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# ON NON-COMMUTING SETS IN FINITE SOLUBLE CC-GROUPS

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**Abstract:** Lower bounds for the number of elements of the largest non-commuting set of a finite soluble group with a CC-subgroup are considered in this paper.

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**Key words:** finite groups, non-commuting sets, CC-subgroups.

A subset of pairwise non-commuting elements in a group $G$ is a subset $T$ of $G$ such that $ab \neq ba$ for all $a, b \in T$. We let $\alpha(G)$ be the order of the largest subset consisting of pairwise non-commuting elements of a finite group $G$. Then $\alpha(G)$ is the size of the maximal empty subgraph of $\Gamma = \Gamma(G)$, the commuting graph associated to $G$. Lower bounds for $\alpha(G)$ have been considered by a number of authors. Bertram [**4**] found that for a finite group with a CC-subgroup $\alpha(G) \geq |G|^{1/3}$. (If $G$ is a group with a proper subgroup $M$ satisfying $C_G(x) \leq M$ for each $1 \neq x \in M$ then $M$ is called a CC-subgroup of $G$.) Isaacs found $\alpha(G)$ for extraspecial 2-groups (see [**4**, p. 40]) and Chin [**5**] found upper and lower bounds for extraspecial groups of odd order. Some insoluble groups have also been considered. Abdollahi, Akbari and Maimani [**1**] found $\alpha(G)$ for $G \cong GL(2, q)$ and Azad and Praeger [**3**] found $\alpha(G)$ for $G \cong GL(3, q)$.

Our aim here is to consider finite soluble groups with a CC-subgroup. Bertram [**4**] shows that $\alpha(G) \geq |G|^{1/3}$ for a group with a CC-subgroup. We will show that this bound can be improved for soluble groups by using the classification of finite groups with a CC-subgroup given by Arad and Herfort [**2**]. They prove that a soluble group with a CC-subgroup is either a Frobenius group or a 2-Frobenius group. We will use the description of Frobenius groups given in [**8**]. In a Frobenius group the CC-subgroups are the kernel and complements. A group with a normal CC-subgroup $K$ must be a Frobenius group, with $K$ the Frobenius kernel. We say that a group $G$ is a 2-Frobenius group if $G$ has a normal series $K < L < G$ such that $L$ and $G/K$ are Frobenius groups having kernels $K$ and $L/K$, respectively. As an easy consequence of Frobenius group theory one can show that $G = KAB$, $KA = L$ and $AB$ Frobenius groups. Moreover both $A$ and $B$ are cyclic.

To give an estimate of the size of the largest subset of non-commuting elements of a finite soluble group with a CC-subgroup, we use the structure above, but treat Frobenius and 2-Frobenius groups separately. Note that if $G$ is a Frobenius or 2-Frobenius group with Fitting subgroup $F = F(G)$, then the order of $G/F$ is bounded by the square of the order of the smallest chief factor of $G$ in $F$. Since $G$ contains a conjugacy class of non-commuting elements of order $|F|$ it is not surprising that the number of chief factors of $G$ in $F$ will play a role in our estimate.

**Theorem 1.** *Let $G$ be a finite soluble Frobenius group and let $n$ be the number of chief factors of $G$ below $F(G)$ in a chief series of $G$ passing through $F(G)$. Then $\alpha(G) \geq |G|^{n/(n+1)}$.*

*Proof:* Suppose then that $G$ is a Frobenius group with kernel $K$ and complement $H$. If $M < N \leq K$ and $N/M$ is a chief factor of $G$, then $G/M$ is a Frobenius group with kernel $K/M$ and complement $HM/M$. Hence $H$ acts fixed point freely on $N/M$ and so $|H| < |N/M|$. Since the number of conjugates of $H$ in $G$ is $|K|$, and nontrivial elements of distinct conjugates can not commute, we can choose a nontrivial element from each conjugate to form a non-commuting set. Suppose now the number of chief factors of $G$ below $K$ in a chief series of $G$ through $K$ is $n$. We show that $|K| > |G|^{n/(n+1)}$. Consider $|G|^n = |K|^n|H|^n$. Since $|H|$ is less than the order of any chief factor below $K$, $|H|^n < |K|$ and so $|G|^n < |K|^{n+1}$. □

For the class of Frobenius groups with abelian complement, this lower bound is best possible. We choose a prime $p$ and an integer $n \geq 1$ and then let $K$ be an elementary abelian group of order $p^n$. Let $H$ be a cyclic group of order $p-1$ acting as power automorphisms on $K$ and put $G = KH$. Note that $G$ is the disjoint union of the non-trivial elements of the abelian groups $K$ and the conjugates of $H$ and so the size of a maximal non-commuting subset of $G$ is exactly $|K|+1$. We now have

$$(|K|+1)^{n+1}/|G|^n = (p^n+1)^{n+1}/p^{n^2}(p-1)^n < (p^n+1)^{n+1}/(p-1)^{n(n+1)}$$

$$= \left(\left(1+\frac{1}{p-1}\right)^n + \left(\frac{1}{p-1}\right)^n\right)^{n+1}$$

and so we can make $\alpha(G)$ as close to $|G|^{n/(n+1)}$ as we want by choosing $p$ large enough. Note in particular when $n = 1$ we have $|G| < (|K|+1)^2 <$

$|G|(1+\frac{2}{p-1})^2$ and so there are examples with $\alpha(G)$ as close to $\sqrt{|G|}$ as desired.

For 2-Frobenius groups the situation is more complicated. In this case $G$ has Fitting subgroup $F = F(G)$ complemented by a metacyclic group $H = AB$ with $A$ cyclic, normal, of odd order and a core-free CC-subgroup, $B = \langle b\rangle$ a cyclic subgroup acting faithfully on $A$ and $A$ and $B$ of coprime order. Moreover $FB$ is a CC-subgroup. We may not have $|H|$ less than the order of any chief factor of $G$ in $F$ and so the proof above will not hold. We find a slightly worse bound in this case, using similar ideas to the proof of Theorem 1.

We first analyse the action of $H$ on a chief factor of $G$ in $F$. Since $A$ acts fixed point freely on $F$, each chief factor of $G$ below $F$ is faithful as $H$-module. We begin by assuming that $F$ is a minimal normal subgroup of $p$-power order for some prime $p$. We then have $F$ is free as $B$-module and so $F$ is the direct sum of $r$ copies of the regular module by [**6**, 44.14], [**7**, B, 5.15], [**7**, B, 6.21] and [**7**, B, 5.25]. Thus $|F| = p^{r|B|}$. Let $B = \langle b\rangle$. Suppose that $fb$ and $gb$, $f, g \in F$, commute. We have $1 = [fb, gb] = [f, b]^b[b, g]^b$ if and only if $1 = [b, fg^{-1}]$. Thus $fb$ and $gb$ commute if and only if $f$ and $g$ are equal modulo the centraliser of $B$ in $F$. Since the regular module contains a unique trivial irreducible submodule, $|C_F(B)| = p^r$. Let $\mathcal{S}$ be a transversal for $C_F(B)$ in $F$ and set $\mathcal{B} = \{fb : f \in \mathcal{S}\}$. Then $\mathcal{B}$ is a non-commuting subset of $G$. Further if $1 \neq a \in A$ and $\mathcal{B}^a = \{x^a : x \in \mathcal{B}\}$, then $\mathcal{B} \cap \mathcal{B}^a = \emptyset$ and elements of $\mathcal{B}$ do not commute with elements of $\mathcal{B}^a$. If $\mathcal{B}_1 = \cup_{a\in A}\mathcal{B}^a$ then $\mathcal{B}_1$ is a non-commuting set containing $|A|p^{r(|B|-1)}$ elements. If $A = \langle a\rangle$ then the set $\{a^f : f \in F\}$ is a non-commuting set and no element of this set commutes with any element of $\mathcal{B}_1$. Thus $\alpha(G) \geq |A|p^{r(|B|-1)} + p^{r|B|} = p^{r(|B|-1)}(|A| + p^r)$.

It is now easy to extend this estimate to arbitrary $F$. Since each chief factor is a $|B|^{\text{th}}$ power, we have $|F| = s^{|B|}$ for some $s$ and then $\alpha(G) \geq s^{|B|-1}(|A| + s)$. Note that for small numbers of chief factors, the contribution of elements of order prime to $|A|$ can be significant, but since $s \geq 2^n$ if there are $n$ chief factors in $F$ they do not make a significant contribution for large $n$. This is reflected in the proofs below.

**Theorem 2.** *Let $G$ be a finite soluble 2-Frobenius group and let $n$ be the number of chief factors of $G$ below $F(G)$ in a chief series of $G$ passing through $F(G)$, the Fitting subgroup of $G$. Then*

(i) *if $n \leq 3$ then $\alpha(G) \geq |G|^{2/3}$;*

(ii) *if $n \geq 4$ then $\alpha(G) \geq |G|^{2n/(2n+3)}$.*

*Proof:* We use the notation above.

(i) We prove that $(s^{|B|-1}(|A|+s))^3 \geq |G|^2$ or equivalently $(s^{|B|-1}(|A|+s))^3 \geq s^{2|B|}|A|^2|B|^2$ (note that the proof is independent of $n$ and so this bound holds for all 2-Frobenius groups: (ii) gives better bounds for $n \geq 4$). We first dispose of some small values of $|B|$.

If $|B| = 2$, then we require $(s(|A|+s))^3 \geq 4s^4|A|^2$ or $|A|^3 + 3|A|^2s + 3|A|s^2 + s^3 \geq 4s|A|^2$. If $s \geq |A|$ then $3|A|s^2 + s^3 \geq 4s|A|^2$. If $|A| \geq s$ then $|A|^3 + 3|A|^2s \geq 4s|A|^2$. In either case we have the desired inequality.

If $|B| = 3$, we require $(s^2(|A|+s))^3 \geq 9s^6|A|^2$ or $|A|^3 + 3|A|^2s + 3|A|s^2 + s^3 \geq 9|A|^2$. Since $|A| \geq 7$ we have $|A|^3 + 3|A|^2 \geq 10|A|^2$, giving the required inequality.

If $|B| \geq 4$, we require $s^{3(|B|-1)}(|A|+s)^3 \geq s^{2|B|}|A|^2|B|^2$ or $s^{|B|-3}(|A|^3 + 3|A|^2s + 3|A|s^2 + s^3) \geq |A|^2|B|^2$. It will be enough to show that $s^{|B|-3}(|A|^3 + 3|A|^2s) \geq |A|^2|B|^2$ or equivalently $s^{|B|-3}(|A|+3s) \geq |B|^2$. Since $|A| \geq |B|$ it will be enough to show $s^{|B|-3} \geq \frac{1}{2}|B|$ and $3s^{|B|-2} \geq \frac{1}{2}|B|^2$. Both inequalities are easily checked. This completes the proof of (i).

(ii) Suppose now that $n > 3$. It will be enough to prove that $s^{|B|(2n+3)} \geq s^{2n|B|}|A|^{2n}|B|^{2n}$ or equivalently $s^{3|B|} \geq |A|^{2n}|B|^{2n}$. If $M/N$ is a chief factor of $G$ of smallest order in $F$, then $s^{|B|} \geq |M/N|^n \geq |A|^n$ and so it will be enough to prove that $s^{|B|} \geq |B|^{2n}$. Since $s \geq 2^n$, it will be enough to show $2^{n|B|} \geq |B|^{2n}$ or equivalently $2^{|B|} \geq |B|^2$. This is easily established unless $|B| = 3$ (and is false for $|B| = 3$). If $|B| = 3$, from the observations above either all chief factors in $F$ have order greater than $2^3$ or there is at least one with order $2^3$. In the first case we have $s \geq 3^n$ and then $3^{3n} \geq 2^{3n}$. In the second case we must have $|A| = 7$. Then it will be enough to show $2^{9n} \geq 7^{2n}3^{2n}$ and this is immediate, since $2^9 > (21)^2$. This completes the proof of (ii). □

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## References

[1] A. Abdollahi, S. Akbari, and H. R. Maimani, Non-commuting graph of a group, *J. Algebra* **298(2)** (2006), 468–492. DOI: 10.1016/j.jalgebra.2006.02.015.

[2] Z. Arad and W. Herfort, Classification of finite groups with a CC-subgroup, *Comm. Algebra* **32(6)** (2004), 2087–2098. DOI: 10.1081/AGB-120037209.

[3] A. Azad and C. E. Praeger, Maximal subsets of pairwise noncommuting elements of three-dimensional general linear groups, *Bull. Aust. Math. Soc.* **80(1)** (2009), 91–104. DOI: 10.1017/S0004972709000057.
[4] E. A. Bertram, Some applications of graph theory to finite groups, *Discrete Math.* **44(1)** (1983), 31–43. DOI: 10.1016/0012-365X(83)90004-3.
[5] A. Y. M. Chin, On non-commuting sets in an extraspecial $p$-group, *J. Group Theory* **8(2)** (2005), 189–194. DOI: 10.1515/jgth.2005.8.2.189.
[6] C. W. Curtis and I. Reiner, *"Representation theory of finite groups and associative algebras"*, Pure and Applied Mathematics **XI**, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.
[7] K. Doerk and T. Hawkes, *"Finite soluble groups"*, de Gruyter Expositions in Mathematics **4**, Walter de Gruyter & Co., Berlin, 1992.
[8] B. Huppert, *Endliche Gruppen. I.*, Die Grundlehren der Mathematischen Wissenschaften **134**, Springer-Verlag, Berlin-New York, 1967.

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