

THE K -GROUP OF SUBSTITUTIONAL SYSTEMS

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Abstract

In another article we associated a dynamical system to a non-properly ordered Bratteli diagram. In this article we describe how to compute the K -group K_0 of the dynamical system in terms of the Bratteli diagram. In the case of properly ordered Bratteli diagrams this description coincides with what is already known, namely the so-called dimension group of the Bratteli diagram. The new group defined here is more relevant for non-properly ordered Bratteli diagrams. We use our main result to describe K_0 of a substitutional system.

0. Introduction

An important tool in the study of aperiodic Cantor minimal dynamical systems (X, T) is its K -theory; in particular the K_0 -group $K^0(X, T)$, is an important invariant. After the celebrated Vershik-Herman-Putnam-Skau approach of codifying minimal Cantor dynamical systems by using the so-called ordered Bratteli diagrams, it became relevant to understand the group K_0 directly through diagrams. This is achieved in [HPS, Theorem 5.4 and Corollary 6.3] when properly ordered Bratteli diagrams are employed. Recently, we showed how to associate dynamical systems to non-properly ordered Bratteli diagrams. We generalise the above result of [HPS] by a careful modification (see Definition 3.1) of the notion of dimension group of an ordered Bratteli diagram. In doing this we have employed the “tripling” construction that was first introduced in [EP]. The result which describes the group K_0 in the case of a substitutional system arising from an aperiodic non-proper substitution is described in Theorem 3.11. It may be remarked that a method of computing K_0 even in the case of primitive non-proper substitutions is indicated in [DHS,

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Sections 5, 6, 7]; it relies on showing that the substitutional dynamical system is isomorphic to another one arising from a proper substitution. The proof of [DHS, Proposition 20] and [DHS, Proposition 23] relies heavily on ‘return words’ and ‘derivative sequences’ (*loc.cit*). Our description in Theorem 3.11 for non-proper substitutions is direct and closer in its approach and simplicity to the above cited Herman-Putnam-Skau description for proper substitutions. It eliminates the handicap of having to first work out details in the properly ordered case and then do the job of reducing to one such. Our methods have the advantage of yielding equally well a systematic way of computing the K -group in the much more general case of aperiodic Cantor dynamical systems which are not necessarily minimal and which may not arise from a substitutional rule (Theorem 3.8). The basic tool is a nested sequence of Kakutani-Rokhlin partitions where the heights of the towers at level n go to infinity. The referee has brought to our attention that such nested sequences have been constructed by Bezuglyi, Dooley and Medynets in [BDM, Proposition 3 and Corollary 4] in the much general case falling under the purview of the systems studied in this article. We thank the referee for this observation and further suggestions which greatly improved our presentation. The authors would also like to thank Christian Skau for his interest and suggestions in private communication.

1. Preliminaries

We summarize the key constructions in [EP] for the benefit of the reader following closely the text of the first chapter of [EP]. Some of the basic definitions and concepts in the study of Cantor dynamical systems are also recalled in this section.

A *topological dynamical system* is a pair (X, T) where X is a compact metric space and T is a homeomorphism in X . We say that T is *minimal* if for any $x \in X$, the T -orbit of $x := \{T^n(x) \mid n \in \mathbb{Z}\}$ is dense in X . We say that (X, T) is a *Cantor dynamical system* if X is a Cantor set, i.e. X is totally disconnected without isolated points. (X, T) is a *Cantor minimal dynamical system* if, in addition, T is minimal. The dynamical systems considered in this article are not necessarily minimal. They are aperiodic, i.e., no T -orbit is finite. Some of the basic concepts of the theory are recalled below, mostly from the more detailed sources [DHS] and [HPS].

1.1. Bratteli diagram. A Bratteli diagram is an infinite directed graph (V, E) , where V is the vertex set and E is the edge set. Both V

and E are partitioned into non-empty disjoint finite sets

$$V = V_0 \cup V_1 \cup V_2 \cup \cdots \cup \text{ and } E = E_1 \cup E_2 \cup \cdots .$$

There are two maps $r, s: E \rightarrow V$ the *range* and *source* maps. The following properties hold:

- (i) $V_0 = \{v_0\}$ consists of a single point, referred to as the ‘top vertex’ of the Bratteli diagram.
- (ii) $r(E_n) \subseteq V_n$, $s(E_n) \subseteq V_{n-1}$, $n = 1, 2, \dots$. Also $s^{-1}(v) \neq \phi$, $\forall v \in V$ and $r^{-1}(v) \neq \phi$ for all $v \in V_1, V_2, \dots$.

Maps between Bratteli diagrams are assumed to preserve gradings and intertwine the range and source maps. If $v \in V_n$ and $w \in V_m$, where $m > n$, then a path from v to w is a sequence of edges (e_{n+1}, \dots, e_m) such that $s(e_{n+1}) = v$, $r(e_m) = w$ and $s(e_{j+1}) = r(e_j)$. Infinite paths from $v_0 \in V_0$ are defined similarly. The Bratteli diagram is called *simple* if for any $n = 0, 1, 2, \dots$ there exists $m > n$ such that every vertex of V_n can be joined to every vertex of V_m by a path. However, we are not assuming this for our Bratteli diagrams. The following brief discussion is intended to indicate how general our Bratteli diagrams could be. A condition on a Bratteli diagram weaker than the above condition of ‘simplicity’ is to require that $\forall n, \exists m > n$ and $v \in V_m$ such that every vertex at level n is connected to v by a path. We are not assuming this either. Note that if this condition which is weaker than simplicity does not hold for a Bratteli diagram then $\exists n, \exists u, v \in V_n$ such that no $w \in V_m$ for any $m > n$ is connected to both u and v . We allow our (X, T) to have proper closed subsets invariant under T^\pm , which in addition may have non-empty interior. A condition on Bratteli diagrams which might ensure that (X, T) does not admit T -invariant proper clopen subsets is the following: $\forall n, \forall u, v \in V_n, \exists u_0 (= u), u_1, u_2, \dots, u_k (= v) \in V_n, \exists m > n, \exists w_1, w_2, \dots, w_k \in V_m$ such that u_{i-1} and u_i are both connected to w_i by a path for $i = 1, 2, \dots, k$. If such a possibility of linking two vertices $u, v \in V_n$ as above is not assumed, one is led to consider for each n , an equivalence relation in V_n , (see the proof of Lemma 3.3), where two elements in the same equivalence class can be linked.

1.2. Ordered Bratteli diagram. An *ordered* Bratteli diagram (V, E, \geq) is a Bratteli diagram (V, E) together with a linear order on $r^{-1}(v), \forall v \in V - \{v_0\} = V_1 \cup V_2 \cup V_3 \cup \cdots$. We say that an edge $e \in E_n$ is a *maximal* edge (resp. *minimal* edge) if e is maximal (resp. minimal) with respect to the linear order in $r^{-1}(r(e))$.

Given $v \in V_n$, it is easy to see that there exists a unique path (e_1, e_2, \dots, e_n) from v_0 to v such that each e_i is maximal (*resp.* minimal).

Note that if $m > n$, then for any $w \in V_m$, the set of paths starting from V_n and ending at w obtains an induced (lexicographic) linear order:

$$(e_{n+1}, e_{n+2}, \dots, e_m) > (f_{n+1}, f_{n+2}, \dots, f_m)$$

if for some i with $n+1 \leq i \leq m$, $e_j = f_j$ for $i < j \leq m$ and $e_i > f_i$.

1.3. Proper order. A *properly ordered* Bratteli diagram is a simple ordered Bratteli diagram $(V, E \geq)$ which possesses a unique infinite path $x_{\max} = (e_1, e_2, \dots)$ such that each e_i is a maximal edge and a unique infinite path $x_{\min} = (f_1, f_2, \dots)$ such that each f_i is a minimal edge.

Given a properly ordered Bratteli diagram $B = (V, E, \geq)$ we denote by X_B its infinite path space. So

$$X_B = \{(e_1, e_2, \dots) \mid e_i \in E_i, r(e_i) = s(e_{i+1}), i = 1, 2, \dots\}.$$

For an initial segment (e_1, e_2, \dots, e_n) we define the cylinder sets

$$U(e_1, e_2, \dots, e_n) = \{(f_1, f_2, \dots) \in X_B \mid f_i = e_i, 1 \leq i \leq n\}.$$

By taking cylinder sets to be a basis for open sets X_B becomes a topological space. We exclude trivial cases (where X_B is finite, or has isolated points). Thus, X_B is a Cantor set. X_B is a metric space, where for two paths x, y whose initial segments to level m agree but not to level $m+1$, $d(x, y) = 1/m+1$.

1.4. Vershik map for a properly ordered Bratteli diagram. If $x = (e_1, e_2, \dots, e_n, \dots) \in X_B$ and if at least one e_i is not maximal define

$$V_B(x) = y = (f_1, f_2, \dots, f_j, e_{j+1}, e_{j+2}, \dots) \in X_B$$

where e_1, e_2, \dots, e_{j-1} are maximal, e_j is not maximal and has f_j as successor in the linearly ordered set $r^{-1}(r(e_j))$ and $(f_1, f_2, \dots, f_{j-1})$ is the minimal path from v_0 to $s(f_j)$. Extend the above V_B to all of X_B by setting $V_B(x_{\max}) = x_{\min}$. Then (X_B, V_B) is a Cantor minimal dynamical system.

Next, we describe the construction of a dynamical system associated to a non-properly ordered Bratteli diagram. The Bratteli diagram need not be simple. To motivate this construction, it is perhaps worthwhile to begin by indicating how it works in the case of an ordered Bratteli diagram associated to a nested sequence of Kakutani-Rokhlin partitions of a Cantor dynamical system (X, T) .

1.5. K-R partition. A *Kakutani-Rokhlin partition* of the Cantor minimal system (X, T) is a clopen partition \mathcal{P} of the kind

$$\mathcal{P} = \{T^j Z_k \mid k \in A \text{ and } 0 \leq j < h_k\}$$

where A is a finite set and h_k is a positive integer. The k^{th} tower \mathcal{S}_k of \mathcal{P} is $\{T^j Z_k \mid 0 \leq j < h_k\}$; its *floors* are $T^j Z_k$, ($0 \leq j < h_k$). The *base* of \mathcal{P} is the set $Z = \bigcup_{k \in A} Z_k$.

Let $\{\mathcal{P}_n\}$, ($n \in \mathbb{N}$) be a sequence of Kakutani-Rokhlin partitions

$$\mathcal{P}_n = \{T^j Z_{n,k} \mid k \in A_n, \text{ and } 0 \leq j < h_{n,k}\},$$

with $\mathcal{P}_0 = \{X\}$ and with base $Z_n = \bigcup_{k \in A_n} Z_{n,k}$. We say that this sequence is *nested* if, for each n ,

- (i) $Z_{n+1} \subseteq Z_n$.
- (ii) \mathcal{P}_{n+1} refines the partition \mathcal{P}_n .

For the Bratteli-Vershik system (X_B, V_B) of Sections 1.3–1.4, one obtains a Kakutani-Rokhlin partition \mathcal{P}_n for each n by taking the sets in the partition to be the cylinder sets $U(e_1, e_2, \dots, e_n)$ of Section 1.3 and taking as the base of the partition the union $\bigcup U(e_1, e_2, \dots, e_n)$ over minimal paths (i.e., each e_i is a minimal edge). This is a nested sequence.

1.6. To any nested sequence $\{\mathcal{P}_n\}$, ($n \in \mathbb{N}$) of Kakutani-Rokhlin partitions we associate an ordered Bratteli diagram $B = (V, E, \geq)$ as follows (see [DHS, Section 2.3]): the $|A_n|$ towers in \mathcal{P}_n are in $1-1$ correspondence with V_n , the set of vertices at level n . Let $v_{n,k} \in V_n$ correspond to the tower $\mathcal{S}_{n,k} = \{T^j Z_{n,k} \mid 0 \leq j < h_{n,k}\}$ in \mathcal{P}_n . We refer to $T^j Z_{n,k}$, $0 \leq j < h_{n,k}$ as *floors* of the tower $\mathcal{S}_{n,k}$ and to $h_{n,k}$ as the *height* of the tower. *We will exclude nested sequences of K-R partitions where the infimum (over $k \in A_n$ for fixed n) of the height $h_{n,k}$ does not go to infinity with n .* Let us view the tower $\mathcal{S}_{n,k}$ against the partition $\mathcal{P}_{n-1} = \{T^j Z_{n-1,k} \mid k \in A_{n-1}, \text{ and } 0 \leq j < h_{n-1,k}\}$. As the floors of $\mathcal{S}_{n,k}$ rise from level $j = 0$ to level $j = h_{n,k} - 1$, $\mathcal{S}_{n,k}$ will start traversing a tower \mathcal{S}_{n-1,i_1} from the bottom to the top floor, then another tower \mathcal{S}_{n-1,i_2} from the bottom to the top floor, then another tower \mathcal{S}_{n-1,i_3} likewise and so on till a final segment of $\mathcal{S}_{n,k}$ traverses a tower \mathcal{S}_{n-1,i_m} from the bottom to the top. Note that in this final step the top floor $T^j Z_{n,k}$ for $j = h_{n,k} - 1$ of $\mathcal{S}_{n,k}$ reaches the top floor $T^q Z_{n-1,i_m}$ for $q = h_{n-1,i_m} - 1$ of \mathcal{S}_{n-1,i_m} as a consequence of the assumption $Z_n \subset Z_{n-1}$ and the fact that T^{-1} (union of bottom floors) = union of top floors. Bearing in mind this order in which $\mathcal{S}_{n,k}$ traverses $\mathcal{S}_{n-1,i_1}, \mathcal{S}_{n-1,i_2}, \dots, \mathcal{S}_{n-1,i_m}$ we associate m edges, ordered as $e_{1,k} < e_{2,k} < \dots < e_{m,k}$ and we set the range and source maps for edges

by $r(e_{j,k}) = v_{n,k}$ and $s(e_{j,k}) = v_{n-1,i_j}$. Note that m depends on the index $k \in A_n$ (and that by convention the indexing sets A_n are disjoint). E_n is the disjoint union over $k \in A_n$ of the edges having range in V_n .

1.7. Let us assume that the bases Z_n shrink to $Z = \bigcap_m Z_m$ and that Z has empty interior; make a similar hypothesis for the tops. (See Lemma 1.15 below.) For $x \in X$, we define $x_n \in \mathcal{P}_n^{\mathbb{Z}}$, $n \in \mathbb{N}$ as follows: $x_n = (x_{n,i})_{i \in \mathbb{Z}}$, where $x_{n,i} \in \mathcal{P}_n$ is the unique floor in \mathcal{P}_n to which $T^i(x)$ belongs. If $m > n$, let $j_{m,n}: \mathcal{P}_m \rightarrow \mathcal{P}_n$ be the unique map defined by $j_{m,n}(F) = F'$ if $F \subseteq F'$. (By abuse of notation, we use the same symbol F to denote a point of the finite set \mathcal{P}_m and also to denote the subset of X , in the partition \mathcal{P}_m , which F represents.) An important property of the map

$$X \longrightarrow \prod_n (\mathcal{P}_n^{\mathbb{Z}}), x \longmapsto (x_1, x_2, \dots), x_n = (x_{n,i})_{i \in \mathbb{Z}},$$

defined above is the following:

1.8. If F and TF are two successive floors of a \mathcal{P}_n -tower and if $x_{n,i} = F$ then $x_{n,i+1} = TF$. If $x_{n,i}$ is the top floor of a \mathcal{P}_n -tower, then $x_{n,i+1}$ is the bottom floor of a \mathcal{P}_n -tower. More importantly, given integers K and n , there exist $m > n$ and a single tower $\mathcal{S}_{m,k}$ of level m such that the finite sequence $(x_{n,i})_{-K \leq i \leq K}$ is an interval segment contained in

$$\{j_{m,n}(T^\ell(Z_{m,k})) \mid 0 \leq \ell < h_{m,k}\}.$$

This is a consequence of the assumption that the infimum of the heights of level- n towers goes to infinity. If “height of towers goes to infinity” does not imply the asserted property, choose n, x, K where this fails. This means that if $m > n$, and a \mathcal{P}_m -tower traverses through the floor $x_{n,-K}$ it dies out before reaching $x_{n,K}$. In the same way, a \mathcal{P}_m -tower traversing downwards through the floor $x_{n,K}$ reaches its base before it could traverse through the floor $x_{n,-K}$.

Recall the definition (from Section 1.6) $Z_m =$ union of all bottom floors of \mathcal{P}_m -towers. Then $Z_{m+1} \subset Z_m$ and $Z = \bigcap_m Z_m$ has empty interior; (see Lemma 1.15 below). Similar remarks for tops. {For the usual nested sequence of K-R partitions giving rise to a properly ordered Bratteli diagram, these intersections reduce to singletons. But for the nested sequences of K-R partitions considered by us, these intersections are nowhere dense sets.} Let $W = T^{-1}(Z)$, so that $W =$ the intersection of tops.

Each $x_{n,i}$ for $\{-K \leq i \leq K\}$ is open.

Each $x_{n,i} \cap Z$ is nowhere dense.

Each $x_{n,i} \cap W$ is nowhere dense.

Now, it is evident that there exists y in $x_{n,0}$ such that:

$$\{T^{-K}y, T^{-K+1}y, \dots, y, Ty, \dots, T^Ky\}$$

is disjoint from the above nowhere dense sets. Thus for some sufficiently large m , a single \mathcal{P}_m -tower has to contain $\{T^{-K}y, T^{-K+1}y, \dots, y, Ty, \dots, T^Ky\}$. Observe that if Section 1.7 is done taking y as it was done for x , then $x_{n,i} = y_{n,i}$ for $i \in [-K, K]$. Thus $x_{n,i} = j_{m,n}$ {the \mathcal{P}_m -floor to which $T^i(y)$ belongs}. It is true that $x_{n,i} = j_{m,n}(x_{m,i})$, but the sequence $(x_{m,i})_{-K \leq i \leq K}$ need not be an interval segment of $\{T^\ell(Z_{m,k}) \mid 0 \leq \ell < h_{m,k}\}$.

The foregoing observations in the case of an ordered Bratteli diagram associated to a nested sequence of Kakutani-Rokhlin partitions gives us the hint to define a dynamical system (X_B, T_B) of a non properly ordered Bratteli diagram $B = (V, E, \geq)$ as follows:

Definition 1.9. For each n define $\varpi_n =$ the set of paths from V_0 to V_n . There is an obvious truncation map $j_{m,n}: \varpi_m \rightarrow \varpi_n$ which truncates paths from V_0 to V_m to the initial segment ending in V_n . For each $v \in V_n$, the set $\varpi(v)$ of paths from $\{*\} \in V_0$ ending at v will be called a ' ϖ_n -tower parametrised by v '. Each tower is a linearly ordered set (whose elements may be referred to as floors of the tower) since paths from v_0 to v acquire a linear order (*cf.* Section 1.2). *We will exclude unusual examples of ordered Bratteli diagram where the infimum of the height of level- n towers does not go to infinity, with n (for example like [HPS, Example 3.2]).* Now, we define

Definition 1.10. $X_B = \{x = (x_1, x_2, \dots, x_n, \dots)\}$ where

- (i) $x_n = (x_{n,i})_{i \in \mathbb{Z}} \in \varpi_n^{\mathbb{Z}}$,
- (ii) $j_{m,n}(x_{m,i}) = x_{n,i}$ for $m > n$ and $i \in \mathbb{Z}$, and
- (iii) given n and K there exists m such that $m > n$ and a vertex $v \in V_m$, such that the interval segment $x_n[-K, K] := (x_{n,-K}, x_{n,-K+1}, \dots, x_{n,K})$ is obtained by applying $j_{m,n}$ to an interval segment of the linearly ordered set of paths from v_0 to v .

The condition (iii) is the crucial part of the definition. Without it what one gets is an inverse system.

The condition (iii) implies that a property similar to Section 1.8 holds. Since each ϖ_n is a finite set $\varpi_n^{\mathbb{Z}}$ has a product topology which makes it a compact set — in fact a Cantor set. Likewise, $\prod_n (\varpi_n^{\mathbb{Z}})$ is again a Cantor

set. Thus, $X_B \subseteq \prod_n (\varpi_n^{\mathbb{Z}})$ has an induced topology. The lemma below and the following proposition are analogous to corresponding facts for the Vershik model associated to properly ordered Bratteli diagrams.

The following results Lemma 1.11 and Proposition 1.12 are proved in [EP].

Lemma 1.11. *The topological space X_B is compact.*

Denote by T_B the restriction of the shift operator to X_B . So, if $x = (x_n)_{n \geq 1}$ where $x_n = (x_{n,i})_{i \in \mathbb{Z}} \in \varpi_n^{\mathbb{Z}}$, then $T_B(x) = (x'_1, x'_2, \dots, x'_n, \dots)$, where $x'_n = (x'_{n,i})_{i \in \mathbb{Z}} \in \varpi_n^{\mathbb{Z}}$ and $x'_{n,i} = x_{n,i+1}$.

(X_B, T_B) will be called the dynamical system associated to $B = (V, E, \geq)$.

Proposition 1.12. *If $B = (V, E, \geq)$ is a simple ordered Bratteli diagram, then (X_B, T_B) is a Cantor minimal dynamical system.*

1.13. In Section 1.7, given a nested sequence of Kakutani-Rokhlin partitions of (X, T) , we defined a map from (X, T) to the dynamical system (X_B, T_B) of the associated ordered Bratteli diagram. If the topology of (X, T) is spanned by the collection of the clopen sets belonging to the K-R partitions and if the bases and the tops shrink to nowhere dense sets then the map $(X, T) \rightarrow (X_B, T_B)$ is an isomorphism. In particular, if the Bratteli diagram is properly ordered then the Bratteli-Vershik system is naturally isomorphic to the system given by our construction in Definition 1.10.

1.14. Note that the same term ‘towers’ has been used to denote two different but related objects [in Section 1.5 and Definition 1.9]. For $v \in V_n$, let y be a path from $\{*\}$ to v in (V, E, \geq) . So, y is a ‘floor’ (consisting of the single element y) belonging to the ϖ_n -tower $\varpi(v)$ (a finite set) parametrized by $v \in V_n$ —all in the sense of Definition 1.9. Here, $\varpi(v)$ = all paths from $\{*\}$ to v . Put $\mathcal{F}_y = \{x = (x_1, x_2, \dots, x_n, \dots) \in X_B \mid x_{n,0} = y\}$. \mathcal{F}_y is a clopen set of the Cantor set X_B . Put $\mathcal{P}_n = \{\mathcal{F}_y \mid y \in \varpi(v), v \in V_n\}$. Then, in the sense of Section 1.5 \mathcal{P}_n is a K-R partition of X_B whose base is the union of $\bigcup \mathcal{F}_y$, (y minimal $\in \varpi(v)$, $v \in V_n$). Its towers \mathcal{S}_v are parametrized by $v \in V_n$. The floors of the tower \mathcal{S}_v are $\{\mathcal{F}_y \mid y \in \varpi(v)\}$. (We encountered this K-R partition earlier in the case of the Bratteli-Vershik system at the end of Section 1.5.) The ordered Bratteli diagram obtained from $\{\mathcal{F}_y \mid y \in \varpi(v), v \in V_n\}$ is (V, E, \geq) .

By [BDM, Proposition 3 and Corollary 4] and [M, Theorem 2.1] for the much general context of aperiodic Cantor dynamical systems under the purview of this article there exists a nested sequence of K-R

partitions $\{\mathcal{P}_n\}$, where the heights of the towers of level n go to infinity as n goes to infinity. The purpose of the lemma below is to start with one such and replace it by another nested sequence $\{\mathcal{P}'_n\}$ where in addition the $Z = \bigcap Z_n$ has empty interior and the same holds for the intersection of the tops.

Lemma 1.15. *Let $\{\mathcal{P}_n\}$, ($n \in \mathbb{N}$) be a nested sequence of Kakutani-Rokhlin partitions*

$$\mathcal{P}_n = \{T^j Z_{n,k} \mid k \in A_n, \text{ and } 0 \leq j < h_{n,k}\},$$

with $\mathcal{P}_0 = \{X\}$ and with base $Z_n = \bigcup_{k \in A_n} Z_{n,k}$. Let $h_n = \min\{h_{n,k} \mid k \in A_n\}$ be the infimum of the heights of level- n towers. Assume that h_n goes to infinity with n . Then there exists a nested sequence of K - R partitions

$$\mathcal{P}'_n = \{T^j Z'_{n,k} \mid k \in A'_n, \text{ and } 0 \leq j < h'_{n,k}\},$$

with $\mathcal{P}'_0 = \{X\}$ and with base $Z'_n = \bigcup_{k \in A'_n} Z'_{n,k}$ such that

- (i) \mathcal{P}'_n refines \mathcal{P}_n .
- (ii) $Z'_n \subseteq Z_n$.
- (iii) Let $Z' = \bigcap_n Z'_n$. Then Z' has empty interior.

Proof: If Z has empty interior there is nothing to prove. Assume, therefore, that Z has non-empty interior Z° . Let $x \in Z^\circ$. Suppose that for some $m > 0$, $T^m x \in Z^\circ$; choose n sufficiently large so that $h_n \geq m + 1$. Since $x \in Z_m$, $T^m x$ does not belong to Z_m . This shows that $T^m Z^\circ$, $m = 0, 1, 2, \dots$ are disjoint.

Let W be a non-empty clopen subset of Z° . Choose n sufficiently large such that W is the union of bases of finitely many level- n towers and $T^{-1}(W)$ is the union of tops of finitely many level- n towers; then, $Z_n - W$ is the union of bases of level- n towers and likewise $T^{-1}(Z_n - W)$ is the union of tops of level- n towers. Let $\{T^j Z_{n,k} \mid j = 0, 1, \dots, h_{n,k} - 1\}$ be a level- n tower such that $Z_{n,k} \subseteq W$. Then $T^{h_{n,k}}(Z_{n,k}) \subseteq Z_n - W$, since W and $T^j W$ are disjoint for all $j > 0$.

Next observe that $T^{-1}(W) \subseteq$ union of tops of level- n towers $\{T^j Z_{n,k} \mid j = 0, 1, \dots, h_{n,k} - 1\}$ whose base $Z_{n,k}$ is contained in $Z_n - W$. For otherwise $T^{-1}(W)$ has a non-empty intersection with $\bigcup_{j \geq 0} T^j(W)$. Applying T , W would then intersect $\bigcup_{j > 0} T^j(W)$, which was already ruled out.

These observations show that $X = \bigcup_{j \geq 0} T^j(Z_n - W)$. Hence, by compactness in fact $X = \bigcup_{0 \leq j \leq N} T^j(Z_n - W)$ for some N .

Thus we can apply the first return function $n_{Z_n - W}: Z_n - W \rightarrow \mathbb{N}$ and obtain a K-R partition \mathcal{P}'_n ; further partitioning the bases of the towers thus obtained the partition of X given by \mathcal{P}'_n can be assumed to be finer than that given by \mathcal{P}_n . Choose an increasing sequence of clopen subsets $W_1 \subseteq W_2 \subseteq \dots \subseteq W_j \subseteq \dots$ whose union equals Z° ; choose an increasing sequence $n_1, n_2, \dots, n_j, \dots$ such that W_j , (*resp.* $T^{-1}(W_j)$) is the union of bases (*resp.* tops) of finitely many \mathcal{P}_{n_j} -towers. For n such that $n_j \leq n < n_{j+1}$ apply the above indicated procedure for constructing \mathcal{P}'_n from the first return function on $Z_n - W_{n_j}$. Then, $\{\mathcal{P}'_n\}_n$ is a nested sequence of K-R partitions satisfying the required properties. \square

2. The Bratteli diagram $(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$

2.1. We will now define two nested sequences of K-R partitions of X . For $v \in V_n$, let y be a path from $\{*\}$ to v in (V, E, \geq) . So, y is a ‘floor’ belonging to the ϖ_n -tower $\varpi(v)$ parametrized by v . Put $\mathcal{F}_y = \{x = (x_1, x_2, \dots, x_n, \dots) \in X \mid x_{n,0} = y\}$

$$\mathcal{P}_n = \{\mathcal{F}_y \mid y \in \varpi(v), v \in V_n\}.$$

Then $\{\mathcal{P}_n\}_n$ is a nested sequence of K-R partitions of X . But, the topology of X need not be spanned by the collection of clopen sets $\{\mathcal{F}_y\}$, ($y \in \varpi(v)$, $v \in V_n$, $n \in \mathbb{N}$). In contrast, the topology of X is indeed spanned by the collection of clopen sets in another nested sequence $\{\mathcal{Q}_n\}_n$ of K-R partitions, defined below. Let $\varpi = \varpi(u)$, $\varpi' = \varpi(v)$, $\varpi'' = \varpi(w)$ be three ϖ_n -towers and y a floor of ϖ' . For any $x \in X$ and for any n if $x_{n,i}$ is a floor of a ϖ_n -tower $\overline{\varpi}$, then for some $a, b \in \mathbb{Z}$ such that $a \leq i \leq b$, the segment $x_n[a, b]$ is just the sequence of floors in $\overline{\varpi}$. We define $\mathcal{F}(\varpi, \varpi', \varpi''; y) =$ the clopen subset of \mathcal{F}_y consisting of the elements $x = (x_1, x_2, \dots, x_n, \dots)$ with the property that for some $a_1 < a_2 \leq 0 < a_3 < a_4 \in \mathbb{Z}$, the segment $x_n[a_1, a_2 - 1]$ is the sequence of floors of ϖ , the segment $x_n[a_2, a_3 - 1]$ is the sequence of floors of ϖ' and the segment $x_n[a_3, a_4]$ is the sequence of floors of ϖ'' .

Some of the sets $\mathcal{F}(\varpi, \varpi', \varpi''; y)$ may be empty, but the sets $\mathcal{F}(\varpi, \varpi', \varpi''; y)$, which are non-empty, form a K-R partition which we denote by \mathcal{Q}_n . For fixed $\varpi, \varpi', \varpi''$ the subcollection $\{\mathcal{F}(\varpi, \varpi', \varpi''; y)\}$ as y varies through the floors of ϖ' , is a \mathcal{Q}_n -tower parametrized by $[u, v, w]$. We denote this \mathcal{Q}_n -tower by $\mathcal{S}_{(\varpi, \varpi', \varpi')}$. The floors of the tower $\mathcal{S}_{(\varpi, \varpi', \varpi')}$ are $\{\mathcal{F}(\varpi, \varpi', \varpi''; y)\}$ as y runs through the sequence of floors of ϖ' .

2.2. The tripling of (V, E, \geq) . Let (V, E, \geq) be an arbitrary simple, ordered Bratteli diagram. Define $(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$ as follows: $V_0^{\mathcal{Q}} = \{*\}$, a single point.

$V_n^{\mathcal{Q}}$ consists of triples $(u, v, w) \in V_n \times V_n \times V_n$ such that for some $y \in V_m$ where $m > n$, the level- m tower $\varpi(y)$ passes successively through the level- n tower $\varpi(u)$, then $\varpi(v)$ and then $\varpi(w)$. An edge $\tilde{e} \in E_n^{\mathcal{Q}}$ is a triple (u, e, w) such that e is an edge of (V, E) and $(u, r(e), w) \in V_n^{\mathcal{Q}}$. Let

$\{e_1, e_2, \dots, e_k\}$ be all the edges in $r^{-1}(r(e))$,
 $\{f_1, f_2, \dots, f_\ell\}$ be all the edges in $r^{-1}(u)$ and
 $\{g_1, g_2, \dots, g_m\}$ be all the edges in $r^{-1}(w)$.

The sources of $(u, e_1, w), (u, e_2, w), \dots, (u, e_k, w)$ are defined to be

$(s(f_\ell), s(e_1), s(e_2)), (s(e_1), s(e_2), s(e_3)), \dots, (s(e_{k-1}), s(e_k), s(g_1)))$

respectively. The range of (u, e, w) is of course $(u, r(e), w)$. If $r^{-1}(r(e))$ is ordered as $\{e_1, e_2, \dots, e_k\}$, we declare the ordering of $r^{-1}(r(u, e, w))$ to be $\{(u, e_1, w), (u, e_2, w), \dots, (u, e_k, w)\}$. The ordered Bratteli diagram $(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$ thus defined will be called the tripling of (V, E, \geq) .

The map $\pi: (V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq) \rightarrow (V, E, \geq)$ given by $(u, v, w) \mapsto v$, $(u, e, w) \mapsto e$ enjoys the ‘*unique path lifting*’ property in the following sense. If $m > n \geq 1$, and $(e_n, e_{n+1}, \dots, e_m)$ is a path in (V, E) from V_{n-1} to V_m with $r(e_m) = v$ then for any $(u, v, w) \in V_m^{\mathcal{Q}}$, there is a unique path $(\tilde{e}_n, \tilde{e}_{n+1}, \dots, \tilde{e}_m)$ in $(V^{\mathcal{Q}}, E^{\mathcal{Q}})$ which maps onto $(e_n, e_{n+1}, \dots, e_m)$ under π and such that $r(\tilde{e}_m) = (u, v, w)$. It is quite elementary to check that the map $\pi: (V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq) \rightarrow (V, E, \geq)$ induces an isomorphism between the corresponding dynamical systems given by Definition 1.10 ([EP, 2.17]).

[Two different edges on the left with the same source may map into the same edge on the right. Two different edges on the left with the same range cannot map to the same edge on the right.]

Let $\{n_k\}_{k=0}^{\infty}$ be a subsequence of $\{0, 1, 2, \dots\}$ where we assume $n_0 = 0$. A Bratteli diagram (V', E') is called a ‘*telescoping*’ of (V, E) if $V'_k = V_{n_k}$ and E'_k consists of paths $(e_{n_{k-1}+1}, \dots, e_{n_k})$ from $V_{n_{k-1}}$ to V_{n_k} in (V, E) , the range and source maps being the obvious ones. It is easy to see that tripling is compatible with telescoping.

2.3. Stationary Bratteli diagrams. A Bratteli diagram is stationary if the diagram repeats itself after level 1. (One may relax by allowing a period from some level onwards; but, a telescoping will be stationary in the above restricted sense.) If (V, E, \geq) is an ordered Bratteli diagram and the diagram together with the order repeats itself after level 1, then

(V, E, \geq) will be called a stationary ordered Bratteli diagram. We refer the reader to [DHS, Section 3.3] for the usual definition of a substitutional system and how they give rise to stationary Bratteli diagrams. Some details are recalled below. Let (V, E, \geq) be as above. We have

- (1) an enumeration $\{v_{n,1}, v_{n,2}, \dots, v_{n,L}\}$ of $V_n, \forall n \geq 1$,
- (2) for $n > 1$ and $1 \leq j \leq L$ an enumeration $\{e_{n,j,1}, e_{n,j,2}, \dots, e_{n,j,a_j}\}$ of $r^{-1}(e_{n,j})$ which is assumed to be listed in the linear order in $r^{-1}(v_{n,j})$,
- (3) in the enumerations above, L does not depend on n and a_j depends only on j and not on n . Moreover, the ordering in $r^{-1}(e_{n,j})$ is stationary, i.e., if $n, m > 1$, if $1 \leq j \leq L, 1 \leq k \leq L, 1 \leq i \leq a_j$, then “ $s(e_{n,j,i}) = v_{n-1,k}$ ” \implies “ $s(e_{m,j,i}) = v_{m-1,k}$ ”.

2.4. Substitutional systems. Let A be an alphabet. Write A^+ for the set of words of finite length in the letters of A . Let $\sigma: A \rightarrow A^+$ be an aperiodic non-proper substitution, written, $\sigma(a) = \alpha\beta\gamma\dots$. The stationary ordered Bratteli diagram $B = (V, E, \geq)$ associated to (A, σ) (cf. [DHS, Section 3.3]) can be described as

$$V_n = A, \quad \forall n \geq 1, \quad V_0 = \{*\}$$

$$E_n = \{(a, k, b) \mid a, b \in A, k \in \mathbb{N}, a \text{ is the } k^{\text{th}} \text{ letter in the word } \sigma(b)\}.$$

(The reader who prefers a more carefully evolved notation can consider introducing an extra factor ‘ $\times \{n\}$ ’ so that vertices and edges at different levels are seen to be disjoint.) The source and range maps s and r are defined by $s(a, k, b) = a, r(a, k, b) = b$. In the linear order in $r^{-1}(b)$, (a, k, b) is the k^{th} edge.

To the stationary ordered Bratteli diagram B of (A, σ) (which may not be properly ordered unless σ is a primitive, aperiodic, proper substitution, —see [DHS, Section 3]) we can associate a dynamical system (X_B, T_B) following the construction of Definition 1.10; our assumptions that “heights of level- n towers go to infinity” and “bases shrink to a set Z with empty interior” translate to the following conditions which we assume to hold:

- (i) $\forall a \in A$, the length of $\sigma^n(a)$ goes to infinity with n .
- (ii) $\forall a \in A, \exists b \in A$ such that for a sufficiently large positive integer n , the letter a occurs in the middle (i.e., non-extreme position) of $\sigma^n(b)$.

Then (X_B, T_B) is naturally isomorphic to the substitutional dynamical system (X_σ, T_σ) associated to (A, σ) defined for example in [DHS, Section 3.3.1]. (See [EP, Section 2.5].)

2.5. Tripling for a substitutional system (A, σ) . Let (A, σ) be a substitutional system and suppose $B = (V, E, \geq)$ is the stationary ordered Bratteli diagram associated to (A, σ) . Define $A^\mathcal{Q} = \{(a, b, c) \in A \times A \times A \mid abc \text{ occurs as a subword of } \sigma^n(d) \text{ for some } d \in A \text{ and some } n\}$. Define

$$\sigma^\mathcal{Q}: A^\mathcal{Q} \longrightarrow (A^\mathcal{Q})^+$$

by $\sigma^\mathcal{Q}[(a, b, c)] = (a_m, b_1, b_2) \cdot (b_1, b_2, b_3) \cdots (b_{n-2}, b_{n-1}, b_n) \cdot (b_{n-1}, b_n, c_1)$, where $\sigma(b) = b_1 \cdot b_2 \cdots b_n$, and a_m is the last letter in $\sigma(a)$, while c_1 is the first letter in $\sigma(c)$. Then $(V^\mathcal{Q}, E^\mathcal{Q}, \geq)$ is the stationary ordered Bratteli diagram associated to $(A^\mathcal{Q}, \sigma^\mathcal{Q})$.

3. The groups $K^0(X, T)$, $K_{-0}(V, E, \geq)$ and $K_0(V, E)$

Definition 3.1. Let (X, T) be an aperiodic Cantor dynamical system. Let $C(X, \mathbb{Z})$ be the space of integer valued continuous functions on X . Let

$$K^0(X, T) = C(X, \mathbb{Z}) / \partial_T C(X, \mathbb{Z})$$

where $\partial_T: C(X, \mathbb{Z}) \rightarrow C(X, \mathbb{Z})$ denotes the coboundary operator $\partial_T(f) = f - f \circ T$. A function of the form $f - f \circ T$ is called a coboundary.

Let (V, E) be a Bratteli diagram and (V, E, \geq) the same thing equipped with a linear order on edges which makes it an ordered Bratteli diagram. As usual the group $K_0(V, E)$ is defined to be the inductive limit of the system of groups

$$\mathbb{Z}^{|V_0|} \xrightarrow{A_0} \mathbb{Z}^{|V_1|} \xrightarrow{A_1} \mathbb{Z}^{|V_2|} \xrightarrow{A_2} \mathbb{Z}^{|V_3|} \xrightarrow{A_3} \dots$$

where the positive homomorphism A_n is given by matrix multiplication with the incidence matrix between levels $n-1$ and n .

On the other hand we define the group $K_{-0}(V, E, \geq)$ in the following way. For $\overline{m} = (m_k)_{k \in V_n} \in \mathbb{Z}^{|V_n|}$, $w \in V_l$, ($l > n$) and two paths θ, τ both ranging at w from V_n to V_l denote by $I(\theta, \tau)$ the set of all paths lying between θ and τ (both included) in the linear order on the set of

paths from V_n to V_m ranging at w . Put

$$\sigma_{\bar{m}}(\theta; w; \tau) = \begin{cases} \sum_{\phi} m_{s(\phi)}, & \text{sum over all paths } \phi \text{ in } I(\theta, \tau) \\ & \text{with } \tau \text{ excluded, if } \theta < \tau \\ -\sum_{\phi} m_{s(\phi)}, & \text{sum over all paths } \phi \text{ in } I(\theta, \tau) \\ & \text{with } \theta \text{ excluded, if } \theta > \tau \\ 0, & \text{if } \theta = \tau. \end{cases}$$

Here, $s(\theta)$ denotes the source of the path θ .

Let $\bar{m} = (m_k)_{k \in V_n}$ be an element of $\mathbb{Z}^{|V_n|}$. We define a subset

$$B\mathbb{Z}^{|V_n|} \subseteq \mathbb{Z}^{|V_n|}$$

by declaring that $\bar{m} \in B\mathbb{Z}^{|V_n|}$ if and only if the following condition is satisfied:

$$\sum_{i=1}^N \sigma_{\bar{m}}(\theta_i; w_i; \tau_i) = 0,$$

whenever

- (a) $w_1, w_2, \dots, w_N \in V_l$, ($l > n$).
- (b) θ_i, τ_i are paths from V_n to V_l ranging at w_i .
- (c) $s(\tau_i) = s(\theta_{i+1})$, $i = 1, \dots, N-1$.
- (d) $s(\tau_N) = s(\theta_1)$.

Observe that $A_n(B\mathbb{Z}^{|V_n|}) \subseteq B\mathbb{Z}^{|V_{n+1}|}$. Define $K_{-0}(V, E, \geq)$ to be the inductive limit of the system of groups

$$\frac{\mathbb{Z}^{|V_0|}}{B\mathbb{Z}^{|V_0|}} \xrightarrow{A_0} \frac{\mathbb{Z}^{|V_1|}}{B\mathbb{Z}^{|V_1|}} \xrightarrow{A_1} \frac{\mathbb{Z}^{|V_2|}}{B\mathbb{Z}^{|V_2|}} \xrightarrow{A_2} \frac{\mathbb{Z}^{|V_3|}}{B\mathbb{Z}^{|V_3|}} \xrightarrow{A_3} \dots$$

Theorem 3.2. For $B = (V, E, \geq)$ let (X_B, T_B) be defined as in Definition 1.10. Write $(X, T) = (X_B, T_B)$. Define the tripling $B^{\mathcal{Q}} = (V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$ as in Section 2.2. Then $K^0(X, T)$ is naturally isomorphic to $K_{-0}(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$.

Proof: We recall the notation introduced in Section 2.1. Given $f \in C(X, \mathbb{Z})$, choose n sufficiently large such that $f, \partial_T(f)$ are both constant on the sets of the partition \mathcal{Q}_n . The vertices $V_n^{\mathcal{Q}}$ of the Bratteli diagram $(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$ correspond to towers $\mathcal{S}_{(\varpi, \varpi', \varpi'')}$ of a K-R partition which in turn are partitioned into floors $\{\mathcal{F}(\varpi, \varpi', \varpi''; y)\}$ as y varies through the floors of ϖ' . For f as above, define $\gamma_n(f) \in \mathbb{Z}^{|V_n^{\mathcal{Q}}|}$ by $\gamma_n(f)(\varpi, \varpi', \varpi'') = f(x) + f(Tx) + f(T^2x) + \dots + f(T^{h-1}x)$ where x belongs to the lowest floor of $\mathcal{S}_{(\varpi, \varpi', \varpi'')}$ and h is the height of the tower $\mathcal{S}_{(\varpi, \varpi', \varpi'')}$. Then $A_n^{\mathcal{Q}}(\gamma_n(f)) = \gamma_{n+1}(f)$ and $\gamma_n(\partial_T(f)) \in B\mathbb{Z}^{|V_n^{\mathcal{Q}}|}$. This gives rise to a map

$$\gamma: K^0(X_B, T_B) \longrightarrow K_{-0}(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq).$$

Lemma 3.3. *Let $f \in C(X, \mathbb{Z})$ and suppose that f is constant on the sets of the partition \mathcal{Q}_n . Suppose that $\gamma_n(f) \in B\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$. Then, $f = \partial_T(g)$, for some $g \in C(X, \mathbb{Z})$.*

Define an equivalence relation in $V_n^\mathcal{Q}$ as follows: Let $\mathbf{u}, \mathbf{v} \in V_n^\mathcal{Q}$; we say $\mathbf{u} \sim_{\mathcal{R}} \mathbf{v}$ if there exist

- (a) $\mathbf{w}_1, \dots, \mathbf{w}_N \in V_\ell^\mathcal{Q}$, ($\ell > n$) for some N, ℓ .
- (b) Paths θ_i, τ_i , ($i = 1, \dots, N$) from $V_n^\mathcal{Q}$ to $V_\ell^\mathcal{Q}$ ranging at \mathbf{w}_i such that $s(\theta_1) = \mathbf{u}$, $s(\tau_N) = \mathbf{v}$, $s(\tau_i) = s(\theta_{i+1})$ for $i = 1, \dots, N-1$.

It is evident that \mathcal{R} is an equivalence relation in $V_n^\mathcal{Q}$. We choose and fix a set of representatives $C = \{\mathbf{u}_1, \dots, \mathbf{u}_e\}$ in the equivalence classes of $V_n^\mathcal{Q}$ for \mathcal{R} .

Now let $x \in X_B$. Choose $\mathbf{v} \in V_n^\mathcal{Q}$ and y in the base of the \mathcal{Q}_n -tower represented by \mathbf{v} such that $x = T^j y$ for some $j < h_{\mathbf{v}}$ the height of the tower. Choose

$$\mathbf{w}_1, \dots, \mathbf{w}_N \in V_\ell^\mathcal{Q}, (\ell > n).$$

Paths θ_i, τ_i ($i = 1, \dots, N$) from $V_n^\mathcal{Q}$ to $V_\ell^\mathcal{Q}$ ranging at \mathbf{w}_i such that $s(\tau_i) = s(\theta_{i+1})$, $i = 1, \dots, N-1$ and $s(\theta_1) \in C$, $s(\tau_N) = \mathbf{v}$.

Put

$$g(x) = \sum_{i=1}^N \sigma_{\overline{m}}(\theta_i; \mathbf{w}_i; \tau_i) + \sum_{k=0}^{j-1} f(T^k(y))$$

where $\overline{m} = \gamma_n(f)$. Since $\gamma_n(f) \in B\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$, $g(x)$ is well defined independent of the choices $N, \ell, \mathbf{w}_1, \dots, \mathbf{w}_n, \theta_i, \tau_i$. The function g thus defined has the required property. \square

From Lemma 3.3 one can immediately deduce that the map $\gamma: K^0(X_B, T_B) \rightarrow K_{-0}(V^\mathcal{Q}, E^\mathcal{Q}, \geq)$ defined just before the statement of Lemma 3.3 is an isomorphism.

This completes the proof of Theorem 3.2. \square

3.4. A subgroup of $B\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$. In practice it is quite tedious to determine whether a given element \overline{p} of $\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$ lies in $B\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$. We now begin to describe a subgroup $\Delta\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|} \subseteq B\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$, which is more easily identifiable than $B\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$. Even though, in general, this inclusion is proper we will later see that the distinction disappears when one takes inductive limits. As a consequence, we are able to obtain Theorem 3.8, which yields a feasible method to compute K_0 effectively. Clearly, $\mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$ is the space of integral valued functions on the set $V_n^\mathcal{Q}$. For a function $\varphi: V_n \times V_n \rightarrow \mathbb{Z}$ define $\delta(\varphi) \in \mathbb{Z}^{|\mathcal{V}_n^\mathcal{Q}|}$ by $\delta(\varphi)(a, b, c) = \varphi(b, c) - \varphi(a, b)$.

Lemma 3.5. $\delta(\varphi) \in B\mathbb{Z}^{|V_n^{\mathcal{Q}}|}$.

Proof: Let $\bar{p} \in \mathbb{Z}^{|V_n^{\mathcal{Q}}|}$. Write $\bar{p} = \{p_{(u,v,w)}\}_{(u,v,w) \in V_n^{\mathcal{Q}}}$. Let $\mathbf{w}_1, \dots, \mathbf{w}_N \in V_\ell^{\mathcal{Q}}$, ($\ell > n$) for some N, ℓ . Let θ_i, τ_i , ($i = 1, \dots, N$) be paths from $V_n^{\mathcal{Q}}$ to $V_\ell^{\mathcal{Q}}$ ranging at \mathbf{w}_i such that $s(\theta_1) = s(\tau_N)$, $s(\tau_i) = s(\theta_{i+1})$ for $i = 1, \dots, N-1$. Let $s(\theta_i) = (u_i, v_i, w_i)$ and $s(\tau_i) = (u'_i, v'_i, w'_i)$. If $\bar{p} = \delta(\varphi)$ we see that

$$\sigma_{\bar{p}}(\theta_i, \mathbf{w}_i, \tau_i) = \begin{cases} -\varphi(u_i, v_i) + \varphi(u'_i, v'_i), & \text{if } \theta_i \leq \tau_i \\ -\varphi(u_i, v_i) + \varphi(u'_i, v'_i), & \text{if } \theta_i \geq \tau_i. \end{cases}$$

So, $\sum_{i=1}^N \sigma_{\bar{p}}(\theta_i; \mathbf{w}_i; \tau_i) = 0$.

The condition for \bar{p} to belong to $B\mathbb{Z}^{|V_n^{\mathcal{Q}}|}$ is precisely that the sums of the above type should all vanish. As the foregoing calculation shows this holds whenever $\bar{p} = \delta(\varphi)$ for some $\varphi: V_n \times V_n \rightarrow \mathbb{Z}$. \square

We define $\Delta\mathbb{Z}^{|V_n^{\mathcal{Q}}|}$ to be the subgroup $\delta(\mathbb{Z}^{|V_n \times V_n|})$ of $\mathbb{Z}^{|V_n^{\mathcal{Q}}|}$.

Recall the map $A_n^{\mathcal{Q}}: \mathbb{Z}^{|V_n^{\mathcal{Q}}|} \rightarrow \mathbb{Z}^{|V_{n+1}^{\mathcal{Q}}|}$ given by matrix multiplication by the incidence matrix between the levels $V_n^{\mathcal{Q}}$ and $V_{n+1}^{\mathcal{Q}}$. For a function $\varphi: V_n \times V_n \rightarrow \mathbb{Z}$ define $\varphi': V_{n+1} \times V_{n+1} \rightarrow \mathbb{Z}$ by $\varphi'(u', v') = \varphi(u, v)$ where u (*resp.* v) is the source of the last (*resp.* first) edge ranging at u' (*resp.* v').

Lemma 3.6. *With notation as above, $A_n^{\mathcal{Q}}(\delta(\varphi)) = \delta(\varphi')$. In particular the identity endomorphism of $\mathbb{Z}^{|V_n^{\mathcal{Q}}|}$ induces a map*

$$\frac{\mathbb{Z}^{|V_n^{\mathcal{Q}}|}}{\Delta\mathbb{Z}^{|V_n^{\mathcal{Q}}|}} \xrightarrow{\rho_n} \frac{\mathbb{Z}^{|V_n^{\mathcal{Q}}|}}{B\mathbb{Z}^{|V_n^{\mathcal{Q}}|}}$$

and

$$\begin{array}{ccc} \frac{\mathbb{Z}^{|V_n^{\mathcal{Q}}|}}{B\mathbb{Z}^{|V_n^{\mathcal{Q}}|}} & \xrightarrow{A_n^{\mathcal{Q}}} & \frac{\mathbb{Z}^{|V_{n+1}^{\mathcal{Q}}|}}{B\mathbb{Z}^{|V_{n+1}^{\mathcal{Q}}|}} \\ \rho_n \uparrow & & \rho_{n+1} \uparrow \\ \frac{\mathbb{Z}^{|V_n^{\mathcal{Q}}|}}{\Delta\mathbb{Z}^{|V_n^{\mathcal{Q}}|}} & \xrightarrow{A_n^{\mathcal{Q}}} & \frac{\mathbb{Z}^{|V_{n+1}^{\mathcal{Q}}|}}{\Delta\mathbb{Z}^{|V_{n+1}^{\mathcal{Q}}|}} \end{array}$$

is commutative.

Proof: The proof of the assertion $A_n^{\mathcal{Q}}(\delta(\varphi)) = \delta(\varphi')$ is a straightforward calculation using definitions and notation. The rest follows immediately. \square

We might wish to ask whether every element \bar{p} of $B\mathbb{Z}^{|V_n^{\mathcal{Q}}|}$ is of the form $\delta(\varphi)$ for some function $\varphi: V_n \times V_n \rightarrow \mathbb{Z}$. The Proposition 3.7 below shows that after applying a finite iteration $A_{n+i}^{\mathcal{Q}} \circ \cdots \circ A_{n+1}^{\mathcal{Q}} \circ A_n^{\mathcal{Q}}$ to \bar{p} it will indeed be so.

Let $\bar{p} \in B\mathbb{Z}^{|V_n^{\mathcal{Q}}|}$. Let $g \in C(X, \mathbb{Z})$ be chosen as in Lemma 3.3 so that

- (i) $\partial_T(g)$ is constant on the sets of the partition \mathcal{Q}_n and moreover, $\bar{p} = \gamma_n(\partial_T(g))$.
- (ii) g itself is constant on the sets of the partition \mathcal{Q}_{n+i} for some positive integer i .

Choose a positive integer j such that any \mathcal{Q}_{n+i+j} -tower traverses through at least two \mathcal{Q}_{n+i} -towers. Then, of course, any \mathcal{P}_{n+i+j} -tower traverses through at least two \mathcal{P}_{n+i} -towers.

For $u', v' \in V_{n+i+j}$ let u_a (*resp.* u_{a-1} , *resp.* v_1 , *resp.* v_2) be the source of the *last* (*resp.* *last but one*, *resp.* *first*, *resp.* *second*) path from V_{n+i} to V_{n+i+j} ranging at u' (*resp.* u' , *resp.* v' , *resp.* v'). If there exists $x \in X$ such that

- (a) $x \in$ bottom floor of the \mathcal{Q}_{n+i} -tower represented by (u_a, v_1, v_2) ,
- (b) $T^{-1}x \in$ top floor of the \mathcal{Q}_{n+i} -tower represented by (u_{a-1}, u_a, v_1) ,

define $\varphi'(u', v') = g(x)$; then, $\varphi'(u', v')$ is independent of x . For given $u', v' \in V_{n+i+j}$ if no such x exists, or if either one of (u_a, v_1, v_2) , (u_{a-1}, u_a, v_1) does not belong to $V_{n+i}^{\mathcal{Q}}$ define $\varphi'(u', v')$ arbitrarily.

Proposition 3.7. *With notation as above $A_{n+i+j-1}^{\mathcal{Q}} \circ \cdots \circ A_{n+1}^{\mathcal{Q}} \circ A_n^{\mathcal{Q}}(\bar{p}) = \delta(\varphi')$.*

Proof: $\delta(\phi')(u', v', w') = \phi'(v', w') - \phi'(u', v') = g(T^h y) - g(y)$, if y lies in the lowest floor of the \mathcal{Q}_{n+i+j} -tower of height h represented by (u', v', w') . Also, for $(u, v, w) \in V_n^{\mathcal{Q}}$, $\bar{p}(u, v, w) = \gamma_n(\partial_T(g))(u, v, w) =$ the sum $(g \circ T - g)(z) + (g \circ T - g)(Tz) + (g \circ T - g)(T^2 z) + \cdots + (g \circ T - g)(T^{k-1} z)$, where k is the height of the \mathcal{Q}_n -tower represented by (u, v, w) and z lies in its lowest floor. Thus, $A_{n+i+j-1}^{\mathcal{Q}} \circ \cdots \circ A_{n+1}^{\mathcal{Q}} \circ A_n^{\mathcal{Q}}(\bar{p})(u', v', w')$ is the sum of $g \circ T - g$ taken over all the floors of the \mathcal{Q}_{n+i+j} -tower represented by (u', v', w') . This sum also equals $g \circ T^h(y) - g(y)$. \square

We can therefore give an alternate description of $K^0(X_B, T_B)$ which is more elegant than the description in Theorem 3.2.

As we already observed, $A_n^{\mathcal{Q}}(\delta(\mathbb{Z}^{|V_n \times V_n|})) \subseteq \delta(\mathbb{Z}^{|V_{n+1} \times V_{n+1}|})$.

Theorem 3.8. For $B = (V, E, \geq)$ let (X_B, T_B) be defined as in Definition 1.10 and Lemma 1.11. Write $(X, T) = (X_B, T_B)$. Define the tripling $B^\mathcal{Q} = (V^\mathcal{Q}, E^\mathcal{Q}, \geq)$ as in Section 2.2. Then the map induced between the inductive limits of the two (horizontal) systems of groups in the following diagram is an isomorphism.

$$\begin{array}{ccccccc} \frac{\mathbb{Z}^{|V_0^\mathcal{Q}|}}{B\mathbb{Z}^{|V_0^\mathcal{Q}|}} & \xrightarrow{A_0^\mathcal{Q}} & \frac{\mathbb{Z}^{|V_1^\mathcal{Q}|}}{B\mathbb{Z}^{|V_1^\mathcal{Q}|}} & \xrightarrow{A_1^\mathcal{Q}} & \frac{\mathbb{Z}^{|V_2^\mathcal{Q}|}}{B\mathbb{Z}^{|V_2^\mathcal{Q}|}} & \xrightarrow{A_2^\mathcal{Q}} & \frac{\mathbb{Z}^{|V_3^\mathcal{Q}|}}{B\mathbb{Z}^{|V_3^\mathcal{Q}|}} \xrightarrow{A_3^\mathcal{Q}} \dots \\ \rho_0 \uparrow & & \rho_1 \uparrow & & \rho_2 \uparrow & & \rho_3 \uparrow \\ \frac{\mathbb{Z}^{|V_0^\mathcal{Q}|}}{\Delta\mathbb{Z}^{|V_0^\mathcal{Q}|}} & \xrightarrow{A_0^\mathcal{Q}} & \frac{\mathbb{Z}^{|V_1^\mathcal{Q}|}}{\Delta\mathbb{Z}^{|V_1^\mathcal{Q}|}} & \xrightarrow{A_1^\mathcal{Q}} & \frac{\mathbb{Z}^{|V_2^\mathcal{Q}|}}{\Delta\mathbb{Z}^{|V_2^\mathcal{Q}|}} & \xrightarrow{A_2^\mathcal{Q}} & \frac{\mathbb{Z}^{|V_3^\mathcal{Q}|}}{\Delta\mathbb{Z}^{|V_3^\mathcal{Q}|}} \xrightarrow{A_3^\mathcal{Q}} \dots \end{array}$$

Furthermore, the two inductive limits are both isomorphic to $K^0(X, T)$.

Proof: By Lemma 3.5, $\Delta\mathbb{Z}^{|V_n^\mathcal{Q}|} \subseteq B\mathbb{Z}^{|V_n^\mathcal{Q}|}$. By Proposition 3.7, for sufficiently large K , $A_{n+K-1}^\mathcal{Q} \circ \dots \circ A_{n+1}^\mathcal{Q} \circ A_n^\mathcal{Q} (B\mathbb{Z}^{|V_n^\mathcal{Q}|}) \subseteq \Delta\mathbb{Z}^{|V_{n+K}|}$. Hence, the induced map between the inductive limits is an isomorphism. That $K^0(X, T)$ is isomorphic to the inductive limit of the top horizontals was already proved in Theorem 3.2. Observe that $\mathbb{Z}^{|V_0^\mathcal{Q}|} = \mathbb{Z}$ and $B\mathbb{Z}^{|V_0^\mathcal{Q}|} = \Delta\mathbb{Z}^{|V_0^\mathcal{Q}|} = 0$. The image of $1 \in \mathbb{Z}^{|V_0^\mathcal{Q}|}$ in the inductive limit maps to the order unit u in $K^0(X, T)$ corresponding to the image of the constant function $1 \in C(X, \mathbb{Z})$. \square

Remark. Since it is known that $K^0(X, T)$ is isomorphic to the K_0 -group of the associated C^* -crossed product $C(X) \rtimes_T \mathbb{Z}$, we see that as a corollary to Theorem 3.8, we can effectively compute $K_0(C(X_B) \rtimes_{T_B} \mathbb{Z})$.

Finally, we should point out how these descriptions simplify further for properly ordered Bratteli diagrams and yield the isomorphism $K^0(X, T) \simeq K_0(V, E)$, (see Definition 3.1), proved by Herman, Putnam and Skau [HPS, Theorem 5.4 and Corollary 6.3].

3.9. Let (V, E, \geq) be a properly ordered Bratteli diagram. Telescoping if necessary, assume that every level $n + 1$ -tower traverses through at least two level n -towers. Telescoping further if necessary, (see [HPS, Proposition 2.8]), we can assume that any two maximal edges of E_n have the same source. Similarly, we can assume that any two minimal edges of E_n have the same source. For the rest of the paper we assume that these properties hold. Then for any \mathcal{Q}_{n+2} -tower $\mathcal{S}(u, v, w)$ the first \mathcal{Q}_n -tower traversed by $\mathcal{S}(u, v, w)$ is independent of $(u, v, w) \in V_{n+2}^\mathcal{Q}$. Thus one sees from the definition of $B\mathbb{Z}^{|V_n^\mathcal{Q}|}$ that $A_{n+1}^\mathcal{Q} \circ A_n^\mathcal{Q} (B\mathbb{Z}^{|V_n^\mathcal{Q}|}) = 0$.

As a consequence, the map induced between the inductive limits of the top two horizontal in the following diagram is an isomorphism.

$$\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{A_0^{\mathcal{Q}}} & \frac{\mathbb{Z}^{|V_1^{\mathcal{Q}}|}}{B\mathbb{Z}^{|V_1^{\mathcal{Q}}|}} & \xrightarrow{A_1^{\mathcal{Q}}} & \frac{\mathbb{Z}^{|V_2^{\mathcal{Q}}|}}{B\mathbb{Z}^{|V_2^{\mathcal{Q}}|}} & \xrightarrow{A_2^{\mathcal{Q}}} & \cdots \frac{\mathbb{Z}^{|V_n^{\mathcal{Q}}|}}{B\mathbb{Z}^{|V_n^{\mathcal{Q}}|}} \longrightarrow \cdots \\
1 \uparrow & & \tau \uparrow & & \tau \uparrow & & \tau \uparrow \\
\mathbb{Z} & \xrightarrow{A_0^{\mathcal{Q}}} & \mathbb{Z}^{|V_1^{\mathcal{Q}}|} & \xrightarrow{A_1^{\mathcal{Q}}} & \mathbb{Z}^{|V_2^{\mathcal{Q}}|} & \xrightarrow{A_2^{\mathcal{Q}}} & \cdots \mathbb{Z}^{|V_n^{\mathcal{Q}}|} \longrightarrow \cdots \\
1 \uparrow & & \pi^* \uparrow & & \pi^* \uparrow & & \pi^* \uparrow \\
\mathbb{Z} & \xrightarrow{A_0} & \mathbb{Z}^{|V_1|} & \xrightarrow{A_1} & \mathbb{Z}^{|V_2|} & \xrightarrow{A_2} & \cdots \mathbb{Z}^{|V_n|} \longrightarrow \cdots
\end{array}$$

The map $\pi^*: \mathbb{Z}^{|V_n|} \rightarrow \mathbb{Z}^{|V_n^{\mathcal{Q}}|}$ is induced by the map $\pi: V_n^{\mathcal{Q}} \rightarrow V_n$ given by $(u, v, w) \mapsto v$ and of course commutes with multiplication by the respective incidence matrices (i.e., $A_n, A_n^{\mathcal{Q}}$). Recall that after doing necessary telescoping we have arranged so that the properly ordered Bratteli diagram (V, E, \geq) has the properties described in the beginning of Section 3.9. Now let e be an edge in (V, E, \geq) with range $v \in V_{n+1}$. Let $(u, v, w), (u', v, w') \in V_{n+1}^{\mathcal{Q}}$. Let \tilde{e}, \tilde{e}' be the (unique) lifts of e to $(V^{\mathcal{Q}}, E^{\mathcal{Q}}, \geq)$ with ranges $(u, v, w), (u', v, w')$ respectively. Then, from the description in Section 2.2, \tilde{e}, \tilde{e}' have the same sources in $V_n^{\mathcal{Q}}$. From this it follows that for $\bar{p} \in \mathbb{Z}^{|V_n^{\mathcal{Q}}|}$, $A_n^{\mathcal{Q}}(\bar{p})(u, v, w) = A_n^{\mathcal{Q}}(\bar{p})(u', v, w')$; in other words, $A_n^{\mathcal{Q}}(\bar{p}) \in \pi^*(\mathbb{Z}^{|V_{n+1}|})$. Thus the map induced between the inductive limits of the two bottom horizontal in the above diagram is also an isomorphism.

3.10. Specialization of Theorem 3.8 to substitutional systems.

We recall the notation from Section 2.5. Let (A, σ) be an aperiodic non-proper substitutional system. Let $B = (V, E, \geq)$ be the stationary ordered Bratteli diagram associated to (A, σ) . Define

$$A^{\mathcal{Q}} = \{(a, b, c) \in A \times A \times A \mid abc \text{ subword of } \sigma^n(d) \text{ for some } d \in A \text{ and some } n\}.$$

Define

$$\sigma^{\mathcal{Q}}: A^{\mathcal{Q}} \longrightarrow (A^{\mathcal{Q}})^+$$

by $\sigma^{\mathcal{Q}}[(a, b, c)] = (a_m, b_1, b_2) \cdot (b_1, b_2, b_3) \cdots (b_{n-2}, b_{n-1}, b_n) \cdot (b_{n-1}, b_n, c_1)$, where $\sigma(b) = b_1 \cdot b_2 \cdots b_n$, and a_m is the last letter in $\sigma(a)$, while c_1 is the first letter in $\sigma(c)$. For a function $\varphi: A \times A \rightarrow \mathbb{Z}$ define $\delta(\varphi) \in \mathbb{Z}^{|A^{\mathcal{Q}}|}$ by $\delta(\varphi)(a, b, c) = \varphi(b, c) - \varphi(a, b)$. Let $\Delta\mathbb{Z}^{|A^{\mathcal{Q}}|}$ be the subgroup $\delta(\mathbb{Z}^{|A \times A|})$ of $\mathbb{Z}^{|A^{\mathcal{Q}}|}$.

Let $\beta^{\mathcal{Q}}: \mathbb{Z}^{|A^{\mathcal{Q}}|} \rightarrow \mathbb{Z}^{|A^{\mathcal{Q}}|}$ be given by matrix multiplication by the incidence matrix of the substitution $\sigma^{\mathcal{Q}}$. For a function $\varphi: A \times A \rightarrow \mathbb{Z}$ define $\varphi': A \times A \rightarrow \mathbb{Z}$ by $\varphi'(u', v') = \varphi(u, v)$ where u (*resp.* v) is the last (*resp.* first) letter in the substitution $\sigma(u')$ (*resp.* $\sigma(v')$). Then $\beta^{\mathcal{Q}}(\delta(\varphi)) = \delta(\varphi')$; thus, $\beta^{\mathcal{Q}}[\delta(\mathbb{Z}^{|A \times A|})] \subseteq \delta(\mathbb{Z}^{|A \times A|})$. Hence, $\beta^{\mathcal{Q}}$ induces a homomorphism of ordered groups

$$\frac{\mathbb{Z}^{|A^{\mathcal{Q}}|}}{\Delta \mathbb{Z}^{|A^{\mathcal{Q}}|}} \longrightarrow \frac{\mathbb{Z}^{|A^{\mathcal{Q}}|}}{\Delta \mathbb{Z}^{|A^{\mathcal{Q}}|}}$$

still denoted by $\beta^{\mathcal{Q}}$.

From Theorem 3.8 one deduces immediately

Theorem 3.11. *The inductive limit of the system of groups*

$$\frac{\mathbb{Z}^{|A^{\mathcal{Q}}|}}{\Delta \mathbb{Z}^{|A^{\mathcal{Q}}|}} \xrightarrow{\beta^{\mathcal{Q}}} \frac{\mathbb{Z}^{|A^{\mathcal{Q}}|}}{\Delta \mathbb{Z}^{|A^{\mathcal{Q}}|}} \xrightarrow{\beta^{\mathcal{Q}}} \frac{\mathbb{Z}^{|A^{\mathcal{Q}}|}}{\Delta \mathbb{Z}^{|A^{\mathcal{Q}}|}} \xrightarrow{\beta^{\mathcal{Q}}} \frac{\mathbb{Z}^{|A^{\mathcal{Q}}|}}{\Delta \mathbb{Z}^{|A^{\mathcal{Q}}|}} \xrightarrow{\beta^{\mathcal{Q}}} \dots$$

is isomorphic to $K^0(X_{\sigma}, T_{\sigma})$ where (X_{σ}, T_{σ}) is the substitutional dynamical system associated to (A, σ) .

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