Publ. Mat. **55** (2011), 3–18

GROUP ACTIONS ON ALGEBRAIC CELL COMPLEXES

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Abstract

We establish an algebraic version of the classical result that a G-map f between G-complexes which restricts to a homotopy equivalence f^H on H-fixed sets for all subgroups H of G is a G-homotopy equivalence. This is used to give an alternative proof of a theorem of Bouc. We also include a number of illustrations and applications.

1. Introduction

The following result [**15**, Chapter II, Proposition 2.7] is recorded by tom Dieck.

Theorem 1.1. *If G is a discrete group and $f\colon X \to Y$ is a G-map between G-complexes which induces homotopy equivalences $X^H \to Y^H$ between the H-fixed subspaces for all subgroups $H \leq G$, then f is itself a G-homotopy equivalence.*

We give a more algebraic statement concerning chain complexes over a fixed ring S (associative with 1). Let G be a discrete group. We say that X is a *G-complex* if it is a CW-complex on which G acts permuting the cells and in such a way that the stabilizer of each cell fixes that cell point by point. Brown uses the terminology *admissible* G-complex [**2**, Chapter IX, §10] to emphasise the strict condition on cell stabilizers. Our terminology is consistent with tom Dieck's definition, following Proposition 1.5 of [**15**, Chapter II], of G-complex for arbitrary locally compact groups G. We write $C_*(X;S)$ for the augmented (or reduced) cellular chain complex with coefficients in a ring S, [**11**, Chapter IV, §4]. When the coefficient ring is clear from the context we use the abbreviation $C_*(X)$. For a subgroup H of G, the subcomplex of H-fixed points is denoted by X^H.

2000 *Mathematics Subject Classification.* 57Q05, 20J05.
Key words. Cell complex, group action, equivariant homotopy.

Theorem 1.2. *Let $f\colon X \to Y$ be a G-map between G-complexes X and Y. Suppose that the induced chain maps $f_*^H\colon C_*(X^H; S) \to C_*(Y^H; S)$ are chain homotopy equivalences for all subgroups H of G. Then f_* is an equivariant chain homotopy equivalence.*

We will give a proof which is entirely algebraic and so has the virtue of providing a result at the more general level of chain complexes. If Δ is a G-set then the free S-module $S\Delta$ on Δ is an SG-module: this is the *permutation module* on Δ. Permutation modules arise naturally from any G-complex X because the nth cellular chain group is, in effect, the permutation module on the set of n-cells of X. Given two G-sets Δ and Δ' we shall say that an SG-map $S\Delta \to S\Delta'$ is *admissible* if it carries $S[\Delta^H]$ into $S[\Delta'^H]$ for all subgroups H. While the boundary maps in a cellular chain complex are more subtle than the chain groups, nonetheless these maps are always admissible, and this is essentially all that matters for the validity of Theorem 1.2.

We define a *special* SG-complex to consist of a sequence of G-sets Δ_n for $n \in \mathbb{N} = \{0, 1, 2, 3, \dots\}$ together with admissible maps $S\Delta_n \to S\Delta_{n-1}$ which define a chain complex. If we use a single letter X to denote a special SG-complex, then $C_n(X, S)$ denotes $S\Delta_n$ for $n \geq 0$ and $C_n(X, S) = 0$ for $n < -1$. We shall always consider the *augmented* (also called *reduced*) chain complex, so $C_{-1}(X, S) = S$ and the chain map $C_0 \to C_{-1}$ is defined to be the augmentation. Elements of Δ_n may be called n-cells, but of course, in this generality it is not necessarily possible to realize the complex X as a space. The classical construction in case the complex does arise from a space is explained in [**11**, Chapter IV, §4]. However it is possible, for each subgroup H of G to make sense of the chain complex $C_*(X^H; S)$ and Theorem 1.2 now extends to a statement about admissible SG-maps between special SG-complexes: this is our main result and Theorem 1.2 is a particular case of it.

Theorem 1.3. *Let $f\colon X \to Y$ be an admissible SG-map between special SG-complexes which induces chain homotopy equivalences $C_*(X^H; S) \to C_*(Y^H; S)$ for all subgroups H of G. Then f induces an SG-chain homotopy equivalence from $C_*(X; S)$ to $C_*(Y; S)$.*

There are several applications of this result. Our first, and the original motivation for our results, is a new and fairly short proof of a theorem of Bouc [**1**].

Theorem 1.4. *Let G be a finite group and let S be a ring with no $|G|$-torsion. If X is a finite-dimensional S-acyclic special SG-complex then the augmented cellular chain complex of X is SG-split.*

Bouc states his result for G-simplicial complexes. Our more general formulation, Theorem 1.4, includes the possibility of CW-complexes and this is often convenient in applications. Moroever, by using Theorem 1.3 in the argument instead of Theorem 1.2, we see that our version of Bouc's Theorem is valid for the wider notion of special SG-complex.

Our proof of Bouc's Theorem rests on the following corollary of Theorem 1.3 which may be of independent interest.

Corollary 1.5. *Let S be a ring. If X is a special SG-complex such that X^H is S-acyclic for all subgroups H of G then $C_*(X)$ is SG-split.*

There are many instances where Bouc's Theorem provides a nice insight. One very simple example concerns relative group cohomology as discussed by Brown [**2**, Chapter VI, §2]. Bouc's Theorem has been used by Nucinkis in this context to give information about relative cohomological dimension, [**12**]. Further illustrations abound and we conclude the paper with short sections highlighting some of these.

2. Deduction of Corollary 1.5 from Theorem 1.3

Lemma 2.1. *Let R be a ring. Let C_* and C'_* be chain homotopy equivalent chain complexes of R-modules. If C'_* is R-split then so is C_*.*

Proof: Let $f\colon C_* \to C'_*$ and $g\colon C'_* \to C_*$ be chain maps and let $h\colon C_* \to C_{*+1}$ be a chain homotopy so that $1 - gf = dh + hd$. Let $\ell\colon C'_* \to C'_{*+1}$ be a splitting of C'_*. Then $g\ell f + h\colon C_* \to C_{*+1}$ is a splitting of C_*:

$$\begin{aligned} d(g\ell f + h) + (g\ell f + h)d &= g(d\ell + \ell d)f + (dh + hd) \\ &= gf + (dh + hd) = 1. \end{aligned} \qquad \square$$

Proof of Corollary 1.5: Let $\cdot$ denote a one point space. The map $f : X \to \cdot$ satisfies the hypotheses of Theorem 1.3, and so is an equivariant homotopy equivalence. Since $C_*(\cdot)$ is SG-split, Lemma 2.1 applies with $R = SG$ to the chain complexes $C_*(X)$ and $C_*(\cdot)$ and gives the result. $\square$

3. Proof of Theorem 1.3

Let X and Y be special SG-complexes. We fix the group G and ring S throughout this section, and take S as coefficient group for all chain groups and homology groups. We need to consider G-maps at the level of chains.

For each $i \in \mathbb{Z}$, a G-map

$$\ell\colon C_*(X) \to C_{*+i}(Y)$$

of degree i consists of a sequence of SG-module homomorphisms

$$\ell_n\colon C_n(X) \to C_{n+i}(Y).$$

Then $\ell\colon C_*(X) \to C_{*+i}(Y)$ is *admissible* if it is a G-map of degree i and it carries $C_*(X^H)$ into $C_*(Y^H)$ for all subgroups H of G.

We write d for the boundary map in any chain complex C_*. The boundary map in $C_*(X)$ is an admissible map $C_*(X) \to C_{*-1}(X)$ of degree -1.

An *admissible chain map* $\ell\colon C_*(X) \to C_*(Y)$ is an admissible map of degree 0 such that $d\ell = \ell d$. If $f\colon X \to Y$ is a G-map then the induced map f_* is an admissible chain map.

One could proceed at this point to construct a derived category of admissible G-complexes. We do not need the full strength of this, but continue naively.

An *admissible chain homotopy* between two admissible chain maps ℓ and ℓ' is an admissible map $h\colon C_*(X) \to C_{*+1}(Y)$ of degree $+1$ such that $\ell - \ell' = dh + hd$.

For G-complexes X and Y, the set of admissible chain maps from $C_*(X)$ to $C_*(Y)$ naturally has the structure of an additive group and the subset of those maps which are admissibly homotopic to 0 is a subgroup. If S is commutative then these naturally inherit S-module structures. We write $[X,Y]^G_*$ for the quotient group; this is the group of admissible homotopy classes of admissible chain maps, and it is an S-module when S is commutative.

Our immediate goal is now to establish

Proposition 3.1. *Let $f\colon Y \to Z$ be a G-map between special SG-complexes such that for all subgroups H of G, the restriction $f^H\colon C_*(Y^H) \to C_*(Z^H)$ is a chain homotopy equivalence. Then the map $f_*\colon [\ \ ,Y]^G_* \to [\ \ ,Z]^G_*$ is an isomorphism.*

Theorem 1.3 follows at once from this proposition by Yoneda's Lemma.

The next two results are the key steps needed for the proof of Proposition 3.1.

Proposition 3.2. *Let X, Y and Z be G-complexes and let A be a G-subcomplex of X; write $i\colon A \subset X$ for the inclusion. Suppose given admissible maps*

$$\ell\colon C_*(X) \to C_*(Z), \quad f\colon C_*(Y) \to C_*(Z),$$
$$k\colon C_*(A) \to C_*(Y), \quad h\colon C_*(A) \to C_{*+1}(Z),$$

where

(1) *ℓ, k, f are chain maps, h is a chain homotopy between $\ell \circ C_*(i)$ and fk; that is $\ell \circ C_*(i) - fk = dh + hd$, and*
(2) *for all subgroups H of G, the restriction $f^H\colon C_*(Y^H) \to C_*(Z^H)$ is a chain homotopy equivalence.*

Then k and h can be extended to admissible maps

$$\widehat{k}\colon C_*(X) \to C_*(Y), \quad \widehat{h}\colon C_*(X) \to C_{*+1}(Z),$$

where $\widehat{k}$ is a chain map and $\widehat{h}$ is a chain homotopy between ℓ and $f \circ \widehat{k}$.

Proof: This is a chain complex analogue of [**15**, Chapter II, Proposition (2.5)]. Suppose first that there is a natural number n such that all cells of X not lying in A have dimension n. Let Σ be a set of G-orbit representatives of these cells. For each $\sigma \in \Sigma$ let G_σ denote the stabilizer of σ. Then there exist choices of a chain map $g_\sigma\colon C_*(Z^{G_\sigma}) \to C_*(Y^{G_\sigma})$ and chain homotopies

$$h'_\sigma\colon C_*(Z^{G_\sigma}) \to C_*(Z^{G_\sigma})$$

and

$$h''_\sigma\colon C_*(Y^{G_\sigma}) \to C_*(Y^{G_\sigma})$$

such that

$$1 - fg_\sigma = dh'_\sigma + h'_\sigma d$$

on $C_*(Z^{G_\sigma})$ and

$$1 - g_\sigma f = dh''_\sigma + h''_\sigma d$$

on $C_*(Y^{G_\sigma})$. For each $\sigma \in \Sigma$, $d\sigma$ is a chain of $(n-1)$-cells of X and so can be regarded as an element of the chain complex $C_*(A)$ where k, h are defined. We define $\widehat{k}$ and $\widehat{h}$ on Σ by

$$\widehat{k}(\sigma) := g_\sigma \ell(\sigma) + h''_\sigma k(d\sigma) - g_\sigma h(d\sigma);$$

and

$$\widehat{h}(\sigma) := -h'_\sigma h(d\sigma) + h'_\sigma \ell(\sigma) + (fh''_\sigma - h'_\sigma f)\widehat{k}\sigma.$$

Now for each $\sigma \in \Sigma$ we have

$$\begin{aligned}
d\widehat{k}(\sigma)-k(d\sigma) &= d(g_\sigma\ell(\sigma)+h''_\sigma k(d\sigma)-g_\sigma h(d\sigma))-k(d\sigma)\\
&= g_\sigma\ell(d\sigma)+dh''_\sigma k(d\sigma)-g_\sigma dh(d\sigma)-(g_\sigma f+dh''_\sigma+h''_\sigma d)k(d\sigma)\\
&= g_\sigma(\ell-dh-fk)(d\sigma)-h''_\sigma dk(d\sigma)\\
&= g_\sigma hd(d\sigma)-h''_\sigma kd(d\sigma)\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
h(d\sigma)-\ell\sigma+f\widehat{k}(\sigma) &= h(d\sigma)-\ell\sigma+f(g_\sigma\ell(\sigma)+h''_\sigma k(d\sigma)-g_\sigma h(d\sigma))\\
&= h(d\sigma)-\ell\sigma+fg_\sigma\ell(\sigma)+fh''_\sigma k(d\sigma)-fg_\sigma h(d\sigma)\\
&= (dh'_\sigma+h'_\sigma d)h(d\sigma)-(dh'_\sigma+h'_\sigma d)\ell(\sigma)+fh''_\sigma k(d\sigma)\\
&= dh'_\sigma h(d\sigma)+h'_\sigma(\ell-fk-hd)(d\sigma)\\
&\qquad -(dh'_\sigma+h'_\sigma d)\ell(\sigma)+fh''_\sigma k(d\sigma)\\
&= dh'_\sigma h(d\sigma)-dh'_\sigma\ell(\sigma)+(fh''_\sigma-h'_\sigma f)k(d\sigma)\\
&= dh'_\sigma h(d\sigma)-dh'_\sigma\ell(\sigma)+(fh''_\sigma-h'_\sigma f)d\widehat{k}\sigma\\
&= dh'_\sigma h(d\sigma)-dh'_\sigma\ell(\sigma)-d(fh''_\sigma-h'_\sigma f)\widehat{k}\sigma\\
&= d(h'_\sigma h(d\sigma)-h'_\sigma\ell(\sigma)-(fh''_\sigma-h'_\sigma f)\widehat{k}\sigma)\\
&= -d\widehat{h}(\sigma).
\end{aligned}$$

Thus the unique G-equivariant maps which are determined by these choices on Σ and which extend h, k satisfy the required conclusion.

In the general case, we apply this argument inductively, extending h, k to $A \cup X^n$ where X^n denotes the n-skeleton of X, for each $n \geq 0$. In this way, h, k are extended to the whole of X. □

Proposition 3.3. *Let A, Y and Z be G-complexes. Suppose given admissible maps*

$$f\colon C_*(Y)\to C_*(Z),\quad k\colon C_*(A)\to C_*(Y),\quad h\colon C_*(A)\to C_{*+1}(Z),$$

where f, k, h are chain maps, h is a chain homotopy between fk and 0, and for all subgroups H of G, the restriction

$$f^H\colon C_*(Y^H)\to C_*(Z^H)$$

is a chain homotopy equivalence. Then there exists an admissible chain homotopy $k'\colon C_*(A)\to C_{*+1}(Y)$ between k and 0.

Proof: Let X be the cone on A. Then the augmented cellular chain complex $C_*(X)$ is determined by that of A as follows: $C_n(X) = C_n(A) \oplus C_{n-1}(A)$, with boundary given by the 2×2 matrix $\left(\begin{smallmatrix} d & 1 \\ 0 & -d \end{smallmatrix}\right)$. The inclusion of A into X induces the inclusion $C_*(A) \to C_*(X)$, $x \mapsto (x, 0)$. Define $\ell \colon C_*(X) \to C_*(Z)$ by $\ell \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) := fkx + hy$. Then ℓ is an admissible chain map and $\ell = fk$ on $C_*(A)$. By Proposition 3.2, k can be extended to an admissible chain map $\widehat{k} \colon C_*(X) \to C_*(Y)$ such that $f\widehat{k}$ is homotopic to ℓ. Now observe that $\widehat{k} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = kx + k'y$ for some admissible map k' of degree $+1$, and the fact that $\widehat{k}$ is a chain map implies that k' is a chain homotopy between k and 0 as required. □

Proof of Proposition 3.1: Proposition 3.2 shows that f_* is surjective and Proposition 3.3 that it is injective. □

4. Reductions for the proof of Theorem 1.4

We review the classical theory of extensions which can be found in detail in many texts, for example see [**10**]. For readers' convenience here is a succinct summary without proofs.

Let R be any ring (associative with 1). Given two R-modules A and C there is a bijective correspondence between the elements of $\mathrm{Ext}^1_p(C, A)$ and equivalence classes of extensions $A \rightarrowtail B \twoheadrightarrow C$. The extensions are short exact sequences and two extensions $A \rightarrowtail B \twoheadrightarrow C$ and $A \rightarrowtail B' \twoheadrightarrow C$ are equivalent if and only if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, equal] \\ A \arrow[r, tail] & B' \arrow[r, two heads] & C \end{tikzcd}$$

It is then automatic that the map $B \to B'$ is an isomorphism and the corresponding diagram with this map replaced by its inverse is witness to the symmetry of this relation between extensions. The zero element of $\mathrm{Ext}^1_R(C, A)$ corresponds to the split extension or direct sum $B = C \oplus A$.

Lemma 4.1.

(1) *Given a commutative diagram connecting two short exact sequences,*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, "\phi"] \\ A \arrow[r, tail] & B' \arrow[r, two heads] & C', \end{tikzcd}$$

in which the top row corresponds to $\xi \in \mathrm{Ext}^1_R(C, A)$ and the bottom row to $\xi' \in \mathrm{Ext}^1_R(C', A)$) then we have

$$\phi^*(\xi') = \xi,$$

where ϕ^* is determined by the contravariant functor $\mathrm{Ext}^1_R(\ \ , A)$, that is $\phi^* = \mathrm{Ext}^1_R(\phi, C)$.

(2) *Given a commutative diagram*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\theta"] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, equal] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C \end{tikzcd}$$

in which the top row corresponds to $\eta \in \mathrm{Ext}^1_R(C, A)$ and the bottom row to $\eta' \in \mathrm{Ext}^1_R(C, A')$) then we have

$$\theta_*(\eta) = \eta',$$

where θ_* is determined by the covariant functor $\mathrm{Ext}^1_R(C,\ \ )$, that is $\theta_* = \mathrm{Ext}^1_R(C, \theta)$.

Now let $R = SG$ where S is a ring (associative with 1) and G is a finite group of order $|G|$. We include a proof of the following consequence of the extension theory.

Proposition 4.2. *Let $A \rightarrowtail B \twoheadrightarrow C$ be a short exact sequence of SG-modules in which A has no $|G|$-torsion and C is projective as an S-module. If the induced sequence*

$$A/|G|A \rightarrowtail B/|G|B \twoheadrightarrow C/|G|C,$$

obtained by tensoring with $S/|G|S$, is SG-split then the original sequence is SG-split.

Proof: Note that the sequence $A \rightarrowtail B \twoheadrightarrow C$ is S-split since C is projective. In particular the induced sequence $A/|G|A \rightarrowtail B/|G|B \twoheadrightarrow C/|G|C$ is again a short exact sequence because it can be obtained by tensoring the S-split sequence over S with $S/|G|S$. We view A as a submodule of B. Consider the commutative diagram of R-modules

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, "\pi"] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, equal] \\
A/|G|A \arrow[r, tail] \arrow[d, equal] & B/|G|A \arrow[r, two heads] \arrow[d] & C \arrow[d, "\rho"] \\
A/|G|A \arrow[r, tail] & B/|G|B \arrow[r, two heads] & C/|G|C
\end{tikzcd}$$

The middle row is obtained from the top row simply by factoring out the submodule $A/|G|A$ from A and B. This gives an exact sequence intermediate between the top row and the bottom row. Thus the rows are short exact sequences and we introduce notation for the corresponding extension classes: $\xi \in \mathrm{Ext}^1_R(C, A)$, $\xi' \in \mathrm{Ext}^1_R(C, A/|G|A)$, and $\xi'' \in \mathrm{Ext}^1_R(C/|G|C, A/|G|A)$. We are given that $\xi'' = 0$. By Lemma 4.1, $\pi_*(\xi) = \xi' = \rho^*(\xi'')$. Thus we have $\pi_*(\xi) = 0$.

Since A has no $|G|$-torsion we have a short exact sequence

$$A \overset{\times|G|}{\rightarrowtail} A \overset{\pi}{\twoheadrightarrow} A/|G|A.$$

We will deduce that $\xi = 0$ by using the associated long exact sequence:

$$\cdots \to \mathrm{Ext}^1_R(C, A) \overset{\times|G|}{\to} \mathrm{Ext}^1_R(C, A) \overset{\pi_*}{\to} \mathrm{Ext}^1_R(C, A/|G|A) \to \cdots .$$

A standard spectral sequence argument using the fact that C is S-projective shows that $\mathrm{Ext}^1_R(C, A)$ can be identified with the group cohomology

$$H^1(G, \mathrm{hom}_S(C, A)),$$

and since the first (and higher) cohomology of a finite group of G has exponent dividing its order $|G|$, it follows that the induced map $\times|G|$ in our long exact sequence is zero. Therefore π_* is injective and it follows that $\xi = 0$ because $\pi_*(\xi) = 0$. Thus the extension splits as required. □

Corollary 4.3. *Let G be a finite group and let X be an S-acyclic special SG-complex. Assume also that the ring S has no $|G|$-torsion. If $C_*(X; S/|G|S)$ is SG-split then so is $C_*(X; S)$.*

Proof: For each n, let Z_n denote the kernel of the boundary map $C_n(X) \to C_{n-1}(X)$. Since X is acyclic, we have short exact sequences

$$Z_n \rightarrowtail C_n(X) \twoheadrightarrow Z_{n-1}$$

for all integers n. Asserting that $C_*(X)$ is SG-split is equivalent to asserting that each of these short exact sequence is split. A simple induction shows that each Z_n is projective as S-module and therefore also inherits the property of having no $|G|$-torsion from S. This allows us to apply Proposition 4.2.

The chain complex $C_*(X, S/|G|S)$ can be identified with $C_*(X) \otimes S/|G|S$ and the hypotheses ensures that each of the short exact sequences

$$Z_n/|G|Z_n \rightarrowtail C_n/|G|C_n \twoheadrightarrow Z_{n-1}/|G|Z_{n-1}$$

is SG-split. Hence the result follows from Proposition 4.2. □

5. The proof of Theorem 1.4

We include a sketch of the classical result of Smith theory in a version suited to our setting.

Proposition 5.1. *Let p be a prime and let G be a finite p-group. Let S be a ring of characteristic p and let X be a finite-dimensional S-acyclic special SG-complex. Then X^G is S-acyclic.*

Proof: By using induction on the length of a central composition series of G one reduces to the case when G is cyclic of order p. In the standard proof, one considers the elements $1 - g$ and $\widehat{g} = 1 + g + \cdots + g^{p-1}$ of the group ring SG and then the two short exact sequences of chain complexes:

$$(1-g) \cdot C_*(X) \oplus C_*(X^G) \rightarrowtail C_*(X) \twoheadrightarrow \widehat{g} \cdot C_*(X),$$

and

$$\widehat{g} \cdot C_*(X) \oplus C_*(X^G) \rightarrowtail C_*(X) \twoheadrightarrow (1-g) \cdot C_*(X).$$

The exactness of these sequence relies on two observations. First, the chain groups are permutation modules and there are only two types of orbit. Secondly, in SG we have $(1-g)^{p-1} = \widehat{g}$ and SG is isomorphic to the quotient $S[x]/(x^p)$ of the polynomial ring $S[x]$ via the map $x \mapsto 1-g$. In fact the pair $1-g$, $\widehat{g}$ could be replaced by $(1-g)^i$, $(1-g)^j$ for any natural numbers i and j such that $i + j = p$ without making any difference to the argument. Each short exact sequence of chain complexes gives rise to a long exact sequence of homology and a simple downward induction shows that $H_m(\widehat{g} \cdot C_*(X)) = H_m((1-g) \cdot C_*(X)) = H_m(X^G) = 0$ using the fact that the central chain complex in both sequences is acyclic. The induction begins from the finite-dimensionality hypothesis: in sufficiently large dimensions, the homology of all the chain complexes involved vanishes. □

Lemma 5.2. *Let p be a prime. Let S be a ring with no p-torsion. Let Y be an S/pS-acyclic special S-complex. Then Y is S/p^mS-acyclic for all natural numbers m.*

Proof: The hypothesis that S has no p-torsion ensures that all the sections S/pS, pS/p^2S, p^2S/p^3S are isomorphic via multiplication by p. Now an easy argument using the universal coefficient theorem gives the result. □

Corollary 5.3. *Let p be a prime. Let G be a finite p-group. Let S be a ring with no p-torsion. Let X be a finite-dimensional S/pS-acyclic special SG-complex. Then X^G is $S/|G|S$-acyclic.*

Proof: Proposition 5.1 shows that X^G is S/pS-acyclic. Now Lemma 5.2 applies. □

Proof of Theorem 1.4: Let P be a Sylow subgroup of G. Corollary 5.3 shows that X^Q is $S/|P|S$-acyclic for all subgroups Q of P, and hence by Corollary 1.5, $C_*(X, S/|P|S)$ is SP-split. Corollary 4.3 shows that $C_*(X)$ is SP-split. Let $p_1, \ldots, p_m$ be the distinct prime divisors of $|G|$ and let P_i be a Sylow p_i-subgroup. Let $h_i\colon C_*(X) \to C_{*+1}(X)$ be a SP_i-splitting. Let T_i be a transversal to P_i in G, so that G is the disjoint union of the cosets P_it with $t \in T_i$. Define $\widehat{h}_i\colon C_*(X) \to C_{*+1}(X)$ by $\widehat{h}_ix := \sum_{t\in T_i}(h_i(xt^{-1}))t$. Then each $\widehat{h}_i$ is a G-map. Choose integers $n_1, \ldots, n_m$ such that $\sum_{i=1}^m n_i|G : P_i| = 1$. Then $h := \sum_{i=1}^m n_i\widehat{h}_i$ is an SG-splitting as required. □

6. Inadmissible actions on CW-complexes

Sometimes one wishes to consider actions of a discrete group on a CW-complex which permute the cells but without insisting that each cell stabilizers fixes that cell point by point. Theorem 1.4 does not hold in this generality as the following interesting example illustrates.

Let X denote the unit disk in the complex plane and let p be a prime. Then X may be endowed with a CW-structure having 0-skeleton the pth roots of unity, 1-skeleton the unit circle and 2-skeleton X itself. There are p 0-cells, p 1-cells and there is just one 2-cell. Let the group $\mathbb{Z}/p\mathbb{Z}$ act on X by rotations; that is, $m+p\mathbb{Z}$ acts as multiplication by $e^{2m\pi i/p}$. Let G be a group which acts on X via a surjective homomorphism $\pi\colon G \to \mathbb{Z}/p\mathbb{Z}$. Let H be the kernel of π. The augmented cellular chain complex of X is a four term exact sequence of permutation modules for G:

$$0 \to \mathbb{Z} \to \mathbb{Z}[H\backslash G] \to \mathbb{Z}[H\backslash G] \to \mathbb{Z} \to 0.$$

Since this begins and ends with the trivial module $\mathbb{Z}$ it represents in a classical way an element of $\mathrm{Ext}^2_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) = H^2(G,\mathbb{Z})$. Which cohomology class is this? It is $\beta\pi$, the image of the representation $\pi \in \hom(G,\mathbb{Z}/p\mathbb{Z}) = H^1(G,\mathbb{Z}/p\mathbb{Z})$ under the Bockstein β, (the Bockstein being the connecting homomorphism in the long exact sequence associated to the short exact sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ of coefficients). The class is non-zero, and the cellular chain complex is non-split.

7. A connection between work of Oliver and Dress

For primes p and q let $\mathcal{G}_p^q$ denote the class of finite groups G possessing normal subgroups $H \subseteq K$ such that H is a p-group, K/H is cyclic and G/K is a q-group. Let $\mathcal{G}$ denote the union of all the classes $\mathcal{G}_p^q$ as p and q vary through all primes.

Theorem 7.1 (Oliver)**.** *Let G be a finite group which does not belong to $\mathcal{G}$. Then there is a contractible finite G-complex with no global fixed point.*

This theorem is a special case of Oliver's work [**13**]. It is described in the introduction of Oliver's paper and in detail it follows from [**13**, Corollary to Theorem 3 and Corollary to Theorem 5]. See also the proof of [**13**, Theorem 7] for further explanation.

Compare this theorem with the following consequence of Dress' work [**3**] and [**4**].

Theorem 7.2. *The induction map*

$$\bigoplus_H K_0(\mathcal{L}(H)) \to K_0(\mathcal{L}(G))$$

is surjective, where H runs through the $\mathcal{G}$-subgroups of G.

This theorem refers to the Grothendieck group of $\mathbb{Z}G$-lattices defined to be the free abelian group on the set of $\mathbb{Z}G$-lattices modulo relations which force the equations $[A \oplus B] = [A] + [B]$ for any $\mathbb{Z}G$-lattices A and B. One can use Theorem 1.4 to deduce Dress' result from Oliver's as follows. Let G be a finite group which does not belong to $\mathcal{G}$. By Oliver's theorem, G admits a fixed-point-free action on a disk X. Triangulate (or cellulate) X as a G-complex. By Theorem 1.4 the cellular chain complex of X is $\mathbb{Z}G$-split and hence we have the equation

$$[\mathbb{Z}] = \sum_{n=0}^{\infty}(-1)^n[C_n(X, \mathbb{Z})]$$

between elements of $K_0(\mathcal{L}(G))$. This shows that $[\mathbb{Z}]$ belongs to the image of the induction map

$$\bigoplus_H K_0(\mathcal{L}(H)) \to K_0(\mathcal{L}(G)),$$

where H runs through the set of subgroups of G which are point stabilizers for the action on X. Since these are all proper subgroups of G, Dress' result follows by induction.

8. Classical results on quotient complexes

The following result is an analogue for G-complexes of results of Floyd, see for example [**6**, Theorem 2.12] and the discussions in [**7**], [**5**].

Theorem 8.1. *Let G be a locally finite group and let X be a finite-dimensional acyclic G-complex. Then the quotient complex X/G is again acyclic.*

Proof: Notice that X/G is the direct limit of the quotients X/H as H runs through the finite subgroups of G, because a locally finite group is the directed union of its finite subgroups. Therefore we only need to prove that each X/H is acyclic, and we may assume that G is finite. Now $C_*(X,\mathbb{Z})$ is $\mathbb{Z}G$-split and so the complex $H_0(G, C_*(X,\mathbb{Z})) = C_*(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} \mathbb{Z}$ is again acyclic. Since there is a natural isomorphism between $H_0(G, C_*(X,\mathbb{Z}))$ and $C_*(X/G,\mathbb{Z})$, X/G is acyclic. □

This is a nice way to prove the following fact about the class of $\mathbf{H}\mathfrak{F}$-groups introduced in [**8**]. We refer the reader to that paper and to [**9**] for details and definitions.

Corollary 8.2. *Let G be an $\mathbf{H}\mathfrak{F}$-group and let K be a locally finite normal subgroup of G. Then G/K belongs to $\mathbf{H}\mathfrak{F}$.*

Proof: Let $\mathfrak{X}$ be the class of groups G such that for all normal locally finite subgroups K, G/K belongs to $\mathbf{H}\mathfrak{F}$. Clearly $\mathfrak{X}$ contains all finite groups. Therefore it suffices to prove that $\mathfrak{X}$ is $\mathbf{H}$-closed, that is $\mathbf{H}\mathfrak{X} = \mathfrak{X}$, because $\mathbf{H}\mathfrak{F}$ is defined to be the smallest such class. By definition, $\mathfrak{F}$ is contained in $\mathfrak{X}$. □

9. Webb's formula for cohomology

Webb's formula [**16**] concerns the p-part of the cohomology of a finite group. Let G be a finite group and let p be a prime which divides its order. Let $\mathcal{S}_p(G)$ denote the poset of non-trivial finite p-subgroups of G on which G acts through its action by conjugation on p-subgroups. Let M be any finitely generated $\mathbb{Z}G$-module. Then the formula asserts that for any $n \geq 1$,

$$H^n(G,M)_p = \sum_{\sigma} (-1)^{|\sigma|} H^n(G_\sigma, M)_p \qquad (\dagger)$$

where σ runs through G-orbit representatives of simplices in $|\mathcal{S}_p(G)|$. Here the subscript p indicates the p-primary component and the equation should be interpreted in the Grothendieck group of finite abelian p-groups. The poset $\mathcal{S}_p(G)$ has the following property:

(‡) For every non-trivial finite p-subgroup H of G, the fixed point subcomplex $|\mathcal{S}_p(G)|^H$ is contractible.

This is an easy consequence of the fact that if K (a p-subgroup) is fixed by H then H normalizes K and

$$H \leq HK \geq K,$$

thus showing that $\mathcal{S}_p(G)^H$ is conically contractible in the sense of Quillen, [**14**, §1.5]. By gluing finitely many free G-orbits of cells to $|\mathcal{S}_p(G)|$ we can embed $|\mathcal{S}_p(G)|$ into a G-complex X which is finite, has dimension $d \leq \max\{3, \dim \mathcal{S}_p(G)\}$ and is $\max\{1, d-1\}$-connected. In particular there is an exact sequence

$$0 \to H_d(X) \to C_d(X) \to \cdots \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0,$$

obtained by including the solitary homology group at the tail of the cellular chain complex of X over $\mathbb{Z}$. Let P be a Sylow p-subgroup of G. The property (‡) of $\mathcal{S}_p(G)^H$ is inherited by X in the sense that X^H is contractible for each non-trivial subgroup H of P. Therefore the homology group $H_d(X)$ is projective as a $\mathbb{Z}P$-module by, for example, Proposition 6.2 of [**9**]. If we work over the ring $\mathbb{Z}_p$ of p-adic integers instead of $\mathbb{Z}$ then these statements hold over the group ring $\mathbb{Z}_pG$: that is, the augmented chain complex

$$0 \to H_d(X, \mathbb{Z}_p) \to C_d(X, \mathbb{Z}_p) \to \cdots \to C_1(X, \mathbb{Z}_p) \to C_0(X, \mathbb{Z}_p) \to \mathbb{Z}_p \to 0$$

is exact, $H_d(X, \mathbb{Z}_p)$ is projective as a $\mathbb{Z}_pG$-module and it is a direct summand of $C_d(X, \mathbb{Z}_p)$. Theorem 1.3 now shows that this chain complex is $\mathbb{Z}_pG$-split and this gives rise to an equation

$$\mathbb{Z}_p = \sum_{i=0}^{n+1} C_i(X, \mathbb{Z}_p) + (-1)^{n+2} H_{n+1}(|\mathcal{S}_p(G)|, \mathbb{Z}_p),$$

in the Grothendieck group, or Green ring, of $\mathbb{Z}_pG$-lattices. Webb's formula (†) follows on applying the functors $\operatorname{Ext}^j_{\mathbb{Z}_pG}(\ \ , M)$ where M is any $\mathbb{Z}_pG$-module and $j > 0$. The functors vanish on the projective homology group, and also on the chain of our added free orbits of cells. It should be noted that Webb's formula has been extended in a number of ways, see for example [**17**].

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Primera versió rebuda el 4 de desembre de 2007,
darrera versió rebuda el 6 de juliol de 2010.