be difficult in general when $\mathcal{B}$ is a division algebra, because the newforms in $S_2(\Gamma_\psi)$ cannot be expressed as a Fourier expansion at the cusps. In the next section, and modeling on the classical case of newforms in $S_2(\Gamma_0)$, we will see that the points P_τ defined in §5.1 can be computed via p-adic uniformization, instead of complex uniformization.

6. p-adic uniformization and CM points

If p is a prime dividing N^- the abelian varieties $J_0 = \mathrm{Jac}(X_0)$ and $J_\psi = \mathrm{Jac}(X_\psi)$ admit rigid analytic uniformizations at p. That is to say, there exist free groups of finite rank Λ_0, S_0,Λ_ψ, S_ψ together with isomorphisms

$$(6.1) \qquad J_0(\mathbb{C}_p) \simeq \mathrm{Hom}(S_0, \mathbb{C}_p^\times)/\Lambda_0, \quad J_\psi(\mathbb{C}_p) \simeq \mathrm{Hom}(S_\psi, \mathbb{C}_p^\times)/\Lambda_\psi.$$

In this section we use the p-adic uniformization of Čerednik–Drinfel'd, in the explicit formulation provided by Bertolini–Darmon, in order to give a p-adic analytic formula for the points $\pi_f(D)$ for D a degree 0 divisor in J_ψ of (5.2). The main feature of this formula, in contrast with that of (5.6), is that it is well suited for numerical computations, thanks to the explicit algorithms of [**FM**].

6.1. Čerednik–Drinfel'd uniformization. The main reference for this part is [**BC**, §5]. Recall the indefinite quaternion algebra $\mathcal{B}/\mathbb{Q}$ of discriminant N^- and $\mathcal{R}_0 \subset \mathcal{B}$ the Eichler order or level N^+ that we fixed in §4. Now let $B/\mathbb{Q}$ be the definite quaternion algebra obtained from $\mathcal{B}$ by interchanging the invariants p and ∞. That is to say, its set of ramification primes is

$$\mathrm{ram}(B) = \{\ell : \ell \neq p \text{ and } \ell \mid N^-\} \cup \{\infty\}.$$

For every $\ell \mid pN^+$ fix an isomorphism

$$i_\ell \colon B \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{M}_2(\mathbb{Q}_\ell).$$

Let R_0 be a $\mathbb{Z}[\frac{1}{p}]$-Eichler order of level N^+ in B, which is unique up to conjugation by elements in $B^\times$. In fact, we can choose R_0 in such a way that is locally isomorphic to $\mathcal{R}_0$ at every prime $\ell \neq p$. Let $\Gamma_0^{(p)} = (R_0)_1^\times$ denote the group of norm 1 units, and let

$$R_\psi = \{\gamma \in R_0 : \gamma \in \ker(\psi \circ \eta)\},$$

where $\eta \colon R_0 \to \mathbb{Z}_{N^+}/N^+\mathbb{Z}_{N^+}$ denotes the map that sends γ to the upper left entry of $i_{N^+}(\gamma)$ taken modulo N^+. Set $\Gamma_\psi^{(p)} = (R_\psi)_1^\times$.

Both groups $\Gamma_0^{(p)}$ and $\Gamma_\psi^{(p)}$ act on the p-adic upper half plane $\mathcal{H}_p$ by means of i_p, and the quotients $\Gamma_0^{(p)}\backslash\mathcal{H}_p$ and $\Gamma_\psi^{(p)}\backslash\mathcal{H}_p$ are rigid analytic varieties. In the following statement we collect some particular cases of the Čerednik–Drinfel'd theorem. We denote by $\mathbb{Q}_{p^i}$ the unramified extension of $\mathbb{Q}_p$ of degree i.

Theorem 6.1.   (1) $X_0 \otimes \mathbb{Q}_{p^2} \simeq \Gamma_0^{(p)} \backslash \mathcal{H}_p(\mathbb{Q}_{p^2})$.

(2) *If p is split in F then* $X_\psi \otimes \mathbb{Q}_{p^2} \simeq \Gamma_\psi^{(p)} \backslash \mathcal{H}_p(\mathbb{Q}_{p^2})$.

(3) *If p is inert in F then* $X_\psi \otimes \mathbb{Q}_{p^4} \simeq \Gamma_\psi^{(p)} \backslash \mathcal{H}_p(\mathbb{Q}_{p^4})$.

Proof: Part (1) is well know. As for parts (2) and (3), it follows from the Čerednik–Drinfel'd theorem that $X_\psi \otimes \mathbb{C}_p \simeq \Gamma_\psi^{(p)} \backslash \mathcal{H}_p$. The only thing that we need to check is that the isomorphism takes place after extending scalars to $\mathbb{Q}_{p^2}$ if p splits in F, and after extending scalars to $\mathbb{Q}_{p^4}$ if p is inert in F. This follows from the discussion in [**BC**, Remark 3.5.3.1]. Indeed, observe that $i_p(R_\psi) \subset \mathrm{M}_2(\mathbb{Q}_p)$, contains $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix} \right)$ if $\psi(p) = 1$ (i.e., if p splits in F), but only contains $\left(\begin{smallmatrix} p^2 & 0 \\ 0 & p^2 \end{smallmatrix} \right)$ if $\psi(p) = -1$ (i.e., if p is inert in F). By [**BC**, Remark 3.5.3.1] the curve $\Gamma_\psi^{(p)} \backslash \mathcal{H}_p$ and the isomorphism to X_ψ are defined over $\mathbb{Q}_{p^2}$ and over $\mathbb{Q}_{p^4}$, respectively. $\square$

6.2. Explicit p-adic uniformization. The main reference for this part is [**D2**, §5]. Let Γ be either $\Gamma_0^{(p)}$ or $\Gamma_\psi^{(p)}$. The group Γ acts on $\mathcal{H}_p = \mathbb{C}_p \backslash \mathbb{Q}_p$ with compact quotient. We can speak of $S_2(\Gamma)$, the space of *rigid analytic modular forms of weight 2 on* Γ. It is the set of all rigid analytic functions $h \colon \mathcal{H}_p \to \mathbb{C}_p$ such that

$$h(\gamma \cdot \tau) = (c\tau + d)^2 h(\tau) \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma.$$

Let $\mathrm{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}_p)$ denote the space of $\mathbb{C}_p$-valued measures of $\mathbb{P}^1(\mathbb{Q}_p)$ with total measure 0. The group Γ acts on it in the following way: if $\mu \in \mathrm{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}_p)$ and $\gamma \in \Gamma$ then $(\gamma \cdot \mu)(U) = \mu(\gamma^{-1} U)$. There is an isomorphism, due to Amice–Velu and Vishik [**DT**, Corollary 2.3.4]

$$(6.2) \qquad S_2(\Gamma) \simeq \mathrm{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}_p)^\Gamma,$$

where the superscript denotes the elements fixed by Γ.

Let $\mathcal{T}$ denote the Bruhat–Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$. Its set of vertices $\mathcal{V}(\mathcal{T})$ is identified with the set of homothety $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. Its set of oriented edges $\mathcal{E}(\mathcal{T})$ consists of ordered pairs of vertices (v_1, v_2) that can be represented by lattices Λ_1, Λ_2 such that $\Lambda_1 \subset \Lambda_2$ with index p. For an oriented edge $e = (v_1, v_2)$, we denote $\bar{e} = (v_2, v_1)$, $s(e) = v_1$, and $t(e) = v_2$. An *harmonic cocycle* is a function

$$h \colon \mathcal{E}(\mathcal{T}) \longrightarrow \mathbb{C}_p$$

such that $h(e) = -h(\bar{e})$ for all $e \in \mathcal{E}$, and such that for all $v \in \mathcal{V}(\mathcal{T})$

$$\sum_{s(e)=v} h(e) = 0.$$

The group Γ acts on $\mathbb{Q}_p^2$ via i_p, and this induces an action on $\mathcal{E}(\mathcal{T})$. The space of Γ-invariant measures can be identified with the set of Γ-invariant harmonic cocycles. This gives an integral structure

$$\mathrm{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z})^\Gamma \subset \mathrm{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{C}_p)^\Gamma$$

given by the $\mathbb{Z}$-valued harmonic cocycles. It thus gives rise, via the isomorphism (6.2), to an integral structure $S_2(\Gamma, \mathbb{Z}) \subset S_2(\Gamma)$.

If $\mu \in \mathrm{Meas}_0(\mathbb{P}^1(\mathbb{Q}_p), \mathbb{Z})$ and r is a continuous function on $\mathbb{P}^1(\mathbb{Q}_p)$ then the *multiplicative integral* of r against μ is defined as

$$\fint_{\mathbb{P}^1(\mathbb{Q}_p)} r(t) \, d\mu(t) = \lim_{\{U_a\}} \prod r(t_a)^{\mu(U_a)},$$

where the limit is defined over increasingly fine disjoint covers $\{U_a\}$ of $\mathbb{P}^1(\mathbb{Q}_p)$ and $t_a \in U_a$ is any sample point.

If $h \in S_2(\Gamma, \mathbb{Z})$ and $z_1, z_2 \in \mathcal{H}_p$ the *multiplicative line integral* $\fint_{z_1}^{z_2} h(z) \, dz$ is defined to be

$$\fint_{z_1}^{z_2} h(z) \, dz := \fint_{\mathbb{P}^1(\mathbb{Q})} \left(\frac{t - z_1}{t - z_2} \right) d_{\mu_h}(t),$$

where μ_h is the measure attached to h by the isomorphism (6.2). This is used to define the p-adic Abel–Jacobi map

$$\Phi_{\mathrm{AJ}} \colon \mathrm{Div}^0(\mathcal{H}_p) \longrightarrow \mathrm{Hom}(S_2(\Gamma, \mathbb{Z}), \mathbb{C}_p^\times) \simeq (\mathbb{C}_p^\times)^g$$

$$z_1 - z_2 \longmapsto \left(h \mapsto \fint_{z_1}^{z_2} h(z) \, dz \right),$$

where g denotes the genus of $\Gamma \backslash \mathcal{H}_p$. The group of degree 0 divisors in $\mathcal{H}_p$ that become trivial on $\Gamma \backslash \mathcal{H}_p$ are mapped by Φ_{AJ} to a lattice $\Lambda_\Gamma \subset \mathrm{Hom}(S_2(\Gamma, \mathbb{Z}), \mathbb{C}_p^\times)$. This gives

$$\phi_{\mathrm{AJ}} \colon \mathrm{Div}^0(\Gamma \backslash \mathcal{H}_p) \longrightarrow \mathrm{Hom}(S_2(\Gamma, \mathbb{Z}), \mathbb{C}_p^\times)/\Lambda_\Gamma \simeq \mathrm{Jac}(X_\Gamma)(\mathbb{C}_p).$$

By particularizing this to the groups $\Gamma_0^{(p)}$ and $\Gamma_\psi^{(p)}$ one obtains an explicit expression for the rigid analytic uniformizations of (6.1):

$$\mathrm{Div}^0(\Gamma_0^{(p)} \backslash \mathcal{H}_p) \simeq J_0(\mathbb{C}_p) \xrightarrow{\Phi_{\mathrm{AJ}}} \mathrm{Hom}(S_2(\Gamma_0^{(p)}, \mathbb{Z}), \mathbb{C}_p^\times)/\Lambda_0,$$

$$\mathrm{Div}^0(\Gamma_\psi^{(p)} \backslash \mathcal{H}_p) \simeq J_\psi(\mathbb{C}_p) \xrightarrow{\Phi_{\mathrm{AJ}}} \mathrm{Hom}(S_2(\Gamma_\psi^{(p)}, \mathbb{Z}), \mathbb{C}_p^\times)/\Lambda_\psi.$$

6.3. CM points and the p-adic uniformization. Let $\mathrm{CM}_0^p(c) \subset \Gamma_0^{(p)} \backslash \mathcal{H}_p$ (resp. $\mathrm{CM}_\psi^p(c) \subset \Gamma_\psi^{(p)} \backslash \mathcal{H}_p$) denote the set of points corresponding to $\mathrm{CM}_0(c) \subset X_0$ (resp. $\mathrm{CM}_\psi(c) \subset X_\psi$) under the isomorphism $X_0(\mathbb{C}_p) \simeq \Gamma_0 \backslash \mathcal{H}_p$ (resp. $X_\psi(\mathbb{C}_p) \simeq \Gamma_\psi \backslash \mathcal{H}_p$).

Bertolini and Darmon give in [**BD1**] an explicit description of $\mathrm{CM}_0^p(c)$ in terms of certain optimal embeddings of the order of conductor c into B. Next, we use this in order to derive the corresponding description of $\mathrm{CM}_\psi^p(c)$.

Let R_0 be an Eichler order of B of level N^+ as in §6.1. Let $\varphi \colon \mathcal{O}_c[\frac{1}{p}] \hookrightarrow R_0$ be an optimal embedding of $\mathbb{Z}[\frac{1}{p}]$-algebras. It has a single fixed point $\tau_\varphi \in \mathcal{H}_p$ satisfying

$$\alpha \begin{pmatrix} \tau_\varphi \\ 1 \end{pmatrix} = i_p(\varphi(\alpha)) \begin{pmatrix} \tau_\varphi \\ 1 \end{pmatrix}$$

for all $\alpha \in \mathcal{O}_c[\frac{1}{p}]$. As before, we can define the notion of *normalized embedding*: the isomorphism

$$i_{N^+} \colon R_0 \simeq \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in M_2(\mathbb{Z}_{N^+}) : c \in N^+ \mathbb{Z}_{N^+} \}$$

allows us to define the homomorphism

$$\eta \colon R_0 \longrightarrow \mathbb{Z}/N^+\mathbb{Z}$$

that sends each element x to the upper left entry of i_{N^+} modulo N^+. Then we say that an optimal embedding $\varphi \colon \mathcal{O}_c[\frac{1}{p}] \hookrightarrow R_0$ is normalized with respect to $\mathfrak{N}^+$ if $\ker(\eta \circ \varphi) = \mathfrak{N}^+$. The explicit description of $\mathrm{CM}_0^p(c)$ given by Bertolini–Darmon is then:

$$\mathrm{CM}_0^p(c) = \{ [\tau_\varphi] \in \Gamma_0^{(p)} \backslash \mathcal{H}_p : \varphi \in \mathcal{E}(c, R_0) \}.$$

Therefore, the set $\mathrm{CM}_\psi^p(c)$ is given by:

$$\mathrm{CM}_\psi^p(c) = \{ [\tau_\varphi] \in \Gamma_\psi^{(p)} \backslash \mathcal{H}_p : \varphi \in \mathcal{E}(c, R_0) \}.$$

As a consequence of Proposition 4.3 we see that $\Phi_{\mathrm{AJ}}(\mathrm{Div}^0(\mathrm{CM}_\psi^p(c)))$ is contained in $J_\psi(L_c)$.

6.4. A p-adic analytic formula for ATR points on $\mathbb{Q}$-curves.
Recall the modular form $f \in S_2(\Gamma_0(N), \psi)$ corresponding to E. There exists a rigid analytic modular form $h \in S_2(\Gamma_\psi, \mathbb{C}_p)$ which is an eigenvector for the Hecke operators, and has the same system of eigenvalues as f. Since the eigenvalues of f are defined over the quadratic imaginary field K_f we can identify h with an harmonic cocycle with values in the ring of integers of K_f, and we denote by $\overline{h}$ the complex conjugated cocycle. Then $h_0 := (h + \overline{h})/2$ and $h_1 := (h - \overline{h})/2i$ belong to $S_2(\Gamma_\psi, \mathbb{Z})$. Let $\Phi^{(p)}$ be the map

$$\Phi^{(p)} \colon \mathrm{Div}^0(\mathcal{H}_p) \longrightarrow \mathbb{C}_p^\times \times \mathbb{C}_p^\times$$

$$z_2 - z_1 \longmapsto \left(\fint_{z_1}^{z_2} h_0(z)\, dz, \fint_{z_1}^{z_2} h_1(z)\, dz \right).$$

The image of the divisors whose image under $\Phi^{(p)}$ becomes trivial in $\Gamma_\psi^{(p)}\backslash\mathcal{H}_p$ generates a lattice $\Lambda_f^p \subset \mathbb{C}_p^\times \times \mathbb{C}_p^\times$, and the quotient[1] $\mathbb{C}_p^\times \times \mathbb{C}_p^\times/\Lambda_f^p$ is isogenous to $A_f(\mathbb{C}_p)$. In particular, if $D = \tau_2 - \tau_1 \in \mathrm{Div}^0\,\mathrm{CM}_0^p(c)$ we find the following p-adic analytic formula for the corresponding CM point in A_f:

$$(6.3) \qquad \pi_f(D) = \left(\oint_{\tau_1}^{\tau_2} h_0(z)\,dz, \oint_{\tau_1}^{\tau_2} h_1(z)\,dz \right),$$

which in fact belongs to $A_f(L_c)$. The above formula for $\pi_f(D)$ can be explicitly computed, thanks to a slight modification of the explicit algorithms of [**FM**]. In the next section we give a detailed example on how these algorithms can be used in order to compute in practice $\pi_f(D)$, and therefore also the point $P_\tau \in E(M)$ constructed in §5.

7. An example

The goal of this section is to illustrate with an example the construction carried out above. Let $F = \mathbb{Q}(\sqrt{5})$ and consider the elliptic curve defined over F given as

$$E: \ y^2 = x^3 + (-432\sqrt{5} - 1296)x + (-113184\sqrt{5} - 282960).$$

Remark that E is a $\mathbb{Q}$-curve, and has conductor $39\mathcal{O}_F$. We will take $p = 13$. The modular form f_E attached to E belongs to $S_2(135, \psi)$, where ψ is the unique quadratic character $\psi\colon (\mathbb{Z}/5\mathbb{Z})^\times \to \{\pm 1\}$ of conductor 5. Note that the form f_E has field of coefficients $\mathbb{Q}(\sqrt{-1})$.

We need to construct the quotient of the Bruhat–Tits tree of $\mathrm{GL}_2(\mathbb{Q}_p)$ by the group Γ_ψ. In order to do so, the algorithms of [**FM**] have been adapted to work with congruence subgroups such as Γ_ψ. The main algorithm of [**FM**] returns, given two vertices or edges of the Bruhat–Tits tree, the (possibly empty) set of elements of Γ_0 relating them. One just needs to check whether the intersection of this set with Γ_ψ is empty, which is easily done. The quotient graph that we obtain is represented in Figure 1. It consists of 4 vertices and 28 edges. The numbers next to each side of the square in Figure 1 describe how many edges link each of the two corresponding vertices. For example, there are 8 edges connecting v_0 with v_1. Note that all vertices have valency $14 = p+1$, so all of them have trivial stabilizers.

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[1]In fact, in view of Theorem 6.1 we can even be more precise in the field of definitions and work over $\mathcal{H}_p(\mathbb{Q}_{p^4})$; we have that $\mathbb{Q}_{p^4}^\times \times \mathbb{Q}_{p^4}^\times/\Lambda_f^p$ is isogenous to $A_f(\mathbb{Q}_{p^4})$.
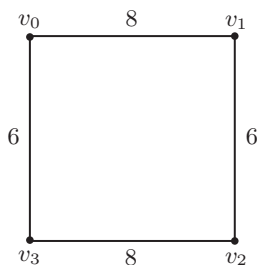
FIGURE 1. Quotient $\Gamma_\psi \backslash \mathcal{E}(\mathcal{T})$.

The space of harmonic cocycles on $\Gamma_\psi \backslash \mathcal{E}(\mathcal{T})$ has dimension 25. Taking the common eigenspace on which T_{19} acts as 4 and T_2 acts as 0 we obtain a 2-dimensional subspace associated to f_E. An integral basis of this subspace is given by the harmonic cocycles h_0 and h_1, which we proceed to describe. The harmonic cocycle h_0 has support on four edges, and takes values in ± 1 there. In fact, it takes the value $+1$ and -1 once on two edges connecting v_0 and v_3, and the value $+1$ and -1 on two edges connecting v_1 and v_2. The harmonic cocycle h_1 can be described exactly as h_0, but they have disjoint supports.

Moreover, T_3 satisfies:

$$T_3(h_0) = -h_1 \text{ and } T_3(h_1) = h_0.$$

Define also $\alpha = 2\sqrt{5} - 1$, and let $M = F(\sqrt{\alpha})$, which is ATR. In this case, the resulting field is $K = \mathbb{Q}(\sqrt{-19})$, which has class number 1. Let g be a root in $\mathbb{C}_p$ of the polynomial $x^2 - x + 5$, and let

$$\tau = (6g + 1) + (8g + 12)13 + (7g + 11)13^2 + (3g + 3)13^3$$
$$+ (12g + 9)13^4 + (6g + 1)13^5 + \cdots$$

be a fixed point under an embedding φ of the maximal order of K into the Eichler order $R_0(1)$ of the quaternion algebra $B = (-3, -1)$, having basis:

$$R_0(1) = \langle 1, j, 5/2j + 5/2k, 1/2 + 1/2i - 3/2j - 3/2k \rangle.$$

We consider the divisor $D = (\tau) - (\overline{\tau})$ and calculate:

$$J_0 = \oint_{\tau}^{\overline{\tau}} \omega_{h_0} = (8g + 12) + (3g + 1)13 + (7g + 10)13^2 + (8g + 8)13^3$$
$$+ (7g + 1)13^4 + (7g + 6)13^5 + (9g + 8)13^6 + (7g + 7)13^7$$
$$+ (4g + 9)13^8 + (4g + 4)13^9 + (5g + 12)13^{10}$$
$$+ (8g + 1)13^{11} + (11g + 11)13^{12} + \cdots$$

and in fact $J_1 = J_0$.

We calculate the image of J_0 under the Tate uniformization map, to get coordinates $(x, y) \in E(\mathbb{C}_p)$:

$$x = (12h^3 + 3h^2 + 4h + 1) + (9h^3 + 10h^2 + h + 9)13$$
$$+ (6h^3 + 5h^2 + 3h + 9)13^2 + (6h^3 + 8h^2 + 8)13^3$$
$$+ (8h^3 + 2h^2 + 5h + 8)13^4 + (4h^3 + 9h^2 + 4h + 6)13^5 + \cdots$$

and

$$y = (11h^3 + 5h^2 + 2h + 9) + (12h^3 + 12h^2 + h + 10)13$$
$$+ (7h^2 + 10h + 7)13^2 + (2h^3 + 5h^2 + 9h + 7)13^3$$
$$+ (5h^3 + 2h^2 + 4h + 4)13^4 + (3h^3 + 3h + 11)13^5 + \cdots$$

Here, h satisfies:

$$h^4 + 3h^2 + 12h + 2 = 0.$$

We have carried out all the calculations to precision 13^{80}, and up to this precision it turns out that x is a root of the irreducible polynomial:

$$P_x(T) = T^4 + 60T^3 + 19728T^2 + 380160T + 40144896$$

and y is a root of the irreducible polynomial:

$$P_y(T) = T^8 - 1166400T^6 + 5027006707200T^4 - 321342050396160000T^2$$
$$+ 75899706935371407360000.$$

The polynomial $P_y(T)$ factors as two quartics over F. We let $\mathcal{M}/F$ be the quartic extension generated by one of these two factors, and we remark that $P_y(T)$ splits completely over $\mathcal{M}$, so it is actually the splitting field of $P_y(T)$. Let α be a root of $P_y(T)$ in $\mathcal{M}$. Then the coordinates (x, y) are defined over $\mathcal{M}$ and correspond to the point:

$$((1/12960\sqrt{5} - 1/4320)\alpha^2 + 3/2\sqrt{5} + 15/2, \alpha) \in E(\mathcal{M}).$$

Since $\mathcal{M}$ contains the field M, we can compute the trace of this point down to M, to obtain the point of infinite order

$$P_D = \left(\frac{474\sqrt{5} + 750}{19}, \frac{20412\sqrt{5} + 19440}{361}\sqrt{\alpha} \right) \in E(M).$$

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Xavier Guitart:
Institut für Experimentelle Mathematik
Universität Duisburg-Essen
45326 Essen
Germany
E-mail address: xevi.guitart@gmail.com

Marc Masdeu:
Mathematics Institute
University of Warwick
Coventry, CV4 7AL
United Kingdom
E-mail address: m.masdeu@warwick.ac.uk