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EMBEDDINGS OF LOCAL FIELDS IN SIMPLE ALGEBRAS AND SIMPLICIAL STRUCTURES

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Abstract: We give a geometric interpretation of Broussous–Grabitz embedding types. We fix a central division algebra D of finite index over a non-Archimedean local field F and a positive integer m. Further we fix a hereditary order $\mathfrak{a}$ of $\mathrm{M}_m(D)$ and an unramified field extension $E|F$ in $\mathrm{M}_m(D)$ which is embeddable in D and which normalizes $\mathfrak{a}$. Such a pair $(E,\mathfrak{a})$ is called an embedding. The embedding types classify the $\mathrm{GL}_m(D)$-conjugation classes of these embeddings. Such a type is a class of matrices with non-negative integer entries. We give a formula which allows us to recover the embedding type of $(E,\mathfrak{a})$ from the simplicial type of the image of the barycenter of $\mathfrak{a}$ under the canonical isomorphism, from the set of $E^\times$-fixed points of the reduced building of $\mathrm{GL}_m(D)$ to the reduced building of the centralizer of $E^\times$ in $\mathrm{GL}_m(D)$. Conversely the formula allows to calculate the simplicial type up to cyclic permutation of the Coxeter diagram.

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1. Introduction

Field extensions in Azumaya algebras with respect to local representation theory have been studied in many situations. We want to mention works of Fröhlich [**Frö87**], Broussous and Grabitz [**Gra99**], [**BG00**] among others. More precisely we consider a skew field D of finite index d over a non-Archimedean local field F and pairs $(E,\mathfrak{a})$ consisting of a field extension $E|F$ in $\mathrm{M}_m(D)$ and a hereditary order $\mathfrak{a}$ of $\mathrm{M}_m(D)$ normalized by $E^\times$. Such a pair $(E,\mathfrak{a})$ is called an embedding. In the representation theory of $G:=\mathrm{GL}_m(D)$ on complex vector spaces it has been important to understand when two such embeddings are conjugate to each other under G. Broussous and Grabitz attached in [**BG00**, 2.3.10, 2.3.1] to every embedding a cyclic permutation class of matrices with non-negative integer entries, which only depends on $\mathfrak{a}$ and the maximal unramified field extension $E_D|F$ of $E|F$ embeddable in $D|F$. We call the latter class *embedding type* of $(E,\mathfrak{a})$. It is slightly different to the "type of embedding" in their paper, where they restrict to matrices with d rows. This invariant allows to define certain equivalent embeddings, called pearl embeddings, which are suitable for calculations and which are conjugate to $(E_D, \mathfrak{a})$. The latter is a generalization of a work [**Frö87**] of Fröhlich, where he considered the case of principal orders. Broussous and Grabitz proved:

Proposition 1.1 ([**BG00**, 3.2])**.** *Two embeddings $(E, \mathfrak{a})$ and $(E', \mathfrak{a})$ are conjugate under $\mathrm{GL}_m(D)$ if and only if the field extensions are isomorphic and the embeddings have the same embedding type, i.e. $(E_D, \mathfrak{a})$ and $(E'_D, \mathfrak{a})$ are conjugate under $\mathrm{GL}_m(D)$.*

This rigidity statement plays a role in the classification of supercuspidal representations of G, see for example [**BSS12**].

The main result of this paper is that the embedding type can be obtained geometrically. The strategy is the following. We denote by G_{E_D} the centralizer of E_D in G. Broussous and Lemaire introduced a canonical isomorphism between reduced Bruhat–Tits buildings

$$j_{E_D} : \mathfrak{B}_{\mathrm{red}}(G)^{E_D^\times} \to \mathfrak{B}_{\mathrm{red}}(G_{E_D})$$

which is affine, G_{E_D}-equivariant and which respects the Moy–Prasad filtrations [**MP94**]. Let $(E, \mathfrak{a})$ be an embedding. The barycenter $x_{\mathfrak{a}}$ of $\mathfrak{a}$ in $\mathfrak{B}_{\mathrm{red}}(G)$ is fixed under the action of $E_D^\times$.

In this article we construct an easy combinatorial map $(\ )^c$ of order two on the set of non-zero cyclic vectors with non-negative integer coefficients. A cyclic vector $\langle v\rangle$ is the set of vectors which are equal to v up to cyclic permutation of the coordinates. We prove:

Theorem 1.2 (see Theorem 6.4)**.** *Let M be an element of the embedding type of $(E, \mathfrak{a})$, in particular M is a matrix with non-negative integer coefficients. Write M line-wise into a vector λ. Fix a labeling of $\mathfrak{B}_{\mathrm{red}}(G_{E_D})$ and let μ be the barycentric coordinates of $j_{E_D}(x_{\mathfrak{a}})$. Then $(\langle \mathrm{rank}(\mathfrak{a})[E_D : F]\mu\rangle)^c$ is equal to $\langle\lambda\rangle$.*

Here $\mathrm{rank}(\mathfrak{a})$ is the simplicial rank of $\mathfrak{a}$ seen as a facet of $\mathfrak{B}_{\mathrm{red}}(G)$, see Definition 2.5. The advantage of this approach is the following. On can consider classes of embeddings as classes of certain points with rational coordinates in a chamber of $\mathfrak{B}_{\mathrm{red}}(G_{E_D})$. The class is given by the action of the group of rotations of the Coxeter diagram on $\mathfrak{B}_{\mathrm{red}}(G_{E_D})$. Let h be a Hermitian form on D^m and $\mathrm{U}(h)$ its group of isometries. One can now study examples of embeddings which are invariant under the adjoint involution of h in studying certain points of $j_{E_D}(\mathfrak{B}(\mathrm{U}(h)) \cap \mathfrak{B}_{\mathrm{red}}(G)^{E_D^\times})$ with rational coordinates.

The article is structured as follows. We give general notation and we recall the model of the reduced Bruhat–Tits building of $\mathrm{GL}_m(D)$ in terms of lattice functions in §2 and introduce in §3 cyclic vectors and matrices. In §4 we recall definitions and statements of [**BG00**] on embedding types and pearl embeddings. In relevant cases we recall the description of the map j_E in §5. Finally, §6 explains the strategy to encode the embedding type from barycentric coordinates, followed by the proof of Theorem 6.4. The proof consists of three main ideas:

1. changing the skew field in Lemma 6.8,
2. a rank reduction for the considered facet $\mathfrak{a}$ in Lemma 6.7, and
3. the proof for the case that $\mathfrak{a}$ is a vertex.

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2. The reduced Bruhat–Tits building of $\mathbf{GL}_m(D)$

Let (F, ν) be a non-Archimedean local field with normalized valuation ν, valuation ring o_F, valuation ideal $\mathfrak{p}_F$, a chosen uniformizer π_F, and residue field κ_F. We use similar notation for other skew fields with non-Archimedean valuation. We fix a skew field D of finite index d and central over F, together with a maximal unramified field extension $L|F$ in $D|F$, and a uniformizer π_D which normalizes L such that the map

$$x \mapsto \sigma(x) := \pi_D x \pi_D^{-1}, \quad x \in D,$$

generates $\mathrm{Gal}(L|F)$. For a positive integer $f|d$ we denote by L_f the subfield of degree f over F in L. Further, let V be a right D-vector space of finite dimension m and denote its ring of D-linear endomorphisms by A. Then V is in a natural way a left $A \otimes_F D^{op}$-module. We write G for $A^\times$.

We recall the model of the reduced Bruhat–Tits building $\mathfrak{B}_{\mathrm{red}}(G)$ in terms of lattice functions. For more details we recommend [**BL02**] and [**BT84**]. We adopt the following definitions from [**BL02**].

Definition 2.1. 1. An m-dimensional free o_D-submodule of V is an *o_D-lattice*. We denote the set of all o_D-lattices by $\mathcal{L}(o_D, V)$. It is partially ordered by inclusion.

2. A strictly decreasing map Λ from $\mathbb{Z}$ to $\mathcal{L}(o_D, V)$ is called an *o_D-lattice chain on V* if there is an $r \in \mathbb{N}$ such that $\mathcal{L}_i \pi_D = \mathcal{L}_{i+r}$, $i \in \mathbb{Z}$. Two lattice chains $\mathcal{L}$ and $\mathcal{L}'$ are *$\mathbb{Z}$-translations* of each other if there is an integer m such that $\mathcal{L}_{i+m}$ is equal to $\mathcal{L}_i$, for all integers i. A translation class is denoted by $[\mathcal{L}]$.

3. A decreasing map Λ from $\mathbb{R}$ to $\mathcal{L}(o_D, V)$ is called an *o_D-lattice function on V* if it is
   - left continuous, i.e. $\Lambda(t) = \cap_{s<t}\Lambda(s)$, and
   - satisfies $\Lambda(t)\pi_D = \Lambda(t + \frac{1}{d})$, $t \in \mathbb{R}$.

   The set of o_D-lattice functions is denoted by $\mathrm{Latt}^1_{o_D}(V)$. The map which sends Λ to $(\Lambda(t-s))_{t\in\mathbb{R}}$, which we denote by $\Lambda + s$, is called a *translation by s*. Two o_D-lattice functions are equivalent if they differ by a translation and the set of all classes of o_D-lattice functions is denoted by $\mathrm{Latt}_{o_D}(V)$. For $[\Lambda] \in \mathrm{Latt}_{o_D}(V)$, the square lattice function of Λ is defined to be the following o_F-lattice function in A:

   $$t \mapsto \mathfrak{g}_\Lambda(t) := \{a \in A \mid a(\Lambda(s)) \subseteq \Lambda(s+t) \text{ for all } s \in \mathbb{R}\}.$$

   This attachment only depends on the translation class of Λ and is injective on $\mathrm{Latt}_{o_D}(V)$, i.e. two different translation classes have different square lattice functions, see [**BL02**]. We denote the set of square lattice functions of A by $\mathrm{Latt}^2_{o_F}(A)$.
4. The *hereditary order* $\mathfrak{a}_\Lambda$ corresponding to Λ is the ring $\mathfrak{g}_\Lambda(0)$, which only depends on the translation class of Λ.

Theorem 2.2 ([**BL02**, I 2.4])**.** *There is a unique G-equivariant, affine bijection from $\mathfrak{B}_{\mathrm{red}}(G)$ to $\mathrm{Latt}_{o_D}(V)$.*

Remark 2.3.
1. The apartments of $\mathfrak{B}_{\mathrm{red}}(G)$ carry over to $\mathrm{Latt}_{o_D}(V)$; more precisely the set of apartments in $\mathrm{Latt}_{o_D}(V)$ is in one to one correspondence with the set of frames in V. A *frame* is a set of one dimensional D-sub-vector spaces of V whose direct sum is V. The apartment corresponding to a frame $\mathcal{R}$ is the set $\mathrm{Latt}_{\mathcal{R}}(V)$ of $[\Lambda]$ such that Λ is *split by* $\mathcal{R}$, i.e.

   $$\Lambda(t) = \oplus_{W\in\mathcal{R}}\Lambda(t) \cap W,$$

   for all $t \in \mathbb{R}$.
2. The inherited simplicial structure on $\mathrm{Latt}_{o_D}(V)$ is given as follows: The facet containing $[\Lambda] \in \mathrm{Latt}_{o_D}(V)$ is

   $$\{[\Lambda'] \in \mathrm{Latt}_{o_D}(V) \mid \mathrm{im}(\Lambda) = \mathrm{im}(\Lambda')\}.$$

Theorem 2.4 ([**Rei03**, 39.14])**.** *There are canonical bijections between the set of $\mathbb{Z}$-translation classes of o_D-lattice chains in V, the set of images of elements of $\mathrm{Latt}^1_{o_D}(V)$, and the set of hereditary orders in A:*

$$[\mathcal{L}] \mapsto \mathrm{im}(\mathcal{L}) \text{ and } \mathrm{im}(\Lambda) \mapsto \mathfrak{a}_\Lambda.$$

To be parallel to the notation in [**BL02**] we write $\mathcal{I}$ for $\mathfrak{B}_{\mathrm{red}}(G)$ and identify it with $\mathrm{Latt}_{o_D}(V)$. Its simplicial structure is denoted by Ω. We also call Ω the Euclidean building of G, so in this article a Euclidean building is a simplicial complex and not its geometric realization.

The simplices are denoted by hereditary orders, i.e. the simplex of $x \in \mathcal{I}$ is denoted by $\mathfrak{a}_x$. The square lattice function attached to x is denoted by $\mathfrak{g}_x$.

For a frame $\mathcal{R}$, we denote the appartment of $\mathcal{I}$ corresponding to $\mathcal{R}$ by $\mathcal{I}_{\mathcal{R}}$ and its simplicial structure by $\Omega_{\mathcal{R}}$, more precisely we have

$$\Omega_{\mathcal{R}} = \{\mathfrak{a} \in \Omega \mid \exists\, [\Lambda] \in \mathrm{Latt}_{\mathcal{R}}(V) : \mathfrak{a} = \mathfrak{a}_{\Lambda}\}.$$

We also call $\Omega_{\mathcal{R}}$ the apartment of Ω corresponding to $\mathcal{R}$, i.e. we have apartments for $\mathcal{I}$ and for Ω.

Definition 2.5. Given two simplices $\mathfrak{a}$ and $\mathfrak{a}'$, we write $\mathfrak{a} \leq \mathfrak{a}'$ if $\mathfrak{a} \supseteq \mathfrak{a}'$. A *vertex of* Ω is a simplex which is minimal with respect to $\leq$. Let $\mathfrak{b}$ be a vertex of Ω and $\mathfrak{a}$ a simplex. We call $\mathfrak{b}$ a vertex of $\mathfrak{a}$ if $\mathfrak{b} \leq \mathfrak{a}$. We define the *simplicial rank* of $\mathfrak{a}$ as the number of vertices of $\mathfrak{a}$, and denote it by $\mathrm{rank}(\mathfrak{a})$. A simplex of maximal rank is a *chamber of* Ω.

3. Cyclic vectors and matrices

The invariants which are considered in this article are vectors or matrices modulo cyclic permutation.

3.1. Vectors.

Definition 3.1. Let R be an arbitrary non-empty set, $\mathrm{Vec}(R)$ be the set of finite dimensional row-vectors with entries in R, i.e.

$$\mathrm{Vec}(R) := \bigcup_{i=1}^{\infty} R^{1\times i}.$$

We call two elements w, w' of $\mathrm{Vec}(R)$ *equivalent* if they have the same dimension, say s, and if there is a k such that

$$w' = (w_k, \dots, w_s, w_1, \dots, w_{k-1}).$$

The equivalence class is denoted by $\langle w\rangle$ and it is a called a *cyclic vector*. We often skip the round brakets and write $\langle w_1, \dots, w_k\rangle$. The set of equivalence classes of $\mathrm{Vec}(R)$ is denoted by $\mathrm{CVec}(R)$.

We now represent a cyclic vector with entries in $\mathbb{N}_0$ in a different way, to be able to attach a dual cyclic vector. There is a canonical map from

$\mathrm{CVec}(\mathbb{N}_0)$ to $\mathbb{N}_0$ which maps a cyclic vector to its sum of the coordinates, and we write $\mathrm{CVec}(\mathbb{N}_0)_+$ for the preimage of $\mathbb{N}$. Consider the map

$$\mathrm{pairs}\colon\ \mathrm{CVec}(\mathbb{N}_0)_+ \to \mathrm{CVec}(\mathbb{N}^2),$$

defined via

$$\mathrm{pairs}(\langle w\rangle) := \langle (w_{i_0}, i_1 - i_0), (w_{i_1}, i_2 - i_1), \ldots, (w_{i_k}, i_0 + s - i_k)\rangle,$$

where $(w_{i_j})_{0\le j\le k}$ is the subsequence of the non-zero coordinates.

Lemma 3.2. *The map* pairs *is bijective.*

Proof: By the definition of pairs we can rebuild $\langle w\rangle$ directly from $\mathrm{pairs}(\langle w\rangle)$ and thus pairs is injective. The preimage of an element

$$\langle (a_0, b_0), (a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\rangle$$

of $\mathrm{CVec}(\mathbb{N}^2)$ contains the cyclic class of the vector $w = (w_i)_{1\le i\le \sum_{l=0}^k b_l}$ defined via

$$w_i = \begin{cases} a_j, & \text{if } i = 1 + \sum_{l=0}^{j} b_l \\ 0, & \text{else} \end{cases}.$$

Thus pairs is surjective. □

On $\mathrm{CVec}(\mathbb{N}^2)$ there is a canonical bijection T which maps an element

$$\langle (a_0, b_0), (a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\rangle$$

to

$$\langle (b_0, a_1), (b_1, a_2), (b_2, a_3), \ldots, (b_k, a_0)\rangle,$$

and it induces the bijection $(\ )^c := \mathrm{pairs}^{-1} \circ \mathrm{T} \circ \mathrm{pairs}$ of $\mathrm{CVec}(\mathbb{N}_0)_+$. We call $\langle w\rangle^c$ the *dual* of $\langle w\rangle$.

3.2. Matrices.

For $r, s, t \in \mathbb{N}$, $\mathrm{M}_{r,s}(t)$ denotes the set of $r \times s$-matrices with non-negative integer entries, such that

- in every column there is an entry greater than zero, and
- the sum of all entries is t.

For a matrix $M = (m_{i,j}) \in \mathrm{M}_{r,s}(t)$, we define the vector $\mathrm{row}(M) \in \mathbb{N}_0^{1\times rs}$ to be

$$(m_{1,1}, m_{1,2}, \ldots, m_{1,s}, m_{2,1}, \ldots, m_{2,s}, \ldots, m_{r,1}, \ldots, m_{r,s}).$$

Two matrices $M, N \in \mathrm{M}_{r,s}(t)$ are said to be *equivalent* if $\mathrm{row}(M)$ and $\mathrm{row}(N)$ are. The equivalence class, denoted by $\langle M\rangle$, is called a *cyclic matrix*.

Example 3.3.

$$\begin{pmatrix} 2 & 0 \\ 1 & 3 \\ 0 & 1 \end{pmatrix} \backsim \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 0 \end{pmatrix}.$$

4. Embedding types

For a field extension $E|F$ we denote by $E_D|F$ the maximal field extension in $E|F$, which is F-algebra isomorphic to a subfield of L. Its degree is the greatest common divisor of d and the inertia degree of $E|F$.

Definition 4.1. An *embedding* is a pair $(E, \mathfrak{a})$ satisfying

1. E is a field extension of F in A,
2. $\mathfrak{a}$ is a hereditary order of A, normalized by $E^\times$.

Two embeddings $(E, \mathfrak{a})$ and $(E', \mathfrak{a}')$ are said to be *equivalent* if there is an element $g \in G$, such that $gE_Dg^{-1} = E'_D$ and $g\mathfrak{a}g^{-1} = \mathfrak{a}'$.

Remark 4.2. In each equivalence class of embeddings there is a pair such that the field can be embedded in L.

Until the end of this section we fix a D-basis of V and identify A with $\mathrm{M}_m(D)$.

Definition 4.3. Let $f|d$ and $r \leq m$. A matrix with f rows and r columns is called an *embedding datum* if it belongs to $\mathrm{M}_{f,r}(m)$. Given an embedding datum $\lambda = (\lambda_{i,j})_{1\leq i\leq f,\, 1\leq j\leq r}$, we define the pearl embedding as follows. The *pearl embedding* of λ (with respect to the fixed D-basis of V) is the embedding $(E_\lambda, \mathfrak{a}_\lambda)$, which satisfies the following conditions:

1. E_λ is the image of the monomorphism
$$x \in L_f \mapsto \mathrm{diag}(D_1(x), D_2(x), \ldots, D_r(x)) \in \mathrm{M}_m(D),$$
where
$$D_j(x) = \mathrm{diag}(\sigma^0(x)\mathbb{1}_{\lambda_{1,j}}, \sigma^1(x)\mathbb{1}_{\lambda_{2,j}}, \ldots, \sigma^{f-1}(x)\mathbb{1}_{\lambda_{f,j}})$$
and $\mathbb{1}_k$ is the identity matrix with k rows, in particular $E_\lambda|F$ is unramified of degree f.
2. $\mathfrak{a}$ is a hereditary order in standard form according to the partition $m = n_1 + \cdots + n_r$ where $n_j := \sum_{i=1}^f \lambda_{i,j}$, i.e. $\mathfrak{a}$ is the set of block matrices such that the (i, j)-th block has size $n_i \times n_j$ and has all its entries in o_D if $i \geq j$ and in $\mathfrak{p}_D$ if $i < j$, respectively.

Theorem 4.4 ([**BG00**, 2.3.3 and 2.3.10]). 1. *Two pearl embeddings are equivalent if and only if the embedding data are equivalent.*
2. *In every class of embeddings there is a pearl embedding.*

Definition 4.5. Let $(E, \mathfrak{a})$ be an embedding. It is equivalent to a pearl embedding $(E_\lambda, \mathfrak{a}_\lambda)$, by the above theorem. The cyclic matrix $\langle\lambda\rangle$ is called the *embedding type* of $(E, \mathfrak{a})$. This definition does not depend on the choice of the basis by the Skolem–Noether Theorem. If $(E, \mathfrak{a})$ has embedding type $\langle\lambda\rangle$ with $\lambda \in \mathrm{M}_{f,r}(m)$, then $r = \mathrm{rank}(\mathfrak{a})$ and $f = [E_D : F]$.

5. The map j_E

Notation 5.1. For this section let $E|F$ be a field extension in A and we set A_E to be the centralizer of E in A, i.e.

$$A_E := \mathrm{Z}_A(E) := \{a \in A \mid ab = ba \,\forall\, b \in E\}.$$

We denote the Euclidean building of $A_E^\times$ by Ω_E and its geometric realization by $\mathcal{I}_E$.

The next results are taken from [**BL02**].

Theorem 5.2 ([**BL02**, Theorem II 1.1.]). *There exists a unique map*

$$j_E \colon \mathcal{I}^{E^\times} \to \mathcal{I}_E,$$

such that for any $x \in \mathcal{I}^{E^\times}$ and $t \in \mathbb{R}$, we have $\mathfrak{g}_{j_E(x)}(t) = A_E \cap \mathfrak{g}_x(e(E|F)t)$. The map j_E satisfies the following properties:

1. *it is bijective,*
2. *it is $A_E^\times$-equivariant, and*
3. *it is affine.*

Moreover its inverse j_E^{-1} is the only map $\mathcal{I}_E \to \mathcal{I}$ such that 2 and 3 hold.

We briefly give Broussous and Lemaire's description of j_E in terms of lattice functions but only in the case where $E|F$ is isomorphic to a sub-extension $L_f|F$ of $L|F$. Then $E \otimes_F L \cong \bigoplus_{k=0}^{f-1} L$ coming from the decomposition $1 = \sum_{k=0}^{f-1} 1_k$ labeled such that the $\mathrm{Gal}(L|F)$-action to the second factor gives $\sigma(1_k) = 1_{k-1}$ for $k \geq 1$ and $\sigma(1_0) = 1_{f-1}$. Applying it on the $E \otimes_F L$-module V, we get $V = \bigoplus_k V_k$, where $V_k := 1_k V$.

Remark 5.3. In this situation, $A_E \cong \mathrm{End}_{\mathrm{Z}_D(L_f)}(V_0) \cong \mathrm{M}_m(\mathrm{Z}_D(L_f))$.

Theorem 5.4 ([**BL02**, II 3.1.]). *In terms of lattice functions, j_E satisfies $j_E^{-1}([\Theta]) = [\Lambda]$, with*

$$\Lambda(s) := \bigoplus_{k=0}^{f-1} \Theta\left(s - \frac{k}{d}\right) \pi_D^k, \quad s \in \mathbb{R},$$

where Θ is an $o_{\mathrm{Z}_D(L_f)}$-lattice function on V_0.

6. Embedding types through barycentric coordinates

In this section we keep the notation of Section 5. We need a notion of orientation on Ω_{E_D} to order the barycentric coordinates of a point in $\mathcal{I}_{E_D}$.

Definition 6.1. Let e and e' be vertices of Ω joined by an edge, and let $\mathcal{L}$ and $\mathcal{L}'$ lattice chains corresponding to e and e', respectively. The edge between e and e' is *oriented towards* e', if there are lattices $\Gamma \in \mathrm{im}(\mathcal{L})$ and $\Gamma' \in \mathrm{im}(\mathcal{L}')$, such that $\Gamma \supseteq \Gamma'$ with the quotient having κ_D-dimension 1, i.e. κ_F-dimension d. We write $e \to e'$. If x is a point in $\mathcal{I}$ then there is a chamber $C \in \Omega$ such that x lies in the closure of $|C|$, i.e. in

$$\bigcup_{S \leq C} |S|.$$

The vertices of C can be given as

$$e_1 \to e_2 \to \cdots \to e_m \to e_1.$$

If (μ_i) are the barycentric coordinates of x with respect to (e_i), i.e.

$$x = \sum_i \mu_i e_i,$$

then the class $\langle \mu \rangle$ is called the *cyclic (simplicial) type* of x in Ω.

Remark 6.2. 1. In general, not every edge has an orientation.
2. Definition 6.1 applies for $\mathcal{I}_{E_D}$ as well. The skew field is then $\mathrm{Z}_D(L_{[E_D:F]})$ instead of D and one has to substitute d by $\frac{d}{[E_D:F]}$. The above chosen orientation is just the choice of one of the directions of rotation of the Coxeter diagram of Ω.

Proposition 6.3. *The notion of cyclic type does not depend on the choice of the chamber C and the starting vertex e_1.*

Proof: Let x be a point of $\mathcal{I}$ contained in $|C|$ and $|C'|$ for two chambers C and C'. Let $e_1 \to e_2 \to \cdots \to e_m \to e_1$ and $e'_1 \to e'_2 \to \cdots \to e'_m \to e'_1$ be the vertices of C and C', respectively. It is clear that the cyclic type does not depend on the choice of the starting vertex e_1, and we can thus assume without loss of generality that $e_1 = e'_1$. Let (μ_i) be the barycentric coordinates of x w.r.t. (e_i). We have to show that they are also the barycentric coordinates w.r.t. (e'_i).

Let (μ'_i) be the barycentric coordinates of x w.r.t. (e'_i). By $e_1 = e'_1$, we have $\mu_1 = \mu'_1$. Without loss of generality let μ_1 be non-zero. If $\mu_1 = 1$, then the other coordinates are zero and we are done. In case of $\mu_1 < 1$ let i_2 and i'_2 be the first indexes greater than 1, such that μ_{i_2} and $\mu'_{i'_2}$ are non-zero, respectively. Without loss of generality assume $i'_2 \leq i_2$. Let j be the index such that e_j is equal to $e'_{i'_2}$. Then μ_j is equal to $\mu'_{i'_2}$ and $j \geq i_2$ by the definition of i_2. Thus we have to prove that the inequalities

$$i'_2 \leq i_2 \leq j$$

are equalities. Let $\mathcal{L}$ and $\mathcal{L}'$ be lattice chains of e_1 and e_j, respectively, and choose $L \in \mathrm{im}(\mathcal{L})$ and $L' \in \mathrm{im}(\mathcal{L}')$ such that $L \supset L' \supset L\pi_D$. Then the κ_D-dimension of L/L' is equal to $j-1$ and $i'_2 - 1$. Thus j is equal to i'_2. The result follows by induction. □

We denote by $x_{\mathfrak{a}}$ the barycenter of $\mathfrak{a}$ in $\mathcal{I}$.

Theorem 6.4. *Let $(E, \mathfrak{a})$ be an embedding of A with embedding type $\langle\lambda\rangle$ and suppose $\mathfrak{a}$ has rank r. If $\langle\mu\rangle$ is the cyclic type of $j_{E_D}(x_{\mathfrak{a}})$, then the following hold.*

1. $[E_D : F]r\mu \in \mathbb{N}_0^m$, *and*
2. $\langle \mathrm{row}(\lambda)\rangle = \langle [E_D : F]r\mu\rangle^c$.

Remark 6.5. With Theorem 6.4 we can calculate the embedding type from the cyclic type. For example take $r = 2$, $f = 6$, $m = 7$ and assume that $j_{E_D}(x_{\mathfrak{a}})$ is

$$\frac{3}{12}e_1 + \frac{2}{12}e_2 + \frac{1}{12}e_3 + \frac{0}{12}e_4 + \frac{0}{12}e_5 + \frac{4}{12}e_6 + \frac{2}{12}e_7,$$

and thus

$$\mathrm{pairs}(\langle 12\mu\rangle) = \mathrm{pairs}(\langle 3,2,1,0,0,4,2\rangle) = \langle (3,1),(2,1),(1,3),(4,1),(2,1)\rangle.$$

From the dual

$$\begin{aligned}\langle 12\mu\rangle^c &= \mathrm{pairs}^{-1}(\langle (1,2),(1,1),(3,4),(1,2),(1,3)\rangle \\ &= \langle 1,0,1,3,0,0,0,1,0,1,0,0\rangle\end{aligned}$$

applying Theorem 6.4 we can deduce the embedding type of $(E, \mathfrak{a})$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For the proof we can restrict to the case where $E = E_D$ because $(E, \mathfrak{a})$ is equivalent to $(E_D, \mathfrak{a})$ and the statement of Theorem 6.4 just uses E_D instead of E. We put $f := [E : F]$, i.e.

$$E \cong L_f \subseteq L$$

and

$$F \subseteq E \subseteq A_E \subseteq A.$$

Firstly we need some lemmas. The action of G on square lattice functions by conjugation induces the following maps. For $g \in G$ we have:

1. $m_g \colon \Omega \to \Omega$, $\mathfrak{a} \mapsto g\mathfrak{a}g^{-1}$,
2. $|m_g| \colon \mathcal{I} \to \mathcal{I}$, where $|m_g|(x)$ is defined to be the element z of $\mathcal{I}$, such that the square lattice function satisfies
$$\mathfrak{g}_z(t) = g\mathfrak{g}_x(t)g^{-1}, \quad t \in \mathbb{R},$$
and
3. $c_g \colon \mathcal{I}_E \to \mathcal{I}_{gEg^{-1}}$ defined via
$$\mathfrak{g}_{c_g(y)}(t) = g\mathfrak{g}_y(t)g^{-1}, \quad t \in \mathbb{R},$$
i.e. in terms of square lattice functions c_g is a map from $\mathrm{Latt}^2_{o_E}(A_E)$ to $\mathrm{Latt}^2_{o_{gEg^{-1}}}(gA_Eg^{-1})$.

We say that a map between partially oriented graphs preserves orientations if an oriented edge is mapped to an oriented edge such that the direction is preserved.

Lemma 6.6. *The maps $|m_g|$ and c_g are affine bijections which induce orientation preserving isomorphisms on the simplicial structures of the Euclidean buildings. In particular, m_g preserves the embedding type, c_g the cyclic type, and the following diagram commutes:*

$$\begin{tikzcd} \mathcal{I}^{E^\times} \arrow[r, "|m_g|"] \arrow[d, "j_E"] & \mathcal{I}^{(gEg^{-1})^\times} \arrow[d, "j_{gEg^{-1}}"] \\ \mathcal{I}_E \arrow[r, "c_g"] & \mathcal{I}_{gEg^{-1}} \end{tikzcd}$$

Proof: The lemma follows directly from the definitions of the maps involved. □

The following lemma gives a geometric interpretation of the map

$$\{\text{cyclic matrices}\} \to \{\text{cyclic matrices with only one column}\}$$
$$\langle\lambda\rangle \mapsto \langle\text{row}(\lambda)^T\rangle.$$

Lemma 6.7 (Rank reduction lemma)**.** *Assume there is a field extension $K|F$ of degree s in $E|F$, where $2 \leq s \leq m$. Suppose $\mathfrak{a}$ is a vertex in $\Omega^{E^\times}$ such that $\mathfrak{a} \cap \mathrm{Z}_A(K)$ is a facet of rank s in $\Omega_K^{E^\times}$ and assume $(E,\mathfrak{a})$ has embedding type $\langle\lambda\rangle$ and $(E, \mathfrak{a} \cap \mathrm{Z}_K(A))$ has embedding type $\langle\lambda'\rangle$. Then we get*

$$\text{row}(\lambda) \sim \text{row}(\lambda'),\ \textit{i.e.}\ \lambda \sim \text{row}(\lambda')^T.$$

Proof: By Lemma 6.6 it is enough to show the result only for one embedding equivalent to $(E,\mathfrak{a})$. For simplicity of exposition, we restrict ourselves to the case of $s = 2$. The argument for $s > 2$ is similar. We fix a D-basis of V. Then $(E,\mathfrak{a})$ is equivalent to the pearl embedding $(E_\lambda, \mathfrak{a}_\lambda)$ of λ, and moreover $\mathfrak{a}_\lambda$ is $\mathrm{M}_m(o_D)$. Now we apply a permutation p on $(E_\lambda, \mathfrak{a}_\lambda)$ such that the odd exponents of σ in $pE_\lambda p^{-1}$ are behind all even exponents, i.e. $pE_\lambda p^{-1}$ is the image of

$$x \in L_f \mapsto \text{diag}(M_{n_1}(x), M_{n_2}(x)), \quad n_1 := \sum_{i\ \text{odd}} \lambda_i, \quad n_2 := \sum_{i\ \text{even}} \lambda_i$$
$$\text{where } M_{n_1}(x) = \text{diag}(\sigma^0(x)\mathbb{1}_{\lambda_1}, \sigma^2(x)\mathbb{1}_{\lambda_3}, \ldots, \sigma^{f-2}(x)\mathbb{1}_{\lambda_{f-1}})$$
$$\text{and } M_{n_2}(x) = \text{diag}(\sigma^1(x)\mathbb{1}_{\lambda_2}, \sigma^3(x)\mathbb{1}_{\lambda_4}, \ldots, \sigma^{f-1}(x)\mathbb{1}_{\lambda_f}).$$

We conjugate $p(E_\lambda, \mathfrak{a}_\lambda)p^{-1}$ by the matrix $\text{diag}(\mathbb{1}_{n_1}, \pi_D^{-1}\mathbb{1}_{n_2})$ to obtain an embedding $(E', \mathfrak{a}')$ with the following properties. Let $K'|F$ be the field extension of degree two in $E'|F$.

- K' is the image of the diagonal embedding of L_2 in $\mathrm{M}_m(D)$ and its centralizer is $\mathrm{M}_m(D_{K'})$, where $D_{K'} := \mathrm{Z}_D(L_2)$. This follows because even powers of π_D commute with L_2.
- The intersection of $\mathfrak{a}'$ with $\mathrm{M}_m(D_{K'})$ is a hereditary order in standard form with invariant $\langle n_1, n_2\rangle$. The positivity of the integers n_i follows from the assumption that this intersection is a facet of rank 2.

Since $\pi_{D_{K'}} := \pi_D^2$ is a prime element of $D_{K'}$ which normalizes L and since the powers of σ occurring in the description of E' are even we can read the embedding type of $(E', \mathfrak{a}' \cap \mathrm{M}_m(D_{K'}))$ directly. It is the class of

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \\ \vdots & \vdots \\ \lambda_{f-1} & \lambda_f \end{pmatrix}.$$

Thus the result follows. □

The next lemma shows that changing the skew field does not change the embedding type.

Lemma 6.8 (Changing skew field lemma)**.** *Let D' be a skew field central and of finite index d over a non-Archimedean local field F'. Suppose further that $L'|F'$ is a maximal unramified field extension in $D'|F'$ normalized by a uniformizer $\pi_{D'}$ of D'. Let V' be an m-dimensional right D'-vector space. Denote the Euclidean building of $\mathrm{GL}_m(D')$ by Ω' and let Σ, Σ' be the apartments of Ω, Ω' corresponding to the standard bases (v_i), (v'_i), respectively. Then Σ' is fixed by the image E' of the diagonal embedding of L'_f in $\mathrm{M}_m(D')$. Assume further that E is the image of the diagonal embedding of L_f in $\mathrm{M}_m(D)$. Under these assumptions the map defined by*

$$\left[t \mapsto \bigoplus_i v_i \mathfrak{p}_D^{[d(t+\alpha_i)]+}\right] \mapsto \left[t \mapsto \bigoplus_i v'_i \mathfrak{p}_{D'}^{[d(t+\alpha_i)]+}\right]$$

from $|\Sigma|$ to $|\Sigma'|$ is the geometric realization $|\phi|$ of an isomorphism ϕ of simplicial complexes which preserves orientations and embedding types. The latter means that if $\mathfrak{a}'$ is the image of a hereditary order $\mathfrak{a}$ under ϕ then the embedding types of $(E, \mathfrak{a})$ and $(E', \mathfrak{a}')$ equal.

We want to remark that there is no condition about how F is related to F', so they could have different residue characteristics, but the map ϕ in the statement is of course only a map between apartments, and not between buildings.

Proof: We define ϕ to map the class of a lattice chain $\mathfrak{L}$ with

$$\mathfrak{L}_j = \bigoplus_i v_i \mathfrak{p}_D^{\nu_{i,j}}$$

to the class of $\mathfrak{L}'$ with

$$\mathfrak{L}'_j = \bigoplus_i v'_i \mathfrak{p}_{D'}^{\nu_{i,j}}.$$

We only show that the embedding type is preserved. The other properties are verified easily. We take the two lattice chains $\mathfrak{L}$ and $\mathfrak{L}'$ with corresponding hereditary orders $\mathfrak{a}$ and $\mathfrak{a}'$. Applying from the left an appropriative permutation matrix P and an appropriative diagonal matrix T (resp. T'), whose non-zero entries are powers of the corresponding prime element, we obtain simultaneously lattice chains corresponding to hereditary orders $\mathfrak{b}$, $\mathfrak{b}'$ in the same standard form. More precisely T' is obtained from T by substituting $\pi_{D'}$ for π_D. Thus, $(TPEP^{-1}T^{-1}, \mathfrak{b})$ and $(T'PE'P^{-1}T'^{-1}, \mathfrak{b}')$ have the same embedding type, and thus, by conjugating back, $(E, \mathfrak{a})$ and $(E', \mathfrak{a}')$ have the same embedding type. $\square$

We now fix a D-basis $v_1, \ldots, v_m$ of V and therefore a frame

$$\mathcal{R} := \{v_i D \mid 1 \leq i \leq m\}$$

and $\Sigma = \Omega_{\mathcal{R}}$ the apartment of Ω corresponding to $\mathcal{R}$. The algebra A is identified with $\mathrm{M}_m(D)$. By the affine bijection $|\Sigma| \cong \mathbb{R}^{m-1}$ which maps

$$[\Lambda] \text{ with } \Lambda(t) = \bigoplus_i \mathfrak{p}_D^{[(t+\alpha_i)d]+} \text{ to } ((\alpha_1 - \alpha_2)d, \ldots, (\alpha_{m-1} - \alpha_m)d),$$

we can introduce affine coordinates on $|\Sigma|$. We denote the points of $|\Sigma|$ corresponding to the vectors $0, (f, 0, \ldots, 0), (0, f, 0, \ldots, 0), \ldots, (0, \ldots, 0, f)$ by $Q_1, Q_2, \ldots, Q_m$.

Remark 6.9. The vertices of Σ are exactly the simplices corresponding to the points of

$$Q_1 + \sum_{i=2}^{m} \frac{1}{f}\mathbb{Z}(Q_i - Q_1).$$

Remark 6.10. For an element $g \in \cap_{i=1}^m (\mathrm{End}_D(v_i D))^\times$, i.e. a diagonal matrix, the map $|m_g|$ induces an affine bijection of $|\Sigma|$. If g is $\mathrm{diag}(1, \ldots, 1, \pi_D^k, 1, \ldots, 1)$, with π_D^k in the i-th row, then $|m_g|$ is of the form

$$x \mapsto x + \frac{k}{f}(Q_{i+1} - Q_i),$$

where Q_{m+1} is understood to mean Q_1. To prove this statement it is enough to prove it for elements x of $|\Sigma|$ corresponding to vertices and group elements g of the above simple form with $k = 1$. The latter is an easy calculation using a lattice chain corresponding to x.

Example 6.11. Let us assume E is the image of the diagonal embedding of L_f in $\mathrm{M}_m(D)$, i.e.

$$E = \{\mathrm{diag}(x, \ldots, x) \mid x \in L_f\}.$$

Then A_E and j_E simplify, i.e.

1. $A_E \cong \mathrm{End}_{D_E}(W)$ with $D_E := \mathrm{Z}_D(L_f)$ and $W := \bigoplus_i v_i D_E$.

2. The geometric realization of Σ is a subset of $\mathcal{I}^{E^\times}$.
3. For $[\Lambda]\in\mathcal{I}$ we have
$$j_E([\Lambda])=[\Lambda\cap W],$$
where $\Lambda\cap W$ denotes the lattice function
$$x\mapsto\Lambda(x)\cap W.$$
4. The image of $j_E|_{|\Sigma|}$ is the geometric realization of the apartment Σ_E which belongs to the frame $\{v_iD_E\mid 1\le i\le m\}$ and in affine coordinates the map has the form
$$x\in\mathbb{R}^{m-1}\mapsto\frac{1}{f}x\in\mathbb{R}^{m-1}.$$
5. The vertices of Σ_E are the points of $|\Sigma_E|$ with affine coordinate vectors in $\mathbb{Z}^{m-1}$. Specifically the points $P_i:=j_E(Q_i)$ are vertices of a chamber of Σ_E.
6. The edge between P_i and P_{i+1} is oriented towards P_{i+1}.

Proof of Example 6.11: The statement 1 is trivial and 5 and 6 follow from 4.

For 2: We have $|\Sigma|\subseteq\mathcal{I}^{E^\times}$ because, for an o_D-lattice function Λ split by $\mathcal{R}$, the action of an element of $E^\times$ on Λ is the multiplication of every lattice $\Lambda(t)$ by a fixed element $x\in D^\times$.

For 3 and 4: We use the decomposition
$$V=W\otimes_{D_E}D=W\oplus W\pi_D\oplus W\pi_D^2\oplus\cdots\oplus W\pi_D^{f-1}.$$
The function $\mathcal{I}_E\to\mathcal{I}$ which maps $[\Gamma]\in\mathcal{I}_E$ to $[\Lambda]\in\mathcal{I}$ defined by
$$\Lambda(t):=\bigoplus_{i=0}^{f-1}\Gamma\left(t-\frac{i}{d}\right)\pi_D^i,$$
is affine and $A_E^\times$-equivariant. By Theorem 5.2 it has to be j_E^{-1} and thus $j_E([\Lambda])$ is equal to $[\Lambda\cap W]$. The appearance of j_E in terms of coordinates follows now from
$$\mathfrak{p}_D^{[t]+}\cap D_E=\mathfrak{p}_{D_E}^{[\frac{t}{f}]+},$$
for $t\in\mathbb{R}$. □

Proof of Theorem 6.4: By Lemma 6.6 and Theorem 4.4 we can assume that we are in the situation of Example 6.11 above and that there is a diagonal matrix h consisting of powers of π_D with non-negative exponents less than f such that
$$(hEh^{-1},h\mathfrak{a}h^{-1})$$
is the pearl embedding of λ. We consider two cases for the proof.

Case 1: $\mathfrak{a}$ has rank 1, i.e.

$$h\mathfrak{a}h^{-1} = \mathrm{M}_m(o_D) = \mathfrak{a}_{Q_1},$$

and λ has only one column. We get $x_{\mathfrak{a}}$ from Q_1 by applying $m_{h^{-1}}$ which is a composition of maps m_g where g differs from the identity matrix by only one diagonal entry π_D^k. Now Remark 6.10 gives

$$x_{\mathfrak{a}} = Q_1 - \sum_{j=1}^{m} \frac{a_j}{f}(Q_{j+1} - Q_j),$$

where $a_j = k - 1$ if

$$\sum_{i=1}^{k-1} \lambda_i < j \leq \sum_{i=1}^{k} \lambda_i.$$

Thus in barycentric coordinates $j_E(x_{\mathfrak{a}})$ has the form

$$\frac{f - a_m + a_1}{f} P_1 + \frac{a_2 - a_1}{f} P_2 + \cdots + \frac{a_m - a_{m-1}}{f} P_m,$$

and therefore

$$\mu := \left(\frac{f - a_m + a_1}{f}, \frac{a_2 - a_1}{f}, \ldots, \frac{a_m - a_{m-1}}{f} \right)$$

satisfies part 1 of the theorem. If $(\lambda_{i_l})_{1 \leq l \leq s}$ is the subsequence of non-zero entries we define the indices

$$j_l := \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_{l-1}} + 1, \quad 2 \leq l \leq s,$$

and $j_1 := 1$. This are the indices where the μ_j are non-zero, more precisely from

$$j_l = \sum_{i=1}^{i_l - 1} \lambda_i + 1 \leq \sum_{i=1}^{i_l} \lambda_i$$

we obtain for a_j the following values:

$$a_j = a_{j_l} = i_l - 1,\ j_l \leq j < j_{l+1}, \quad \text{and} \quad a_j = a_{j_s} = i_s - 1,\ j_s \leq j \leq m,$$

and thus the subsequence of non-zero entries of $f\mu$ is

$$(f\mu_{j_l}) = (f - i_s + i_1, i_2 - i_1, i_3 - i_2, \ldots, i_s - i_{s-1}).$$

Therefore pairs$(\langle f\mu \rangle)$ is equal to

$$\langle (f - i_s + i_1, \lambda_{i_1}), (i_2 - i_1, \lambda_{i_2}), (i_3 - i_2, \lambda_{i_3}), \ldots, (i_s - i_{s-1}, \lambda_{i_s}) \rangle$$

and this is precisely pairs$(\langle \mathrm{row}(\lambda) \rangle^c)$.

Case 2: Assume the rank r of $\mathfrak{a}$ is not 1. Here we want to use rank reduction. We fix an unramified field extension $L'|F$ of degree rd in an algebraic closure of F. Denote by D' a skew field which is a central cyclic algebra over F with maximal field L' and an L'-normalizing prime element $\pi_{D'}$, i.e.

$$D' = \bigoplus_{i=0}^{dr-1} L'\pi_{D'}^{i}, \quad \pi_{D'}L'\pi_{D'}^{-1} = L', \quad \text{and} \quad \pi_{D'}^{dr} = \pi_F.$$

The images of L'_r, L'_{rf} under the diagonal embedding of L' in $\mathrm{M}_m(o_{D'})$ are denoted by F', E', respectively, and the apartment of the Euclidean building Ω' of $\mathrm{GL}_m(D')$ corresponding to the standard basis is denoted by Σ', i.e. we have a field tower

$$E' \supseteq F' \supseteq F,$$

apartments $\Sigma'_{E'}$, $\Sigma'_{F'}$, Σ' of $\Omega'_{E'}$, $\Omega'_{F'}$, Ω' and reduced Bruhat–Tits buildings $\mathcal{I}'_{E'}$, $\mathcal{I}'_{F'}$, $\mathcal{I}'$, respectively. We then obtain a commutative diagram of bijections, where the lines are induced by isomorphisms of chamber complexes which preserve orientations,

$$\begin{array}{ccc} |\Sigma| & \xrightarrow{|\phi_F|} & |\Sigma'_{F'}| \\ \downarrow{\scriptstyle j_E} & & \downarrow{\scriptstyle j_{E'}} \\ |\Sigma_E| & \xrightarrow{|\phi_E|} & |\Sigma'_{E'}| \end{array}$$

The map $|\phi_F|$ is given by

$$\left[x \mapsto \bigoplus_{i=0}^{m-1} v_i \mathfrak{p}_D^{[d(x+\alpha_i)]+}\right] \mapsto \left[x \mapsto \bigoplus_{i=0}^{m-1} v'_i \mathfrak{p}_{\mathrm{Z}_{D'}(L'_r)}^{[d(x+\alpha_i)]+}\right]$$

and $|\phi_E|$ analogously. Here (v'_i) is the standard basis of D'^m. Because of Lemma 6.8, the map $|\phi_F|$ preserves the embedding type and thus we can finish the proof by applying Lemma 6.7 to

$$\Sigma' \to \Sigma'_{F'} \to \Sigma'_{E'}.$$

More precisely, $\phi_F(\mathfrak{a})$ is a facet of rank r in $\Sigma'_{F'}$. Its barycenter has affine coordinates in $\frac{1}{r}\mathbb{Z}^{m-1}$ and therefore its preimage under $j_{F'}$ is a point y with integer affine coefficients, i.e. it corresponds to a vertex of Ω'. To emphasize the base field we write field extensions as the index of j. Because

$$j_{E'|F'}(x_{\phi_F(\mathfrak{a})})) = j_{E'|F'}(j_{F'|F}(y)) = j_{E'|F}(y),$$

the theorem follows now from the rank reduction lemma and Case 1. $\square$

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