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# NICE ELONGATIONS OF PRIMARY ABELIAN GROUPS

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*Abstract*

Suppose $N$ is a nice subgroup of the primary abelian group $G$ and $A = G/N$. The paper discusses various contexts in which $G$ satisfying some property implies that $A$ also satisfies the property, or visa versa, especially when $N$ is countable. For example, if $n$ is a positive integer, $G$ has length not exceeding $\omega_1$ and $N$ is countable, then $G$ is $n$-summable iff $A$ is $n$-summable. When $A$ is separable and $N$ is countable, we discuss the condition that any such $G$ decomposes into the direct sum of a countable and a separable group, and we show that it is undecidable in ZFC whether this condition implies that $A$ must be a direct sum of cyclics. We also relate these considerations to the study of nice bases for primary abelian groups.

## 0. Introduction and Terminology

By the term "group" we will mean an abelian $p$-group, where $p$ is a prime fixed for the duration. Our group theoretic terminology and notation will generally follow [**11**]. We will say a group $G$ is $\Sigma$-*cyclic* if it is isomorphic to a direct sum of cyclic groups. We will also utilize the language of *valuated groups* and *valuated vector spaces* (see [**22**] and [**12**], respectively); for example, if $Y$ is a valuated group, the letter $v$ will be reserved for its valuation, and for any ordinal $\alpha$, by $Y(\alpha)$ we will mean the subgroup $\{y \in Y : v(y) \geq \alpha\}$. We will implicitly assume that all valuated vector spaces are over $\mathbb{Z}_p$.

If $0 \to X \to G \to A \to 0$ is a short exact sequence, we will routinely identify $X$ with an actual subgroup of $G$ and $A$ with the quotient group $G/X$; we will say that $G$ is an *elongation* of $A$ by $X$. We then say that $G$ is a *nice-elongation* of $A$ if $X$ is a nice subgroup (i.e., for every $y \in G$, the coset $y + X$ has an element of maximum height). Further,

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if $X$ is countable, then we will say $G$ is an $\aleph_0$-*elongation* of $A$ by $X$. We can clearly combine these and speak of *nice*-$\aleph_0$-*elongations*.

The main purpose of this paper is to investigate how certain properties of groups are preserved under nice-$\aleph_0$-elongations, though we will occasionally prove results that extend beyond the countable case. For example, if $\lambda \leq \omega_1$ is an ordinal, then a group $G$ is a $C_\lambda$-*group* if for every $\alpha < \lambda$, if $H$ is a $p^\alpha$-high subgroup of $G$ (i.e., $H$ is maximal with respect to the property that $H \cap p^\alpha G = \{0\}$), then $H$ is a dsc-group (i.e., a *direct sum of countable groups*). We verify that if $G$ is a nice-$\aleph_0$-elongation of $A$, then $G$ is a $C_\lambda$-group iff $A$ is a $C_\lambda$-group (Corollary 2.4).

In another direction, a valuated group $Y$ will be said to be *Honda* if it is the ascending union of subgroups $X_m$ whose value spectra $\{v(x) : x \in X_m\}$ are finite (i.e., they are *value-finite*). Note that if $Y$ is actually countable, then it is clearly Honda, since it is the ascending union of finite subgroups.

If $G$ is a group with a subgroup $Y$, then restricting the height function on $G$ to $Y$ gives a valuation on $Y$. A group $G$ is *summable* if $G[p]$ is isometric to a free valuated vector space. More generally, following [**8**], if $n$ is a positive integer, a reduced group $G$ is $n$-*summable* if $G[p^n]$ is isometric to a valuated direct sum of countable valuated groups, and $G$ is $n$-*Honda* if $G[p^n]$ is Honda as a valuated group. In [**8**] it was shown that if $G$ is $n$-summable, then (a) it is summable, so that $p^{\omega_1}G = \{0\}$ (see, for example, [**11**, Theorem 84.3]); (b) the valuated group $G[p^n]$ is determined up to isometry by its Ulm function; (c) $G$ is $n$-Honda iff it is $n$-summable and has countable length (i.e., $p^\alpha G = \{0\}$ for some countable $\alpha$). In fact, these results were proven in the category of *valuated* $p^n$-*socles* (which extends the idea of a valuated vector space). Note that (c) generalizes the classical criterion (due to Honda) for the summability of a group of countable length (see, for example, [**11**, Theorem 84.1]).

If $G$ is a nice-elongation of $A$ by $N$, we show that if $N$ is countable and $G$ is $n$-summable, then $A$ is also $n$-summable (Theorem 1.1); and conversely, if $N$ is Honda (as a valuated group using the height valuation from $G$), $p^{\omega_1}G = \{0\}$ and $A$ is $n$-summable, then $G$ is $n$-summable (Theorem 1.2). It follows that if $G$ is a nice-$\aleph_0$-elongation of $A$ and $p^{\omega_1}G = \{0\}$, then $G$ is $n$-summable iff $A$ is $n$-summable (Corollary 1.3).

If $A$ is a separable group and $G$ is an elongation of $A$ by $X$, then $X$ will always be nice in $G$, so that an elongation of $A$ is always a nice-elongation. Next, if $X = p^\omega G$ (so that $A \cong G/p^\omega G$ is separable), we will say that $G$ is an $\omega$-*elongation* of $A$ (this terminology agrees with [**20**]), so that any $\omega$-elongation is also a nice-elongation; and if $X = p^\omega G$ is also countable, we will say $G$ is an $\omega$-$\aleph_0$-*elongation* of $A$.

The group $G$ will be said to be *countable plus separable* (or cps for short), if it is isomorphic to $C \oplus S$, where $C$ is countable and $S$ is separable. Note that if $G = D \oplus R$, where $D$ is divisible and $R$ is reduced, then $G$ is cps iff $D$ is countable and $R$ is cps; we may occasionally, therefore, assume that some cps group is actually reduced. We will say that the separable group $A$ has the *cps-elongation property* if whenever the group $G$ is a (nice-)$\aleph_0$-elongation of $A$, then $G$ is cps; similarly we will say $A$ has the *$\omega$-cps-elongation property* if whenever the group $G$ is an $\omega$-$\aleph_0$-elongation of $A$, then $G$ is cps. It is straightforward to show that any $\Sigma$-cyclic group has the cps-elongation property and that any group with the cps-elongation property has the $\omega$-cps-elongations property (Proposition 3.1); the natural question is whether a group with one of these latter two properties must actually be $\Sigma$-cyclic. We show that these questions are independent of the standard set-theoretic axioms (ZFC) by showing that they are consequences of the axiom of constructibility ($V = L$), but that counter-examples can be constructed using Martin's Axiom and the denial of the Continuum Hypothesis (MA + $\neg$CH, Theorem 3.4).

We then apply these results to question of when a group $G$ has a *nice basis*; that is, an ascending sequence of nice subgroups $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ whose union is all of $G$ such that each $X_m$ is $\Sigma$-cyclic (see, for example, [**2**]). In particular, we note that MA + $\neg$CH implies that there is separable group $A$ which is not $\Sigma$-cyclic with the property that whenever $G$ is an $\omega$-$\aleph_0$-elongation of $A$ by $X$, then $G$ has a nice basis (Example 4.7).

Referring to [**6**], a group homomorphism $\varphi\colon G \to A$ is said to be *$\omega_1$-bijective* if both the kernel and the co-kernel of $\varphi$ are countable. Thus, if there exists an $\omega_1$-bijective homomorphism $\varphi\colon G \to A$, then $G/\ker\varphi \cong \varphi(G)$ and $G$ is an $\aleph_0$-elongation of $\varphi(G)$ by $\ker\varphi$. In particular, if $\ker\varphi$ is a countable nice subgroup of $G$ and $\varphi$ is surjective, then $G$ is a nice-$\aleph_0$-elongation of $A$; so there is a natural connection between these two concepts. We also establish some results pertaining to $\omega_1$-bijective homomorphisms of special classes of abelian groups (e.g., Proposition 3.6 and Theorem 5.6).

## 1. $n$-Summable Groups

A subgroup $X$ of a valuated group $Y$ is *nice* if every coset $y + X$ has an element of maximal value; such a $y$ is called *proper with respect to $X$*; and in this case, letting $v(y + X) = v(y)$ makes $Y/X$ into a valuated group.

**Theorem 1.1.** *Suppose $n$ is a positive integer and $G$ is a nice-$\aleph_0$-elongation of $A$ by $N$. If $G$ is $n$-summable, then $A$ is $n$-summable.*

*Proof:* Fix a valuated decomposition $G[p^n] = \oplus_{i\in I} C_i$, where each $C_i$ is a countable valuated group. We claim there is a countable subset $J \subseteq I$ such that:

(a) $N[p^n] \subseteq \oplus_{i\in J} C_i$;

(b) $(N + \oplus_{i\in J} C_i) \| G[p^n]$, that is, for every $x \in N + \oplus_{i\in J} C_i$ and $y \in G[p^n]$ there is a $z \in (N + \oplus_{i\in J} C_i) \cap G[p^n] = \oplus_{i\in J} C_i$ such that $ht_G(x+z) \geq ht_G(x+y)$.

To begin, let $J_0$ be a countable subset of $I$ such that $N[p^n] \subseteq \oplus_{i\in J_0} C_i$. If we have constructed a countable subset $J_k$ of $I$, we want to construct another countable subset $J_{k+1}$ of $I$ containing $J_k$ such that:

(c) for every $x \in N + \oplus_{i\in J_k} C_i$ and $y \in G[p^n]$ there is a $z \in \oplus_{i\in J_{k+1}} C_i$ such that $ht_G(x+z) \geq ht_G(x+y)$.

For a second, fix $x$ to be an element of the countable subgroup $N + \oplus_{i\in J_k} C_i$. We claim that there is countable sequence of elements $y_{x,\ell}$ of $G[p^n]$ such that

$$\sup\{ht_G(x+y) : y \in G[p^n]\} = \sup\{ht_G(x+y_{x,\ell}) : \ell < \omega\}.$$

This (essentially well-known) fact is a consequence of the assumption that $G[p^n]$ is a valuated direct sum of countable valuated groups and such an object will always be complete in the (induced) $\omega_1$-topology: To see this, note that if it failed, we could, for all $\alpha < \omega_1$, choose $y_\alpha \in G[p^n]$ such that $ht(x+y_\alpha) \geq \alpha$. It follows that if $\alpha < \alpha' < \omega_1$, then $ht(y_\alpha - y_{\alpha'}) \geq \alpha$. If for each $\alpha < \omega_1$, $y_\alpha = (z_{\alpha,i})_{i\in I} \in \oplus_{i\in I} C_i$, then $z_{\alpha,i}$ must eventually be constant for each $i \in I$; say it takes on the value $w_i$. We next verify that all but finitely many of the $w_i$ must be 0: If this failed, and $\omega_1 > \alpha > ht(w_i)$ for an infinite number of indices $i$, it would easily follow that $z_{\alpha,i} \neq 0$ for this same infinite set of indices $i$, which cannot be. It follows that $w = (w_i)_{i\in I}$ is in $\oplus_{i\in J} C_i$, and that for all $\alpha < \omega_1$, $ht(x+w) = ht(x + y_\alpha + w - y_\alpha) \geq \alpha$, so that $x + w = 0$; which means that we could let $y_{x,\ell} = w$ for all $\ell < \omega$.

We then let $J_{k+1}$ be a countable subset of $I$ containing $J_k$ such that $y_{x,\ell} \in \oplus_{i\in J_{k+1}} C_i$ for all $x \in N + \oplus_{i\in J_k} C_i$ and $\ell < \omega$.

If we then let $J = \cup_{k<\omega} J_k$, then (a) is immediate and (b) follows from (c).

Let $L = I - J$. If $x$ is a non-zero element of $\oplus_{i\in L} C_i$, we claim that $ht_G(x) = ht_{G/N}(x+N)$: Note that if this failed, then since $N$ is nice in $G$, there would be an element $y \in N$ such that $ht_G(x+y) > ht_G(x)$. Now, using (b), we could then find a $z \in \oplus_{i\in J} C_i$ such that

$ht_G(x+z) \geq ht_G(x+y) > ht_G(x)$, but this contradicts the observation that $G[p^n] = (\oplus_{i\in J} C_i) \oplus (\oplus_{i\in L} C_i)$ is a valuated direct sum.

It follows that $\oplus_{i\in L} C_i = V$ embeds isometrically in $A[p^n]$. By Theorem 2.6 of [**8**], we will be done if we can show that $A[p^n]/V$ is countable. To that end, suppose $P$ is a countable pure subgroup of $G$ containing $N$. Note that if $z + N \in A[p^n]$, then $p^n z \in N \subseteq P$, so $p^n z = p^n w$ for some $w \in P$. Therefore, $z - w \in G[p^n] = (\oplus_{i\in J} C_i) \oplus V$, so $z - w = t + u$ where $t \in \oplus_{i\in J} C_i$ and $u \in V$. It follows that $z = (w+t)+u \in (P+\oplus_{i\in J} C_i)+V$, so that $A[p^n] \subseteq (P+\oplus_{i\in J} C_i)/N+V$, and since $(P+\oplus_{i\in J} C_i)/N$ is clearly countable, $A[p^n]/V$ must be countable, as well. □

Recall that a countable valuated group is always Honda, so the following is essentially a converse of the above.

**Theorem 1.2.** *Suppose $n$ is a positive integer, $G$ is a nice-elongation of $A$ by $N$, $p^{\omega_1}G = \{0\}$ and $N$ is Honda (as a valuated group using the height valuation from $G$). If $A$ is $n$-summable, then $G$ is $n$-summable.*

*Proof:* Since the value spectrum of $N$ is countable, there is a countable ordinal $\lambda$ such that $N(\lambda) = N \cap p^\lambda G = \{0\}$. Let $G'$ be a $p^{\lambda+n}$-high subgroup of $G$ containing $N$. Then $p^\lambda G'$ is $p^n$-bounded and there is a decomposition $p^\lambda G = p^\lambda G' \oplus X$ leading to an isometry $G[p^n] = G'[p^n] \oplus X[p^n]$, where the values in $X[p^n]$ are $\lambda$ plus the heights of elements computed in $X$ (cf. [**8**, Lemma 1.9]). Since $N(\lambda) = N \cap p^\lambda G = \{0\}$ and $N$ is nice, the map $p^\lambda G \to p^\lambda A$ is an isomorphism, so that $p^\lambda G$, and hence $X$, is $n$-summable. We therefore only need to show that $G'$ is $n$-summable. Towards this end, note that $G'$ is isotype in $G$ and $N$ is nice in $G'$. It follows that $A' = G'/N$ embeds as an isotype subgroup of $A = G/N$. Since $p^{\lambda+n}A' = p^{\lambda+n}(G'/N) = [p^{\lambda+n}G' + N]/N = \{0\}$, it follows that $A'[p^n]$ is Honda. Without loss of generality, then, assume $G' = G$ has countable length $\mu = \lambda + n$ and $G/N = A$ is $n$-Honda.

Let $M$ be the subgroup of $G$ containing $N$ satisfying $M/N = (G/N)[p^n] = A[p^n]$. The height function on $G$ gives a valuation on $M$ and $M/N$ is the valuated direct sum $\oplus_{i\in I} V_i$, where $V_i$ is countable. For each coset in $M/N$, choose an element which is proper with respect to $N$ and let $M_p \subseteq M$ be the collection of all these proper elements.

Let $O_j$ for $j < \omega$ be an ascending sequence of finite sets of ordinals with union $\mu$.

For each $i \in I$, suppose $F_{i,j}$, for $j < \omega$ is an ascending chain of finite subgroups of $V_i$ whose union is all of $V_i$ such that the value spectrum of $F_{i,j}$ is contained in $O_j$. Let $F'_{i,j} = \langle x \in M_p : x + N \in F_{i,j} \rangle$, so that $F'_{i,j}$ is a finite group.

Suppose $N$ is the ascending union of the subgroups $X_j$ such that the value spectrum of $X_j$ is contained in $O_j$. Define

$$Z_j = X_j + \langle F'_{i,k} : i \in I,\, k \leq j \text{ and } F'_{i,k} \cap N \subseteq X_j \rangle.$$

We claim that $M$ is the ascending union of these $Z_j$, and that the value spectrum of $Z_j$ is contained in $O_j$.

**Claim 1.** $Z_j \subseteq Z_{j+1}$.

First, note that $X_j \subseteq X_{j+1}$. Second, if $i \in I$, $k \leq j$, and $F'_{i,k} \cap N \subseteq X_j$, then $k \leq j+1$, and $F'_{i,k} \cap N \subseteq X_j \subseteq X_{j+1}$, so that the claim follows.

**Claim 2.** $M = \cup_{j<\omega} Z_j$.

Note $N = \cup_{j<\omega} X_j \subseteq \cup_{j<\omega} Z_j$. Further, if $i \in I$ and $k < \omega$, then since $F'_{i,k}$ is finite, we can find a $j \geq k$ such that $F'_{i,k} \cap N \subseteq X_j$, and it follows that $F'_{i,k} \subseteq Z_j$. It follows that $F_{i,k} \subseteq [\cup_{j<\omega} Z_j + N]/N = [\cup_{j<\omega} Z_j]/N$, and since this happens for all $i$ and $k$, it follows that $M/N = [\cup_{j<\omega} Z_j]/N$, which implies the claim.

**Claim 3.** *The value spectrum of $Z_j$ is contained in $O_j$.*

Suppose

$$z = x + g_{i_1} + \cdots + g_{i_m} \in Z_j$$

where $x \in X_j$, and for $1 \leq \ell \leq m$, each $i_\ell \in I$ and for some $k_\ell \leq j$, $g_{i_\ell} \in F'_{i_\ell,k_\ell}$, where $F'_{i_\ell,k_\ell} \cap N \subseteq X_j$. Note that each $g_{i_\ell}$ is congruent modulo $F'_{i_\ell,k_\ell} \cap N \subseteq X_j$ to some element of $F'_{i_\ell,k_\ell} \cap M_p$, and so we actually assume each $g_{i_\ell}$ is proper with respect to $M$.

Note

$$\begin{aligned} v(z) &\leq v(z+N) \\ &= v(g_{i_1} + \cdots + g_{i_m} + N) \\ &= \min\{v(g_{i_1}+N), \ldots, v(g_{i_m}+N)\} \\ &= \min\{v(g_{i_1}), \ldots, v(g_{i_m})\} \end{aligned}$$

which implies that

$$v(x) = v(z - (g_{i_1} + \cdots + g_{i_m})) \geq v(z),$$

which further implies that

$$v(z) = \min\{v(x), v(g_{i_1}), \ldots, v(g_{i_m})\} \in O_j.$$

It follows from Claims 2 and 3 that $M$ is Honda. We now observe that $G[p^n]$ is contained in $M$, and since $M$ is Honda, so is $G[p^n]$, so that $G$ must be $n$-summable, as required. □

Since a countable valuated group is always Honda, the following is an immediate consequence of Theorems 1.1 and 1.2:

**Corollary 1.3.** *Suppose $n$ is a positive integer and $G$ is a nice-$\aleph_0$-elongation of $A$ with $p^{\omega_1}G = \{0\}$. Then $G$ is $n$-summable iff $A$ is $n$-summable.*

**Example 1.4.** The hypothesis of niceness is necessary in Theorem 1.1.

As in Example 6.8 of [**6**], let $G_0$ be a group satisfying the following:

(a) $G_0$ is summable, $B$ is a countable unbounded $\Sigma$-cyclic group and there is an embedding of the torsion completion $\overline{B}$ in $G_0$ such that $p^\omega G_0 = \overline{B}[p]$ and $G_0/\overline{B}$ is $\Sigma$-cyclic. [To construct such a $G_0$, let $H$ be a dsc-group of length $\omega + 1$ such that there is a group isomorphism $\phi\colon p^\omega H \to \overline{B}[p]$, and let $G_0 = [H \oplus \overline{B}]/\{(x, \phi(x)) : x \in p^\omega H\}$, so $G_0$ is the sum of $H$ and $\overline{B}$ along $\phi$.]

Now, suppose $M$ is a countable group such that there is an isomorphism $f\colon p^\omega M \to B$. Let $G = M \oplus G_0$, $X = \{(x, f(x)) : x \in p^\omega M\} \subseteq G$, and $A = G/X$. It follows that

(b) $G$ is summable (since both $M$ and $G_0$ are summable);

(c) $X$ is countable (since $M$ is countable);

(d) $p^\omega A \cong \overline{B}$, so that $A$ is not summable.

To verify (d), note that clearly $B \cong B' = [(\{0\} \oplus B) + X]/X \subseteq p^\omega A$, and since $A/B' \cong (\overline{B}/B) \oplus (G_0/\overline{B}) \oplus (M/p^\omega M)$, where the first term of this decomposition is divisible and the last two terms are $\Sigma$-cyclic and hence separable, (d) then follows. □

**Example 1.5.** Theorem 1.1 does not hold if the nice subgroup $N$ is only assumed to be Honda, as opposed to countable.

Suppose $A$ is any separable group which is not $\Sigma$-cyclic and $N \to G$ is a pure-projective resolution of $A$ (i.e., $G$ is $\Sigma$-cyclic and $N$ is pure in $G$), then $N$ is clearly Honda (since it is $\Sigma$-cyclic and the height valuation on $G$ and $N$ agree), $G$ is summable, but $A$ is not summable. □

**Example 1.6.** The hypothesis $p^{\omega_1}G = \{0\}$ is necessary in Theorem 1.2.

Suppose $G$ is a totally projective group with $N = p^{\omega_1}G$ countably infinite and $A = G/N$. Then $A$ is a dsc-group, and hence $A$ is summable. Since $p^{\omega_1}G \neq \{0\}$, however, $G$ will not be summable. □

**Example 1.7.** The hypothesis of niceness is necessary in Theorem 1.2.

In Example 2.3 of [**6**], a separable group $G$ was constructed which was not $\Sigma$-cyclic but had a (pure) countable subgroup $X$ such that $A = G/X$ was a dsc-group of length $\omega + 1$. It follows that this $A$ is summable, but $G$ is not; in fact $G$ is $p^{\omega+1}$-projective. □

## 2. Totally projective groups and generalizations

A *nice composition series* for the valuated group $Y$ is an ascending chain of nice subgroups $\{X_i : i \leq \delta\}$ such that

(a) $X_0 = \{0\}$, $X_\delta = Y$;

(b) for all $i < \delta$, $X_{i+1}/X_i \cong \mathbf{Z}_p$;

(c) for all limit ordinals $\lambda \leq \delta$, $X_\lambda = \cup_{i<\lambda} X_i$.

It is reasonably easy to verify that if $Y$ has a nice composition series, then so does $Y/Y(\alpha)$ for every ordinal $\alpha$, and that if $Y$ is Honda (so in particular, if $Y$ is countable), then it has a nice composition series. It is well known that a reduced group $G$ using the height valuation has a nice composition series iff it is totally projective (see, for example, [**11**, Theorem 81.9]). The following (essentially well-known) observation follows as a direct consequence.

**Proposition 2.1.** *Suppose $G$ is a nice-elongation of $A$ by $N$. If $N$ has a nice composition series as a valuated group and $A$ is totally projective, then $G$ is totally projective.*

*Proof:* If $\{X_i\}_{i\leq\delta}$ is a nice composition series for $N$, it can readily be checked that each $X_i$ is also nice in $G$. If $\{Z_j\}_{j\leq\epsilon}$ is a nice composition series for $A = G/N$, and for $j < \mu$ we let $Z'_j$ be the subgroup of $G$ containing $N$ defined by the equation $Z'_j/N = Z_j$, then each $Z'_j$ will also be nice in $G$. It readily follows that $\{X_i\}_{i\leq\delta} \cup \{Z'_j\}_{j\leq\epsilon}$ is a composition series for $G$, so $G$ is totally projective. □

We include the following (also well-known) observation for future reference.

**Corollary 2.2.** *Suppose $G$ is a reduced group which is a nice-$\aleph_0$-elongation of $A$ by $N$. Then $A$ is totally projective iff $G$ is totally projective.*

*Proof:* Sufficiency follows immediately from Proposition 2.1, and necessity follows directly from Proposition 1.1 of [**6**] (the second statement does not require the niceness of $N$ in $G$). □

Recall that if $\lambda \leq \omega_1$, then $G$ is a $C_\lambda$-group if for every $\alpha < \lambda$, one (and so each) $p^\alpha$-high subgroup $H$ of $G$ is a dsc-group. If $\lambda$ is a limit, this is equivalent to requiring that $G/p^\alpha G$ is a dsc-group for all $\alpha < \lambda$, and if $\lambda$ is isolated, this is equivalent to requiring that one (and so each) $p^{\lambda-1}$-high subgroup $H$ of $G$ is a dsc-group (see, for example, the discussion in the first two paragraphs of Section 1 in [**17**]).

**Theorem 2.3.** *Suppose $\lambda \leq \omega_1$, $G$ is a nice-elongation of $A$ by $N$ and $N$ has a nice composition series (using the height function on $G$). If $A$ is a $C_\lambda$-group then $G$ is a $C_\lambda$-group.*

*Proof:* Assume $A$ is a $C_\lambda$-group. First, suppose $\lambda$ is a limit. Now, for all $\alpha < \lambda$ there is a nice short-exact sequence

$$0 \to N/N(\alpha) \to G/p^\alpha G \to A/p^\alpha A \to 0.$$

Since $A/p^\alpha A$ is a dsc-group, it follows from Proposition 2.1 that $G/p^\alpha G$ must be a dsc-group. Since this holds for all $\alpha < \lambda$, it follows that $G$ is a $C_\lambda$-group, as required.

Next, suppose $\lambda = \gamma + 1$ is isolated; so, in particular, $\lambda$ is countable. If $\lambda$ is finite, the result is trivial (since then any group is a $C_\lambda$-group), so suppose $\lambda$ is infinite. We therefore have a nice short-exact sequence:

$$0 \to N/N(\lambda) \to G/p^\lambda G \to A/p^\lambda A \to 0.$$

Since a $p^\gamma$-high subgroup of $G$ maps to a $p^\gamma$-high subgroup of $G/p^\lambda G$, $G$ is a $C_\lambda$-group iff $G/p^\lambda G$ is a $C_\lambda$-group. Without loss of generality, then, replace $N$, $G$ and $A$ by $N/N(\lambda)$, $G/p^\lambda G$ and $A/p^\lambda A$, so that we may assume all these groups have length at most $\lambda$.

Let $Y$ be a $p^\gamma$-high subgroup of $A$, then since $\gamma$ is infinite, we have that $A/Y$ is divisible and

$$(p^\gamma A)[p] = p^\gamma A \to (A/Y)[p]$$

is an isomorphism (see, for example, [**14**, Theorem 92]).

Let $X$ be the subgroup of $G$ containing $N$ such that $X/N = Y$. Note that

$$G/X \cong (G/N)/(X/N) = A/Y$$

is divisible and $(p^\gamma G)[p] \to (G/X)[p]$ can be factored as follows:

$$(p^\gamma G)[p] = p^\gamma G \to p^\gamma A = (p^\gamma A)[p] \cong (A/Y)[p] \cong (G/X)[p]$$

and since these maps are all surjective, it follows that $X$ is $p^\lambda$-pure in $G$ (see, for example, [**14**, Theorem 91]). This means that $X$ is isotype in $G$, so that

$$0 \to N \to X \to Y \to 0$$

is a nice sequence. Since $A$ is a $C_\lambda$-group, $Y$ is a dsc-group. By Proposition 2.1 this means that $X$ is also a dsc-group. Next, observe that $G[p]$ is isometric to a valuated direct sum $X[p] \oplus W$ where $W \subseteq p^\gamma G[p]$. This implies that if $X'$ is a $p^\gamma$-high subgroup of $X$, then $X'$ is also $p^\gamma$-high in $G$. Since $X'$ will be isotype in $X$, it follows that $X'$ is a dsc-group (see, for example, [**14**, Theorem 104]). However, this implies that $G$ is a $C_\lambda$-group, as required. $\square$

**Corollary 2.4.** *Suppose $\lambda \leq \omega_1$ and $G$ is a nice-$\aleph_0$-elongation of $A$. Then $G$ is a $C_\lambda$-group iff $A$ is a $C_\lambda$-group.*

*Proof:* If $A$ is a $C_\lambda$-group, then Theorem 2.3 implies that $G$ is as well; and conversely, if $A$ is a $C_\lambda$-group, then Theorem 3.5 of [**6**] implies that $G$ is as well (the second implication does not require the niceness of $N$ in $G$). $\square$

*Remark* 1. If $\lambda > \omega$, then Example 1.5 shows that in Corollary 2.4, if $N$ is only assumed to be Honda, as opposed to countable, then $A$ may be a $C_\lambda$-group, while $G$ is not.

For the remainder of this section $n$ will denote a fixed positive integer; $G$ is said to be an *$n$-$\Sigma$-group* if $G[p^n]$ is the ascending union of a sequence of subgroups $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ such that for all $i < \omega$, $X_i(i) = X_i \cap p^i G = (p^\omega G)[p^n]$. Utilizing Proposition 3.2 of [**6**], a group is an $n$-$\Sigma$-group iff it is a $C_{\omega+n}$-group. Therefore, these last two results can be reformulated as follows:

**Corollary 2.5.** *Suppose $G$ is a nice-elongation of $A$ by $N$.*

(a) *If $N$ has a nice composition series (using the height function on $G$) and $A$ is an $n$-$\Sigma$-group, then $G$ is an $n$-$\Sigma$-group.*
(b) *If $N$ is countable, then $G$ is an $n$-$\Sigma$-group iff $A$ is an $n$-$\Sigma$-group.*

A group $G$ is said to be *pillared* if $G/p^\omega G$ is $\Sigma$-cyclic. Since this is true iff for all $n < \omega$, $G/p^{\omega+n}G$ is a dsc-group, it follows that a group is pillared iff it is a $C_{\omega\cdot 2}$-group. Letting $\lambda = \omega \cdot 2$, then, we have the following:

**Corollary 2.6.** *Suppose $G$ is a nice-elongation of $A$ by $N$.*

(a) *If $N$ has a nice composition series (using the height function on $G$) and $A$ is pillared, then $G$ is pillared.*

(b) *If $N$ is countable, then $G$ is pillared iff $A$ is pillared.*

A group $G$ is said to be $p^{\omega+n}$-*projective* if there is a subgroup $P \subseteq G[p^n]$ such that $G/P$ is $\Sigma$-cyclic (e.g., [**21**]). By a classical result of Fuchs (see [**13**]), if $G_1$ and $G_2$ are $p^{\omega+n}$-projective, then $G_1$ and $G_2$ are isomorphic iff $G_1[p^n]$ and $G_2[p^n]$ are isometric. On the other hand if $P$ is any valuated group with $p^nP = \{0\}$ which is separable in the sense that $P(\omega) = \{0\}$, then there is a separable group $G$ containing $P$ such that the valuation on $P$ agrees with the height function on $G$, $G$ is separable and $G/P$ is $\Sigma$-cyclic; note that this $G$ will be $p^{\omega+n}$-projective.

More generally, imitating [**3**], a group $G$ is called $p^{\omega+n}$-*totally projective* if $p^\omega G$ is totally projective and $G/p^\omega G$ is $p^{\omega+n}$-projective. Clearly both $p^{\omega+n}$-projective groups and totally projective groups are themselves $p^{\omega+n}$-totally projective. By the same token, we define a group $G$ to be $p^{\omega+n}$-*summable* if $p^\omega G$ is summable and $G/p^\omega G$ is $p^{\omega+n}$-projective. Observe that $p^{\omega+n}$-projective groups are $p^{\omega+n}$-summable, whereas summable groups need not be $p^{\omega+n}$-summable since there exists a summable group with unbounded torsion-complete first Ulm factor and it is known that torsion-complete $p^{\omega+n}$-projective groups are bounded (see, for instance, [**16**]).

**Proposition 2.7.** *Suppose $G$ is a nice-$\aleph_0$-elongation of $A$ by $N$. We then have:*

(a) *$G$ is $p^{\omega+n}$-totally projective iff $A$ is $p^{\omega+n}$-totally projective.*

(b) *If $p^{\omega_1}G = \{0\}$, then $G$ is $p^{\omega+n}$-summable iff $A$ is $p^{\omega+n}$-summable.*

*Proof:* Observe that $p^\omega G$ is a nice-$\aleph_0$-elongation of $p^\omega A$ by $N(\omega) = N \cap p^\omega G$, and $G/p^\omega G$ is a nice-$\aleph_0$-elongation of $A/p^\omega A$ by $N/N(\omega)$. By Theorem 4.2 of [**6**], $G/p^\omega G$ is $p^{\omega+n}$-projective iff $A/p^\omega A$ is $p^{\omega+n}$-projective.

Therefore, (a) follows from Corollary 2.2 applied to $p^\omega G$ and $p^\omega A$, and (b) follows from Corollary 1.3 applied to the same. □

## 3. CPS groups and $\omega_1$-separable groups

**Proposition 3.1.** *Suppose $A$ is a separable group.*

(a) *If $A$ is $\Sigma$-cyclic, then it has the cps-elongation property.*

(b) *If $A$ has the cps-elongation property then it has the $\omega$-cps-elongation property.*

*Proof:* In (a), if $A$ is $\Sigma$-cyclic and $G$ is an $\aleph_0$-elongation of $A$ by $X$, then there is clearly a countable pure subgroup $Y$ of $G$ containing $X$ such that $Y/X$ is a summand of $A$. Since $G/Y \cong A/(Y/X)$ will be $\Sigma$-cyclic, it follows that $G \cong Y \oplus S$, for some $\Sigma$-cyclic group $S$, so that $G$ is cps. It follows that $A$ has the cps-elongation property.

The implication in (b) is obvious. $\square$

We now translate a notion that has been of considerable importance in the study of (torsion-free) free groups (see [**10**]) into the language of valuated vector spaces. We will say the valuated vector space $V$ is $\aleph_1$-*coseparable* if it is separable (i.e., $V(\omega) = \{0\}$), and for all subspaces $W$ of $V$ with $V/W$ countable there is a closed subspace $U$ of $V$ such that $U \subseteq W$ and $V/U$ is countable —note that this implies that the quotient valuated vector space $V/U$ is separable and hence free, so that $V$ is isometric to $U \oplus F$ where $F$ is countable and free. Note also that if $V/W$ is unbounded (i.e., $(V/W)(m) \neq \{0\}$ for all $m < \omega$), then the same will be true of $F$.

We pause to review the following standard construction (see, for example, [**14**, Theorem 106]).

**Lemma 3.2.** *Suppose $A$ is a separable group, $D \subseteq A[p]$ is a dense subsocle and $X$ is a group such that $A[p]/D \cong X/pX$. Then there is an $\omega$-elongation $G$ of $A$ by $X$ such that $D = [G[p] + p^\omega G]/p^\omega G \subseteq A[p]$.*

*Proof:* Let $Y$ be a pure and dense subgroup of $A$ such that $Y[p] = D$, so that $A/Y$ is divisible; and let $Z$ be a divisible hull for $X$, so that $Z/N$ is also divisible. The isomorphism $A[p]/D \cong X/pX$ implies that there is an isomorphism $\phi\colon A/Y \to Z/X$, and we can let $G = \{(a, z) \in A \oplus Z : f(a+Y) = z+X\}$. The assignment $x \to (0, x)$ gives an injection $X \to G$, and the assignment $(a, z) \to a$ gives a surjection $G \to A$, and it can be checked that these conditions will imply the conclusions. $\square$

**Theorem 3.3.** *The following conditions are equivalent (in ZFC):*

(a) *Every separable group $A$ with the $\omega$-cps-elongation property is $\Sigma$-cyclic.*

(b) *Every separable $p^{\omega+1}$-projective group $A$ with the $\omega$-cps-elongation property is $\Sigma$-cyclic.*

(c) *Every $\aleph_1$-coseparable valuated vector space is free (as a valuated vector space).*

*Proof:* It is clear that (a) implies (b). Suppose next that (b) holds and that $V$ is $\aleph_1$-coseparable; we want to show that $V$ must be free. Let $A$ be a separable group containing $V$ as a subgroup where $A/V$ is $\Sigma$-cyclic, so that $A$ is $p^{\omega+1}$-projective. Since there is a valuated injection of the valuated quotient space $A[p]/V$ into $(A/V)[p]$ and the latter is Honda, so is $A[p]/V$, and hence free as a valuated vector space. It follows that $A[p]$ is isometric to $V \oplus F$ where $F$ is a free valuated vector space. In fact, after possibly adding to $A$ a summand which is an infinite $\Sigma$-cyclic group, we may assume that every Ulm invariant of $F$ is infinite.

We now show that this $A$ has the $\omega$-cps-elongation property. To this end, let $G$ be an $\omega$-$\aleph_0$-elongation of $A$, so that $A = G/p^\omega G$; we need to show $G$ is cps. We begin by letting $W' = [G[p] + p^\omega G]/p^\omega G \subseteq A[p]$. It can easily be checked that $\phi(x + p^\omega G) = px + p^{\omega+1}G$ gives a well-defined homomorphism $A[p] \to p^\omega G/p^{\omega+1}G$ whose kernel is $W'$. It follows that $A[p]/W'$ embeds in the countable group $p^\omega G/p^{\omega+1}G$, and hence $A[p]/W'$ is also countable.

Now let $W = W' \cap V$. Since $V/W$ embeds in $A[p]/W'$, it follows that $V/W$ is countable. Let $U$ be a closed subspace of $V$ contained in $W$ such that $V/U$ is countable. Finally, let $P$ be a subgroup of $G[p]$ such that $G[p] \to W'$ maps $P$ isomorphically onto $U$; since the homomorphism $G[p] \to W' \subseteq A[p]$ preserves all finite heights, it follows that $P$ also maps isometrically onto $U$.

From this construction we can conclude that $[p^\omega G \oplus P]/p^\omega G = U$. Finally, let $Q = p^\omega G \oplus P \subseteq G$, so that $G/Q \cong (G/p^\omega G)/([p^\omega G \oplus P]/p^\omega G) = A/U$.

**Claim.** *The decomposition $Q = p^\omega G \oplus P \subseteq G$ is valuated, $G/Q$ is $\Sigma$-cyclic and, for all $m < \omega$, the relative Ulm function $f_{G,Q}(m)$ is infinite.*

Since $p^\omega G \cap P = \{0\}$, the first statement readily follows. Since $G/Q \cong A/U$, we must show that the latter is $\Sigma$-cyclic. Observe that $V/U$ is a free, separable valuated vector space, so there is a valuated embedding $V/U \to J$, where $J$ is a $\Sigma$-cyclic group using the height valuation. Since $V$ is nice in $A$ and $A/V$ is $\Sigma$-cyclic, it follows that this embedding extends to a homomorphism $f\colon A \to J$, for which the kernel of $f\big|_V$ is $U$. It follows that the kernel of the obvious map $A \to (A/V) \oplus J$ is $U$, and since $(A/V) \oplus J$ is $\Sigma$-cyclic, so is $A/U$.

Observe that if $m < \omega$, then $f_{G,Q}(m) = f_{A,U}(m)$. Since $F$ has infinite Ulm invariants, $U \subseteq V$, and $A[p]$ is isometric to $V \oplus F$, the claim follows.

Let $C$ be a countable group whose Ulm function is defined by

$$f_C(\alpha) = \begin{cases} f_G(\alpha), & \text{when } \omega \le \alpha \le \infty; \\ 1, & \text{when } \alpha < \omega. \end{cases}$$

Note this implies that $p^\omega C$ and $p^\omega G$ are isomorphic, even when they are not reduced. Let $H = C \oplus A$, so that $Q' = p^\omega C \oplus U$ is nice in $H$, and $H/Q' \cong (C/p^\omega C) \oplus (A/U)$ is $\Sigma$-cyclic. In addition, Q′ and Q are both isometric to $p^\omega G \oplus U$ and so they are isometric to each other. Next, it is readily checked that if $\alpha \geq \omega$, $f_{G,Q}(\alpha) = 0 = f_{H,Q'}(\alpha)$ and if $m < \omega$,

$$f_{H,Q'}(m) = f_{G,Q}(m) + 1 = f_{G,Q}(m).$$

Therefore, by the fundamental result of Hill on extending isometries on nice valuated subgroups of groups (see, for example, [**11**, Theorem 83.4]), it follows that there is an isomorphism $G \to H = C \oplus A$. It therefore follows that $G$ is cps, so that $A$ has the $\omega$-cps-elongation property. By hypothesis, then, $A$ must be $\Sigma$-cyclic, and this implies that $A[p]$ is free as a valuated vector space, and since $V$ is a subspace of $A[p]$, it is also free, as required.

We now need to show that (c) implies (a), so suppose every $\aleph_1$-coseparable valuated vector space is free and $A$ is a separable group with the $\omega$-cps-elongation property, and let $V = A[p]$; it follows trivially that $V$ is separable. Let $W$ be a subspace of $V$ such that $V/W$ is countable. Note that if $\overline{W}$ is the $p$-adic closure of $W$ in $V$, then $\overline{W}/W = (V/W)(\omega)$. Since $V/W$ is countable, there is a subspace $W'$ of $V$ containing $W$ such that $V/W$ is isometric to the valuated direct sum $(W'/W) \oplus (\overline{W}/W)$. Note that $W$ is closed (and hence, nice) in $W'$ and $W'/W$ is countable (and hence, free), so that $W' = W \oplus F$ for some countable, free subspace $F$.

Note that $W'$ is dense in $V$. By Lemma 3.2 there is an $\omega$-elongation $G$ of $A$ such that $p^\omega G$ is countable, $G/p^\omega G = A$ and $[G[p] + p^\omega G]/p^\omega G = W'$.

By hypothesis, $G = C \oplus S$, where $C$ is countable and $S$ is separable. First, note that we can identify $A[p] = V$ with $(C/p^\omega C)[p] \oplus S[p]$, so that we can view $S[p]$ as a subgroup of $W' \subseteq A[p] = V$. We now let $U = S[p] \cap W$. Since both $V/S[p]$ and $V/W$ are countable, and $V/U$ embeds in $(V/S[p]) \oplus (V/W)$, it follows that $V/U$ is countable.

To show that $U$ is closed, let $u_i$ for $i < \omega$ be a sequence in $U$ which converges in the $p$-adic topology to $z \in V$. Since $A[p] = (C/p^\omega C)[p] \oplus S[p]$, $S[p]$ is closed in $V$, so that $z \in S[p] \subseteq W'$. In the decomposition $W' = W \oplus F$, each $u_i \in W$, so that $z \in W$, as well. It follows that $z \in S[p] \cap W = U$.

Note that the arguments in the last two paragraphs imply that $V$ is $\aleph_1$-coseparable. It follows, therefore, that $V = A[p]$ is free, so that $A$ is $\Sigma$-cyclic, as required. □

**Theorem 3.4.** *The following hold:*

(a) *Assuming (MA + $\neg$CH), there is a separable group $A$ which has the cps-elongation property (and hence it also has the $\omega$-cps-elongation property), but is not $\Sigma$-cyclic.*

(b) *Assuming ($V = L$), if $A$ is a separable group which has the $\omega$-cps-elongation property (in particular, if $A$ has the cps-elongation property), then $A$ is $\Sigma$-cyclic.*

Before proceeding, note that the last result implies the following.

**Corollary 3.5.** *The following statements are independent of ZFC:*

(a) *Every separable group with the cps-elongation property is $\Sigma$-cyclic.*

(b) *Every separable group with the $\omega$-cps-elongation property is $\Sigma$-cyclic.*

(c) *Every $\aleph_1$-coseparable valuated vector space is free.*

*Proof:* By Theorem 3.4, both (a) and (b) fail in a model of MA + $\neg$CH and both are true in $V = L$. By Theorem 3.3, (b) is equivalent to (c). ☐

Recall that a group $A$ is $\omega_1$-*separable* iff it is separable and every countable subgroup of $A$ is contained in a countable summand of $A$. Using a (by now) standard construction, it can be shown in ZFC that there are $\omega_1$-separable groups of cardinality $\aleph_1$ which are not $\Sigma$-cyclic.

*Proof of Theorem 3.4:* Regarding (a), suppose $A$ is an $\omega_1$-separable group of cardinality $\aleph_1$ which is not $\Sigma$-cyclic. If $G$ is an $\aleph_0$-elongation of $A$ by $X$, then let $P$ be a countable pure subgroup of $G$ containing $X$. Since $A = G/X$ is $\omega_1$-separable, we can find a countable subgroup $C$ of $G$ containing $P$ such that $C/X$ is a summand of $G/X = A$.

We claim that $C$ will be pure in $G$: Note that $C/X$ is clearly pure in $G/X$, which implies that $C/P \cong (C/X)/(P/X)$ is pure in $G/P \cong (G/X)/(P/X)$. Since $P$ is pure in $G$, it follows that $C$ is pure in $G$, as required.

If we let $A = A' \oplus (C/X)$, then the countability of $C/X$ and the fact that $A$ is $\omega_1$-separable readily implies that $A'$ is also $\omega_1$-separable. In the presence of MA + $\neg$CH, by Theorem 3.1 of [**18**], we can conclude that $\mathrm{Pext}(A', C) = \{0\}$. However, since $C$ is pure in $G$ and $G/C \cong (G/X)/(C/X) \cong A'$, we can conclude that $G \cong A' \oplus C$, as required.

Regarding (b), note that if $G$ is an $\omega$-elongation of $A$ by $\mathbb{Z}_p$, then since $A$ has the $\omega$-cps-elongation property, it follows that $G \cong C \oplus S$, where $C$ is countable and $S$ is separable. If $H_{\omega+1}$ is the generalized Prüfer group of length $\omega + 1$, there is clearly a homomorphism $G \to C \to H_{\omega+1}$ which is non-zero on $p^\omega G = p^\omega C$. In the presence of $V = L$, by Theorem 2.2 of [**20**], $A$ must be $\Sigma$-cyclic. $\square$

Although by Proposition 7.2 of [**4**], $A$ being $\omega_1$-separable implies that $A/C$ is $\omega_1$-separable whenever $C$ is a countable nice subgroup of $A$, we shall see that the converse implication does not hold in some versions of set theory. First, we pause for the following assertion which answers Problem 7.3 from [**4**].

**Proposition 3.6.** *Suppose $A$ is a separable group and $f\colon G \to A$ is an $\omega_1$-bijective homomorphism. If $G$ is $\omega_1$-separable, then $A$ is $\omega_1$-separable.*

*Proof*: Let $I$ be the image of $f$ and $K$ be the kernel of $f$. Since $I$ is a subgroup of $A$, $K$ is a countable and nice subgroup of $G$, so by Proposition 7.2 of [**4**], $I \cong G/K$ is $\omega_1$-separable. Now, if $C$ is a countable subgroup of $A$, then after possibly expanding $C$, we may assume that $A = I + C$. Let $C'$ be a countable subgroup of $I$ containing $I \cap C$ such that $I = I' \oplus C'$; the existence of such a $C'$ follows from the fact that $I$ is $\omega_1$-separable. Then $C \subseteq C + C'$ is countable and one easily checks that $A = I' \oplus (C' + C)$. It follows that $A$ is $\omega_1$-separable, as desired. $\square$

The next result shows that the converse of Proposition 3.6 is undecidable in ZFC for groups of cardinality $\aleph_1$. It settles Problem 7.2 from [**4**].

**Theorem 3.7.** (a) *(MA + $\neg$CH) If the separable group $G$ is an $\aleph_0$-elongation of the $\omega_1$-separable group $A$ and $|A| = \aleph_1$, then $G$ is also $\omega_1$-separable.*

(b) *($V = L$) There is a separable group $G$ which is an $\aleph_0$-elongation of an $\omega_1$-separable group $A$ with $|A| = \aleph_1$ such that $G$ is not $\omega_1$-separable.*

*Proof*: Regarding (a), a separable group $H$ is *weakly $\omega_1$-separable* if for every countable subgroup $C$ of $H$, the $p$-adic closure $\overline{C}$ is also countable. By [**19**], in the presence of MA + $\neg$CH, for groups of cardinality $\aleph_1$, the classes of weakly $\omega_1$-separable and $\omega_1$-separable groups coincide. Finally, in view of Corollary 5.2 of [**6**] we have that $G$ is weakly $\omega_1$-separable iff $A$ is weakly $\omega_1$-separable.

Regarding (b), Megibben [**19**, Theorem 3.2] found in the presence of $V = L$ an $\omega_1$-separable group $A$ of cardinality $\aleph_1$ containing a pure and dense subgroup $G'$ which is not $\omega_1$-separable such that $A/G'$ is countable. If $C$ is a countable subgroup of $A$ such that $G' + C = A$, then let $G = G' \oplus C$ (note this is an *external* direct sum). Since $G'$ is not $\omega_1$-separable, the countability of $C$ easily implies that $G$ is not $\omega_1$-separable. If we then consider the sum map $G = G' \oplus C \to G' + C = A$, defined by $(g, c) \mapsto g + c$, we have our result. □

## 4. Groups with nice bases

Following [**2**], recall that a nice basis for a group $G$ is an ascending sequence of nice subgroups $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ whose union is all of $G$ such that each $X_m$ is $\Sigma$-cyclic.

**Proposition 4.1.** *Let $G$ be a nice-$\aleph_0$-elongation of $A$ by $N$ such that $N \cap p^\omega G = \{0\}$. If $A$ has a nice basis, then $G$ has a nice basis.*

*Proof:* Write $G/N = \cup_{i<\omega}(G_i/N) = (\cup_{i<\omega} G_i)/N$, where $N \leq G_i \subseteq G_{i+1} \leq G$ and all $G_i/N$ are nice in $G/N$ and are $\Sigma$-cyclic groups. Therefore, $G = \cup_{i<\omega} G_i$ with $p^\omega G_i \subseteq N$. Hence $p^\omega G_i \subseteq N \cap p^\omega G = \{0\}$ and we conclude that all $G_i$ are separable. By Corollary 2.2, they are $\Sigma$-cyclic groups. Moreover, utilizing Lemma 79.3 of [**11**], we derive that these $G_i$ are also nice in $G$, as needed. □

The last result can also be deduced from [**4**, Proposition 9.1]; nevertheless we have given another, more smooth, proof.

**Example 4.2.** In Proposition 4.1, the restriction $N(\omega) = N \cap p^\omega G = \{0\}$ is necessary.

Suppose $A$ is any separable thick group and let $G$ be any group such that $p^\omega G$ is countable, reduced inseparable (thus it is not $\Sigma$-cyclic) with $G/p^\omega G \cong A$. Applying Theorem 1.3 of [**7**], $G$ does not has a nice basis (see also Proposition 4.5 below). □

**Example 4.3.** In Proposition 4.1, the niceness of $N$ in $G$ is also necessary.

Let $G$ be a reduced group without a nice basis with a countable basic subgroup $N$ (for example, if $G/p^\omega G$ is torsion-complete with a countable basic subgroup and $p^\omega G$ is not $\Sigma$-cyclic, it follows from [**7**] that $G$ does not have a nice basis). Clearly, $N(\omega) = \{0\}$ is satisfied and $A = G/N$ is divisible, hence it does have a nice basis owing to [**2**]. □

We now investigate the hypothesis that $N(\omega) = \{0\}$ in the last result by contrasting it with the case where $G$ is an $\omega$-$\aleph_0$-elongation of $A$ by $N$; i.e., we are contrasting the situation in which $N \cap p^\omega G = \{0\}$ with the situation in which $N = p^\omega G$. We begin by considering an apparently unrelated notion.

A separable valuated vector space $V$ (i.e., $V(\omega) = \{0\}$) will be called *essentially finitely indecomposable* (or efi for short) if (1) it is *unbounded* (in the sense that $V(m) \neq \{0\}$ for all $m < \omega$); and (2) there is no valuated decomposition $V = W \oplus F$ where $F$ is an unbounded free valuated vector space. For example, if $V$ is complete in the $p$-adic topology (i.e., if it is isometric to the socle of a torsion-complete group), then $V$ is efi. The following topological characterization utilizes ideas from [**7**].

**Proposition 4.4.** *An unbounded separable valuated vector space $V$ is efi iff $V$ cannot be expressed as the ascending union of a sequence of closed subspaces $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$ such that $V/C_m$ is unbounded for every $m < \omega$.*

*Proof:* If there is a decomposition $V = W \oplus F$ and we let $F = \oplus_{i<\omega} B_i$ where $B_i(i) \neq \{0\}$, then we may simply let $C_m = W \oplus (\oplus_{i<m} B_i)$. Therefore, $V/C_m \cong \oplus_{m \leq i} B_i$, so that $C_m$ is closed and $V/C_m$ is unbounded.

Conversely, if we have the collection of closed subgroup $C_m$ satisfying the above, then let $x_m \in V - C_m$ be chosen so that $v(x_m) \geq m$ and $\langle x_m \rangle$ is a valuated summand of $V/C_m$; denote the valuated projection $V/C_m \to \langle x_m \rangle$ by $\pi_m$. If $f\colon V \to \prod_{m<\omega} \langle x_m \rangle$ is given by $f(x) = (\pi_m(x+C_m))_{m<\omega}$, then it is easy to verify that $f$ is a valuated homomorphism whose image is unbounded and is actually contained in the free valuated vector space $\oplus_{m<\omega} \langle x_m \rangle$. If we let $W$ be the kernel of $f$, then $V = W \oplus F$, where $F$ is an unbounded free valuated vector space. $\square$

**Proposition 4.5.** *If $A$ is any unbounded separable group that has a subsocle $P \subseteq A[p]$ which is efi, then there is a group $G$ which is an $\omega$-$\aleph_0$-elongation of $A$, such that $G$ does not have a nice basis.*

*Proof:* Let $N$ be any reduced countable group which is not $\Sigma$-cyclic, i.e., $p^\omega N \neq \{0\}$. Let $D$ be a dense subspace of $A[p]$ such that $A[p]/D$ is countably infinite and $D + P(i) = A[p]$ for all $i < \omega$ [start with a dense subspace $D'$ of $P$ such that $P/D'$ is countably infinite, and if $W$ is a subspace of $A[p]$ such that $P \oplus W = A[p]$, then we can let $D = D' + W$].

By Lemma 3.2 there is a group $G$ which is an $\omega$-elongation of $A$ by $N$ such that $D = [G[p] + p^\omega G]/p^\omega G \subseteq A[p]$. Suppose now that $G$ actually does have a nice basis $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$. If for each $i < \omega$ we let $L_i = [M_i + N]/N$, then $L_i$ is closed in the $p$-adic topology on $A$ and their union is all of $A$. It follows that $P$ is the union of the closed subspaces $C_i = P \cap L_i$. Since $P$ if efi, it follows from Proposition 4.4 that, for some $i$, $P(i) \subseteq P \cap C_i \subseteq L_i$. We therefore have $A[p] \subseteq D + L_i$.

If $x \in N$, then find $y \in G$ such that $py = x$. Since $y + N \in A[p]$ there is a $w \in G[p]$ and $z \in M_i$ such that $y + N = (w + z) + N$. So there is a $u \in N$ such that $y = w + z + u$. This implies that $x = py = pw + pz + pu = pz + pu$. Note that $pz \in M_i(\omega)$, $pu \in pN$, so that we can conclude $N = M_i(\omega) + pN$. This, in turn, implies that $N/M_i(\omega)$ is divisible. Since $M_i$ is nice in $G$, $M_i(\omega)$ is nice in $N$. Since $N$ is reduced and $N/M_i(\omega)$ is divisible, we must have that $N = M_i(\omega) \subseteq M_i$. This is a contradiction, because $M_i$ being a part of a nice basis is $\Sigma$-cyclic, though we assumed $N$ was not. □

**Example 4.6.** There are $p^{\omega+1}$-projective groups $A$ satisfying Proposition 4.5.

If $P$ is any separable valuated vector space which is efi, then there is a separable group $A$ containing $P$ such that the valuation on $P$ agrees with the height function on $A$ and $A/P$ is $\Sigma$-cyclic. □

*Remark* 2. If we start with the group $A$ from Example 4.6 and construct the group $G$ as in Proposition 4.5, then since $G$ does not have a nice basis, it follows from [**2**] that it is not $p^{\omega+1}$-projective. Therefore, in Theorem 4.2 of [**6**], the condition of separability is necessary and cannot be eliminated.

Following [**7**], the separable group $A$ is said to have the *nice basis extension property* if for all $\omega$-elongations $G$ of $A$, if $p^\omega G$ has a nice basis, then so does $G$. By Corollary 2.13 of [**7**], the Continuum Hypothesis (CH) implies that when $A$ has a countable basic subgroup, then $A$ has the nice basis extension property iff it is $\Sigma$-cyclic, and it was asked whether all groups with this property are necessarily $\Sigma$-cyclic. The following is related to this question by restricting to the case of $\omega$-$\aleph_0$-elongations.

**Example 4.7.** In the presence of MA + $\neg$CH, there is a separable group $A$ which is not $\Sigma$-cyclic, with the property that every $\omega$-$\aleph_0$-elongation $G$ of $A$ has a nice basis. This group may be chosen to be $p^{\omega+1}$-projective.

Let $A$ be as in Theorem 3.4(a). If $G$ is an $\omega$-elongation of $A$, then since $A$ has the $\omega$-cps-elongation property, $G \cong C \oplus S$, where $C$ is countable and $S$ is separable. Since $C$ is countable, it is totally projective, and hence it has a nice basis by [**2**]. Since $S$ is separable, it clearly has a nice basis, as well. Therefore, in view of [**2**], $G$ has a nice basis, as required. The last part follows from Theorem 3.3. □

The following completely answers Problem 4 of [**7**].

**Example 4.8.** There are groups $G_1$ and $G_2$ with $p^\omega G_1 \cong p^\omega G_2$ and $G_1/p^\omega G_1 \cong G_2/p^\omega G_2$ such that $G_1$ has a nice basis but $G_2$ does not have a nice basis.

Suppose $A'$ is a separable group with a subsocle $P$ which is efi, and $B$ is an unbounded $\Sigma$-cyclic group. So if $A = A' \oplus B$, then since $P$ is also a subsocle of $A$, by Proposition 4.5, there is an $\omega$-$\aleph_0$-elongation $G_2$ of $A$ by a countable group $N = p^\omega G_2$ which does not have a nice basis. Since $N$ is countable and $B$ is unbounded, there a dsc-group $H$ such that $p^\omega H \cong N$ and $H/p^\omega H \cong B$. If we let $G_1 = H \oplus A'$, then $H$ has a nice basis by [**2**] (since it is a dsc-group and hence totally projective), $A'$ has a nice basis (since it is separable), so that $G_1$ has a nice basis in virtue of [**2**], as required. □

**Question 1.** Is the converse of Proposition 3.1(b) valid in ZFC? That is, if $A$ is a separable group with the $\omega$-cps-elongation property, does it follow in ZFC that $A$ has the cps-elongation property?

**Question 2.** Suppose $A$ is a separable group with the property that every $\omega$-$\aleph_0$-elongation $G$ of $A$ has a nice basis. In $V = L$, does it follow that $A$ must be $\Sigma$-cyclic? (Note that by Example 4.7, this does not hold in a model of MA + $\neg$CH.)

**Question 3.** Suppose $A$ is a separable group with the property that every $\omega$-elongation $G$ by a totally projective group $X = p^\omega G$ has a nice basis. Can we conclude that $A$ is $\Sigma$-cyclic? (If this holds, then all such $G$ will also be totally projective.)

## 5. Almost totally projective groups

A group $G$ is *almost totally projective* if it has a collection of nice subgroups $\mathcal{N}$ with the properties:

(0) $\{0\} \in \mathcal{N}$.

(1) If $\{H_i\}_{i\in I}$ is an ascending chain of subgroups in $\mathcal{N}$, then $\cup_{i\in I} H_i \in \mathcal{N}$.

(2) If $C \subseteq G$ is countable, then there is a countable $M \in \mathcal{N}$ such that $C \subseteq M$.

We pause to review the following.

**Proposition 5.1** ([**15**], [**5**]). *If $K$ is a subgroup of the reduced group $H$ such that $H/K$ is countable and $K$ is almost totally projective, then $H$ is almost totally projective.*

**Proposition 5.2** ([**5**]). *If $K$ is a countable and nice subgroup of the reduced group $H$ such that $H/K$ is almost totally projective, then $H$ is almost totally projective.*

More directly, we can state the following (see [**6**] for totally projective groups).

**Proposition 5.3.** *Suppose $A$ is reduced and $G$ is an $\aleph_0$-elongation of $A$ by $X$. If $G$ is almost totally projective, then $A$ is almost totally projective.*

*Proof:* Let $\mathcal{N}$ satisfy (0), (1), (2) above, and let $\mathcal{N}' = \{0\} \cup \{N/X : N \in \mathcal{N}$ and $X \subseteq N\}$. First, note that if $N/X \in \mathcal{N}'$, then $N$ is nice in $G$, so that $N/X$ is nice in $A$. Next, that $\mathcal{N}'$ satisfies (0) is immediate. As for (1), if $\{H_i/X\}_{i\in I}$ is an ascending chain of subgroups in $\mathcal{N}'$, then $\{H_i\}_{i\in I}$ is an ascending chain of subgroups in $\mathcal{N}$. It follows that $\cup_{i\in I} H_i \in \mathcal{N}$, so that $\cup_{i\in I}(H_i/X) \in \mathcal{N}'$, proving that (1) holds for $\mathcal{N}'$. Finally, if $C/X$ is countable for some $C \leq G$, then so is $C$ and hence there is a countable $M \in \mathcal{N}$ such that $C \subseteq M$. Thus $C/X \subseteq M/X \in \mathcal{N}'$, whence (2) follows as required. □

**Example 5.4.** The converse of Proposition 5.3 need not hold.

By Example 2.3 of [**6**], there is a separable non $\Sigma$-cyclic group $G$ with a countable but not nice subgroup $X$ such that $A = G/X$ is a dsc-group, hence is almost totally projective, with $p^\omega A$ uncountable. If this $G$ were, in fact, almost totally projective, then $X$ would be contained in a countable closed subgroup $N \in \mathcal{N}$. If $\overline{X}$ is the $p$-adic closure of $X$ in $G$, then $p^\omega A = \overline{X}/X \subseteq N/X$ would be countable, contrary to hypothesis. □

We have the following consequence of the above.

**Corollary 5.5.** *Let $G$ be a reduced group which is a nice-$\aleph_0$-elongation of $A$. Then $G$ is almost totally projective iff $A$ is almost totally projective.*

*Proof:* Sufficiency follows from Proposition 5.2 and necessity follows from Proposition 5.3. □

Observe that similar results were proved by Dieudonné [**9**] for $\Sigma$-cyclic groups and by Balof-Keef [**1**] for almost $\Sigma$-cyclic groups.

We conclude with the following result.

**Theorem 5.6.** *Suppose $G$ and $A$ are reduced groups and $f\colon G \to A$ is an $\omega_1$-bijective homomorphism. If $G$ is almost totally projective, then $A$ is almost totally projective.*

*Proof:* Let $I$ be the image of $f$ and $X$ be the kernel of $f$. Note that $G$ is an $\aleph_0$-elongation of $I$ by $X$, so by Proposition 5.3, $I$ is almost totally projective. Since $A/I$ is also countable, by Proposition 5.1, we can conclude that $A$ is almost totally projective, as required. □

The above discussion leads naturally to the following.

**Question 4.** If $f\colon G \to A$ is an $\omega_1$-bijective homomorphism and $A$ is totally projective, what we can say about $G$?

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