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PARABOLIC CURVES FOR DIFFEOMORPHISMS IN $\mathbb{C}^2$

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Abstract

We give a simple proof of the existence of parabolic curves for diffeomorphisms in $(\mathbb{C}^2, 0)$ tangent to the identity with isolated fixed point.

1. Introduction

Let F be a diffeomorphism of $(\mathbb{C}^n, 0)$ tangent to the identity. A *parabolic curve* for F is an injective holomorphic map $\varphi\colon \Omega \to \mathbb{C}^n$, where Ω is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Omega$ such that

(1) φ is continuous at the origin, and $\varphi(0) = 0$.

(2) $F(\varphi(\Omega)) \subset \varphi(\Omega)$ and $F^{\circ k}(p)$ converges to 0 when $k \to +\infty$, for $p \in \varphi(\Omega)$.

We say that φ is *tangent* to $[v] \in \mathbb{P}^{n-1}$ if $[\varphi(\zeta)] \to [v]$ when $\zeta \to 0$. Let us write $F(z) = z + P_k(z) + P_{k+1}(z) + \cdots$, where P_j is a n-dimensional vector of homogeneous polynomials of degree j, and $P_k \not\equiv 0$. A *characteristic direction* for F is a point $[v] \in \mathbb{P}^{n-1}$ such that $P_k(v) = \lambda v$, for some $\lambda \in \mathbb{C}$; it is *nondegenerate* if $\lambda \neq 0$. The integer $\operatorname{ord}(F) := k \geq 2$ is the *tangency order* of F at 0.

The following theorem is analogous to Briot and Bouquet's theorem [**3**] for diffeomorphisms of $(\mathbb{C}^n, 0)$.

Theorem 1.1 (Hakim [**6**]). *Let F be a germ of diffeomorphism of* $(\mathbb{C}^n, 0)$ *tangent to the identity. For any nondegenerate characteristic direction $[v]$ there exist* $\operatorname{ord}(F) - 1$ *disjoint parabolic curves tangent to $[v]$ at the origin.*

When $n = 2$, Abate proved that the nondegeneracy condition can be dismissed.

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Theorem 1.2 (Abate [**1**], [**2**]). *Let F be a germ of diffeomorphism of $(\mathbb{C}^2,0)$ tangent to the identity such that 0 is an isolated fixed point. Then there exist $\operatorname{ord}(F)-1$ disjoint parabolic curves for F at the origin.*

This theorem is analogous to Camacho-Sad's theorem [**4**] of existence of invariant curves for holomorphic vector fields. We show in this note that the analogy is deep enough to prove Theorem 1.2 in a simple way starting with Hakim's theorem.

2. Exponential operator and blow-up transformation

Let $\widehat{\mathfrak{X}}_2(\mathbb{C}^2,0)$ be the module of formal vector fields $X=a(x,y)\frac{\partial}{\partial x}+b(x,y)\frac{\partial}{\partial y}$ of order ≥ 2, i.e., $\min\{\nu(a),\nu(b)\}\geq 2$. We denote by $\widehat{\operatorname{Diff}}_1(\mathbb{C}^2,0)$ the group of formal diffeomorphisms tangent to the identity $F(x,y)=(x+p(x,y),y+q(x,y))$ where $\min\{\nu(p(x,y)),\nu(q(x,y)\}\geq 2$. Let us denote by $\mathfrak{X}_2(\mathbb{C}^2,0)$ and by $\operatorname{Diff}_1(\mathbb{C}^2,0)$ the convergent elements of $\widehat{\mathfrak{X}}_2(\mathbb{C}^2,0)$ and $\widehat{\operatorname{Diff}}_1(\mathbb{C}^2,0)$ respectively.

Let $X\in\widehat{\mathfrak{X}}_2(\mathbb{C}^2,0)$. The exponential operator of X is the application $\exp tX\colon \mathbb{C}[[x,y]]\to\mathbb{C}[[x,y,t]]$ defined by the formula

$$\exp tX(g)=\sum_{j=0}^{\infty}\frac{t^j}{j!}X^j(g)$$

where $X^0(g)=g$ and $X^{j+1}(g)=X(X^j(g))$. Note that, since $\nu(X^j(g))\geq j+\nu(g)$, we can substitute $t=1$ to get the element $\exp X(g)\in\mathbb{C}[[x,y]]$. Moreover, $\exp tX$ gives a homomorphism of $\mathbb{C}$-algebras, in particular, we have

$$\exp tX(fg)=\exp tX(f)\exp tX(g).$$

We get also

Proposition 2.1. *The application*

$$\begin{aligned}\operatorname{Exp}\colon \widehat{\mathfrak{X}}_2(\mathbb{C}^2,0)&\to\widehat{\operatorname{Diff}}_1(\mathbb{C}^2,0)\\ X&\mapsto(\exp X(x),\exp X(y))\end{aligned}$$

is a bijection.

Proof: Let

$$G(x,y)=\Big(x+\sum_{n=2}^{\infty}p_n(x,y),y+\sum_{n=2}^{\infty}q_n(x,y)\Big)$$

and

$$X=\sum_{n=2}^{\infty}\Big(a_n(x,y)\frac{\partial}{\partial x}+b_n(x,y)\frac{\partial}{\partial y}\Big).$$

The identity $\operatorname{Exp}(X) = G$ is equivalent to

$$p_{m+1} = a_{m+1} + HT_{m+1}\left(\sum_{j=2}^{m} \frac{1}{j!} X_m^j(x)\right)$$

$$q_{m+1} = b_{m+1} + HT_{m+1}\left(\sum_{j=2}^{m} \frac{1}{j!} X_m^j(y)\right),$$

where $X_m = \sum_{n=2}^{m}\left(a_n(x,y)\frac{\partial}{\partial x} + b_n(x,y)\frac{\partial}{\partial y}\right)$, and $HT_{m+1}(h)$ is the homogeneous term of h of order $m+1$. These equations determine univocally X if G is given. □

Note that $\operatorname{ord}(G) = \nu(X)$. In general, X may not be convergent for a convergent G. The formal vector field X such that $G = \operatorname{Exp}(X)$ is called the *infinitesimal generator* of G. If $k = \nu(X)$, then $a_k = p_k$ and $b_k = q_k$, thus the characteristic directions of F correspond to the points of the tangent cone of X. Moreover, if $X = fX'$ with $X' \in \widehat{\mathfrak{X}}(\mathbb{C}^2, 0)$ and $f \in \mathbb{C}[[x,y]]$ then $\operatorname{Exp}(X)(x,y) = (x + f(x,y)p(x,y), y + f(x,y)q(x,y))$. The converse statement follows by a process similar to the proof of Proposition 2.1. In particular, 0 is an isolated singular point of X if and only if 0 is an isolated fixed point of F.

Now, let $\pi\colon (M, D) \to (\mathbb{C}^2, 0)$ be the blow up of $\mathbb{C}^2$ at the origin, where $D = \pi^{-1}(0) = \mathbb{P}^1$, thus each characteristic direction determines a point of D.

Proposition 2.2. *Let $F \in \operatorname{Diff}_1(\mathbb{C}^2, 0)$. There exists a unique germ of diffeomorphism $\tilde{F}$ in (M, D) such that $\pi \circ \tilde{F} = F \circ \pi$ and $\tilde{F}|_D = \operatorname{id}|_D$. Moreover, the germ $\tilde{F}_p$ has order $\geq \operatorname{ord}(F)$ for any characteristic direction $p \in D$ and hence $\tilde{F}_p \in \operatorname{Diff}_1(M, p)$.*

Proof: Let $F(x,y) = (x + p_k(x,y) + \cdots, y + q_k(x,y) + \cdots)$ where $k = \operatorname{ord}(F) \geq 2$. We have two charts of $M = U_1 \cup U_2$ such that $\pi|_{U_1}\colon U_1 \to \mathbb{C}^2$, is defined by $\pi(x,v) = (x, xv)$ and $\pi|_{U_2}\colon U_2 \to \mathbb{C}^2$, is defined by $\pi(u,y) = (uy, y)$. We define $\tilde{F}$ in the first chart as

$$\begin{aligned}\tilde{F}(x,v) &= \pi^{-1} \circ F \circ \pi(x,v) = \left(x + p_k(x,xv) + \cdots, \frac{vx + q_k(x,xv) + \cdots}{x + p_k(x,xv) + \cdots}\right) \\ &= (x + x^k(p_k(1,v) + x(\cdots)), v + x^{k-1}(q_k(1,v) - vp_k(1,v) + x(\cdots))).\end{aligned}$$

Observe that $\tilde{F}(0,v) = (0,v)$, thus any point of the divisor is fixed. Moreover, if $q_k(1,v_0) - v_0 p_k(1,v_0) = 0$ we have $dF(0,v_0) = I$, and thus for any characteristic direction $p = (0,v_0) \in D$, $\operatorname{ord}(\tilde{F}_p) \geq \operatorname{ord}(F)$. $\square$

Proposition 2.3. *Let $X \in \widehat{\mathfrak{X}}_2(\mathbb{C}^2,0)$. Let $\tilde{X}$ be the formal vector field in (M,D) such that $D\pi \cdot \tilde{X} = X \circ \pi$. If p is a point of the tangent cone of X then $\tilde{X}_p \in \widehat{\mathfrak{X}}_2(M,p)$.*

Proof: Let $X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$ with $a(x,y) = a_k(x,y) + \cdots$, $b(x,y) = b_k(x,y) + \cdots$ and $k \geq 2$. Let U_1 and U_2 be two charts of $M = U_1 \cup U_2$ as in the proposition above. Then $\tilde{X}$ is given in the chart U_1 by

$$\begin{aligned}\tilde{X}(x,v) &= a(x,xv)\frac{\partial}{\partial x} + \frac{b(x,xv) - va(x,xv)}{x}\frac{\partial}{\partial v}\\ &= x^k(a_k(1,v)+x(\cdots))\frac{\partial}{\partial x} + x^{k-1}((b_k(1,v) - va_k(1,v)) + x(\cdots))\frac{\partial}{\partial y}.\end{aligned}$$

Now, if $p = (0,v_0) \in D$ is such that $b_k(1,v_0) - v_0 a_k(1,v_0) = 0$, then $\nu_p(a(x,xv)) \geq k$ and $\nu_p\big(\frac{b(x,xv)-va(x,xv)}{x}\big) \geq k$ so $\tilde{X}_p \in \widehat{\mathfrak{X}}_2(M,p)$. $\square$

We say that the singular point p is *strictly singular* if any time we write $X = fX'$, then p is a singular point of X'. Note that in the above statement any strictly singular point of $\tilde{X}$ is in the tangent cone of X. Let us also recall that Seidenberg's reduction of singularities [**7**] is done by blowing-up at strictly singular points.

Lemma 2.4. *Let $F \in \operatorname{Diff}_1(\mathbb{C}^2,0)$ and $X \in \widehat{\mathfrak{X}}_2(\mathbb{C}^2,0)$ such that $F = \operatorname{Exp}(X)$. Let $\tilde{X}$ be as in the proposition above. Then for any $p \in D$*

$$\tilde{F}_p = \operatorname{Exp}(\tilde{X}_p).$$

Proof: Let $U \simeq \mathbb{C}^2$ be a chart of M such that $\pi|_U \colon U \to \mathbb{C}^2$ is defined by $\pi(x,v) = (x,xv)$ and $p \in U \cap D = \{(0,v) \in U\}$ be a point on the divisor. Without loss of generality, applying a linear change of coordinates, we can suppose that $p = (0,0) \in U$. Since $F(x,y) = \operatorname{Exp}(X) = (\exp X(x), \exp X(y))$, using the definition of $\tilde{F}$, we have

$$\begin{aligned}\tilde{F}(x,v) &= \left(\exp X(x), \frac{\exp X(xv)}{\exp X(x)}\right) = \left(\exp \tilde{X}(x), \frac{\exp \tilde{X}(xv)}{\exp \tilde{X}(x)}\right)\\ &= \left(\exp \tilde{X}(x), \frac{\exp \tilde{X}(x)\exp \tilde{X}(v)}{\exp \tilde{X}(x)}\right) = (\exp \tilde{X}(x), \exp \tilde{X}(v))\\ &= \operatorname{Exp}(\tilde{X}_p)(x,v).\end{aligned}$$

$\square$

3. Existence of parabolic curves

We need the following formal version of Camacho-Sad's theorem [**4**] whose proof goes exactly as the original one (see also [**5**]).

Theorem 3.1 (Camacho and Sad)**.** *Take* $X \in \widehat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ *with an isolated singularity at the origin. There is a desingularization morphism* $\sigma\colon (\tilde{M}, \tilde{D}) \to (\mathbb{C}^2, 0)$ *composition of a finite sequence of blow-ups with centers at strictly singular points and a point* $p \in \tilde{D}$ *satisfying the following property: There are local coordinates* (u, v) *at* p *such that* $\tilde{D}_p = (u = 0)$ *and the transform* X^* *of* X *at* p *is of the form:*

$$X^*(u,v) = u^m \left((\lambda u + u^2(\cdots))\frac{\partial}{\partial u} + (\mu v + u(\cdots))\frac{\partial}{\partial v} \right)$$

where $\lambda \neq 0$, $\frac{\mu}{\lambda} \notin \mathbb{Q}_{>0}$ *and* $m \geq \nu(X) - 1$.

Remark 3.2. The statement above is also valid when X has a dicritical desingularization; we just need to consider one of the infinitely many nondegenerate characteristic directions on a dicritical divisor.

Let us prove Theorem 1.2. Take X the infinitesimal generator of F, and consider X^* and p as in Camacho-Sad's Theorem. By Lemma 2.4 we have

$$F_p^*(u,v) = \mathrm{Exp}(X_p^*) = (u + \lambda u^{m+1} + O(u^{m+2}), v + \mu u^m v + O(u^{m+1}))$$

so F_p^* is a diffeomorphism tangent to the identity, with $[1, 0]$ as a nondegenerate characteristic direction. By Hakim's Theorem, there exist $\mathrm{ord}(F_p^*) - 1$ disjoint parabolic curves $\varphi_j\colon \Omega_j \to \tilde{M}$ for F_p^* tangent to the direction $[1, 0]$ at p. Since this direction is transversal to the divisor, it follows that $\overline{\varphi_j(\Omega_j)} \cap \tilde{D} = \{p\}$ and thereby $\sigma \circ \varphi_j$ is also a parabolic curve for F. This ends the proof.

Remark 3.3. In the case $X = x^k X'$ and $S = (x = 0)$ invariant by X', Camacho-Sad's index of X at 0 along S is exactly Abate's residual index of F at 0 along S. Furthermore, according to J. Cano's proof [**5**] of Camacho-Sad's theorem, to find the points $p \in \tilde{D}$ that satisfy Camacho-Sad's theorem, it is enough to follow after the first blow up, the singularities with Camacho-Sad's index not in $\mathbb{Q}_{\geq 0}$. Thus, there exist parabolic curves for any characteristic direction of F that gives at the divisor Abate's residual index not in $\mathbb{Q}_{\geq 0}$ (see Corollary 3.1 in [**1**]).

References

[1] M. Abate, The residual index and the dynamics of holomorphic maps tangent to the identity, *Duke Math. J.* **107(1)** (2001), 173–207.
[2] M. Abate, F. Bracci, and F. Tovena, Index theorems for holomorphic self-maps, *Ann. of Math. (2)* **159(2)** (2004), 819–864.
[3] C. A. Briot and J. C. Bouquet, Recherches sur les propriétés des fonctions définies par des équations différentielles, J. Ecole Polytechnique **XXI** (1856), 133–198.
[4] C. Camacho and P. Sad, Invariant varieties through singularities of holomorphic vector fields, *Ann. of Math. (2)* **115(3)** (1982), 579–595.
[5] J. Cano, Construction of invariant curves for singular holomorphic vector fields, *Proc. Amer. Math. Soc.* **125(9)** (1997), 2649–2650.
[6] M. Hakim, Analytic transformations of $(\mathbb{C}^p, 0)$ tangent to the identity, *Duke Math. J.* **92(2)** (1998), 403–428.
[7] A. Seidenberg, Reduction of singularities of the differential equation $A\,dy = B\,dx$, *Amer. J. Math.* **90** (1968), 248–269.

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