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BILIPSCHITZ MAPPINGS WITH DERIVATIVES OF BOUNDED VARIATION

Stanislav Hencl

Abstract

Let $\Omega \subset \mathbb{R}^n$ be open and suppose that $f\colon \Omega \to \mathbb{R}^n$ is a bilipschitz mapping such that $Df \in BV_{\text{loc}}(\Omega, \mathbb{R}^{n^2})$. We show that under these assumptions the inverse satisfies $Df^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbb{R}^{n^2})$.

1. Introduction

Suppose that $\Omega \subset \mathbb{R}^n$ is an open set and let $f\colon \Omega \to f(\Omega) \subset \mathbb{R}^n$ be a homeomorphism. In this paper we address the issue of the regularity of f^{-1} under regularity assumptions on f. The starting point for us is the following very recent result from [**4**] (see Preliminaries for the definition of the space BV).

Theorem 1.1. *Let $\Omega, \Omega' \subset \mathbb{R}^2$ be open and suppose that $f\colon \Omega \to \Omega'$ is a homeomorphism. Then $f \in BV_{\text{loc}}(\Omega; \mathbb{R}^2)$ if and only if $f^{-1} \in BV_{\text{loc}}(\Omega'; \mathbb{R}^2)$. Moreover, both f and f^{-1} are differentiable almost everywhere.*

In the same paper we also studied the conditions that guarantee that $f^{-1} \in BV_{\text{loc}}(\Omega, \mathbb{R}^n)$ in higher dimensions. With some additional assumptions (namely that f is a mapping of finite distortion) it is moreover possible to prove that $f^{-1} \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ (see [**2**], [**3**] and [**5**]).

In this paper we want to address the issue of regularity of the second derivative of f^{-1}. The classical inverse function theorem states that if f is C^2 and $J_f(x_0) \neq 0$ then there is a small neighborhood of x_0 where f is homeomorphism and f^{-1} is C^2. We will assume that f is a bilipschitz mapping and show that $Df^{-1} \in BV_{\text{loc}}$ provided that $Df \in BV_{\text{loc}}$. This resembles the result from [**6**] that the inverse of bilipschitz Delta-convex mapping is again Delta-convex.

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Theorem 1.2. *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open and suppose that $f\colon \Omega \to \Omega'$ is a bilipschitz mapping such that $Df \in BV_{\mathrm{loc}}(\Omega; \mathbb{R}^{n^2})$. Then $Df^{-1} \in BV_{\mathrm{loc}}(\Omega'; \mathbb{R}^{n^2})$.*

It is moreover possible to show that Df^{-1} belongs to the Sobolev space $W^{1,p}_{\mathrm{loc}}$ if $Df \in W^{1,p}_{\mathrm{loc}}$.

Theorem 1.3. *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open, $p \geq 1$ and suppose that $f\colon \Omega \to \Omega'$ is a bilipschitz mapping such that $Df \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^{n^2})$. Then $Df^{-1} \in W^{1,p}_{\mathrm{loc}}(\Omega'; \mathbb{R}^{n^2})$.*

Let us make a comment on our assumptions. Let $\alpha > 1$ and consider the function $f\colon (-1,1) \to \mathbb{R}$ defined as $f(x) = |x|^\alpha \operatorname{sgn} x$. Then it is easy to check that f is Lipschitz, homeomorphism, $Df \in W^{1,1}((-1,1))$, but $Df^{-1} \notin BV_{\mathrm{loc}}((-1,1))$. Thus the assumption that f^{-1} is Lipschitz cannot be omitted.

In Section 4 we give an example which shows that Theorem 1.2 is not valid in dimension $n \geq 4$ without the assumption that f is Lipschitz. If $n = 1$, then $Df \in BV$ implies that Df is bounded and that f is Lipschitz, and thus this assumption is redundant. We would like to know if a homeomorphism $f\colon \Omega \to \Omega'$ such that f^{-1} is Lipschitz and $Df \in BV_{\mathrm{loc}}(\Omega; \mathbb{R}^{n^2})$ must satisfy $Df^{-1} \in BV_{\mathrm{loc}}(\Omega'; \mathbb{R}^{n^2})$ in dimensions $n = 2$ and $n = 3$. Unfortunately our method of the proof and our counterexample do not provide an answer to this question.

2. Preliminaries

By $\mathbf{e}_1, \ldots, \mathbf{e}_n$ we denote the canonical basis in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ we write $x_1, \ldots, x_n$ for its coordinates, i.e. $x = \sum_{i=1}^n x_i \mathbf{e}_i$. The euclidean distance of $x, y \in \mathbb{R}^n$ is denoted by $|x-y|$ and the norm of the n times n matrix A is denoted by $\|A\|$.

In the whole paper Ω will denote an open subset of $\mathbb{R}^n$. We say that $F\colon \Omega \to \mathbb{R}^n$ is a Lipschitz map if there is a constant $K > 0$ such that

$$|F(x) - F(y)| \leq K|x - y|$$

for every $x, y \in \Omega$. Further F is said to be bilipschitz if it is an invertible mapping and both $F\colon \Omega \to \mathbb{R}^n$ and $F^{-1}\colon F(\Omega) \to \mathbb{R}^n$ are Lipschitz.

The Lebesgue measure of a set $A \subset \mathbb{R}^n$ is denoted by $\mathcal{L}_n(A)$. A mapping $f\colon \Omega \to \mathbb{R}^n$ is said to satisfy the Lusin condition (N) if $\mathcal{L}_n(f(A)) = 0$ for every $A \subset \Omega$ such that $\mathcal{L}_n(A) = 0$.

Let $\Omega \subset \mathbb{R}^n$ be open and $m \in \mathbb{N}$. A function $h \in L^1(\Omega)$ is of bounded variation, $h \in BV(\Omega)$, if the distributional partial derivatives of h are measures with finite total variation in Ω: there are Radon (signed) measures $\mu_1, \dots, \mu_n$ defined in Ω so that for $i = 1, \dots, n$, $|\mu_i|(\Omega) < \infty$ and

$$\int_\Omega h D_i \varphi \, dx = - \int_\Omega \varphi \, d\mu_i$$

for all $\varphi \in C_0^\infty(\Omega)$. We say that $f \in L^1(\Omega, \mathbb{R}^m)$ belongs to $BV(\Omega, \mathbb{R}^m)$ if the coordinate functions of f belong to $BV(\Omega)$. Analogously we define the Sobolev space: $f \in W^{1,1}(\Omega, \mathbb{R}^m)$ if $f \in L^1(\Omega, \mathbb{R}^m)$ and the distributional derivatives of the coordinate functions are in $L^1(\Omega, \mathbb{R}^n)$. Further, $f \in BV_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (or $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$) requires that $f \in BV(\Omega', \mathbb{R}^m)$ (or $f \in W^{1,1}(\Omega', \mathbb{R}^m)$) for each open $\Omega' \Subset \Omega$. For an introduction to the theory of BV and $W^{1,1}$ spaces see [**1**], [**7**]. The function $h \colon \Omega \to \mathbb{R}^m$ is said to be a representative of $g \colon \Omega \to \mathbb{R}^m$ if $h = g$ almost everywhere with respect to Lebesgue measure.

For function $f \colon (a,b) \to \mathbb{R}^m$ we define

$$V\big(f,(a,b)\big) := \sup \left\{ \sum_{i=1}^{k} |f(a_i) - f(b_i)| : (a_i, b_i) \text{ are pairwise disjoint intervals in } (a,b) \right\}.$$

The function f is said to have finite variation if $V\big(f,(a,b)\big) < \infty$.

It is a well-known fact (see e.g. [**1**, Section 3.11]) that a mapping $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is in $BV_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (or in $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$) if and only if there is a representative which has bounded variation (or is an absolutely continuous function) on almost all lines parallel to coordinate axes and the variation on these lines is integrable. More precisely, let $i \in \{1, 2, \dots, n\}$ and denote by π_i the projection on the hyperplane perpendicular to the x_i-axis. Suppose that $Q(c,r) := (c_1 - r, c_1 + r) \times \dots \times (c_n - r, c_n + r) \subset \Omega$ for some $c \in \mathbb{R}^n$, $r > 0$ and set $Q^i(c,r) = \pi_i(Q(c,r))$. Let $y \in Q^i(c,r)$ and denote

$$u_{i,y}(t) = u(y + t\mathbf{e}_i) \text{ for } t \in (c_i - r, c_i + r).$$

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$.*

(i) *Then $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ if and only if the following happens. For every cube $Q(c,r) \Subset \Omega$ and for every $i \in \{1, \dots, n\}$ there is a representative $\tilde{u}$ of u such that the function $\tilde{u}_{i,y}(t)$ is absolutely continuous on $(c_i - r, c_i + r)$ (i.e. each coordinate function is absolutely* *continuous) for $\mathcal{L}_{n-1}$ almost every $y \in Q^i(c,r)$ and moreover*

$$\int_{Q^i(c,r)} \int_{c_i-r}^{c_i+r} |\nabla \tilde{u}_{i,y}(t)|\, dt\, dy < \infty. \tag{2.1}$$

(ii) *Then $u \in BV_{\rm loc}(\Omega, \mathbb{R}^m)$ if and only if the following happens. For every cube $Q(c,r) \Subset \Omega$ and for every $i \in \{1, \dots, n\}$ there is a representative $\tilde{u}$ of u such that the function $\tilde{u}_{i,y}(t)$ has bounded variation on $(c_i - r, c_i + r)$ for $\mathcal{L}_{n-1}$ almost every $y \in Q^i(c,r)$ and moreover*

$$\int_{Q^i(c,r)} V\big(\tilde{u}_{i,y}, (c_i - r, c_i + r)\big)\, dy < \infty. \tag{2.2}$$

We shall also need that the composition of BV function and a homeomorphism with Lipschitz inverse is in BV (see [**1**, Theorem 3.16 and Corollary 3.19]).

Theorem 2.2. *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open and let $u\colon \Omega \to \mathbb{R}^m$. Suppose that $F\colon \Omega \to \Omega'$ is Lipschitz and homeomorphism.*

(i) *If $u \in BV_{\rm loc}(\Omega, \mathbb{R}^m)$, then $u \circ F^{-1} \in BV(\Omega', \mathbb{R}^m)$.*

(ii) *If $u \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^m)$ and F^{-1} is Lipschitz, then $u \circ F^{-1} \in W^{1,1}_{\rm loc}(\Omega', \mathbb{R}^m)$ and*

$$Du \circ F^{-1}(y) = Du(F^{-1}(y))DF^{-1}(y) \text{ for almost every } y \in \Omega'.$$

3. Regularity of the inverse

Proof of Theorem 1.2: We want to show that Df^{-1} has bounded variation on almost all lines parallel to coordinate axes and therefore $Df^{-1} \in BV_{\rm loc}$ (see Theorem 2.1 (ii)). Fix $Q(c,r) \Subset f(\Omega)$ and $i \in \{1, \dots, n\}$.

From Theorem 2.2 we know that $Df \circ f^{-1} \in BV_{\rm loc}$. Denote by h a good representative of $Df \circ f^{-1}$ from Theorem 2.1 (ii) and set $h_{i,y}(t) := h(y + t\mathbf{e}_i)$ for $y \in Q^i(c,r)$. From (2.2) we have

$$\int_{Q^i(c,r)} V\big(h_{i,y}, (c_i - r, c_i + r)\big)\, dy < \infty. \tag{3.1}$$

Denote

$$A = \big\{x \in Q(c,r) : h(x) = Df \circ f^{-1}(x),\ f^{-1} \text{ is differentiable at } x \text{ and } f \text{ is differentiable at } f^{-1}(x)\big\}.$$

Lipschitz functions are differentiable almost everywhere and map Lebesgue null sets to Lebesgue null sets and therefore $\mathcal{L}_n(A) = \mathcal{L}_n(Q(c,r))$. From the definition of A we have

$$Df^{-1}(x)Df(f^{-1}(x)) = I \text{ for every } x \in A. \tag{3.2}$$

Fix $y \in Q^i(c,r)$ and let $\{(a_j,b_j)\}_{j=1}^k$ be a system of pairwise disjoint subintervals of (c_i-r,c_i+r) such that $A_j := y+a_j\mathbf{e}_i \in A$ and $B_j := y+b_j\mathbf{e}_i \in A$ for every j. Plainly $\|Df^{-1}(x)\| \le K$ where K denotes the Lipschitz constant of f^{-1}. Together with (3.2) this imply

$$
\begin{aligned}
&\sum_{j=1}^k \|Df^{-1}(A_j)-Df^{-1}(B_j)\| \\
&\quad= \sum_{j=1}^k \|Df^{-1}(A_j)\big(Df(f^{-1}(B_j))-Df(f^{-1}(A_j))\big)Df^{-1}(B_j)\| \\
&\quad\le K^2\sum_{j=1}^k \|Df(f^{-1}(B_j))-Df(f^{-1}(A_j))\| \\
&\quad\le CV\big(h_{i,y},(c_i-r,c_i+r)\big).
\end{aligned}
\tag{3.3}
$$

From (3.1) and $\mathcal{L}_n(A)=\mathcal{L}_n(Q(c,r))$ we know that $V\big(h_{i,y},(c_i-r,c_i+r)\big)<\infty$ and $\mathcal{L}_1\big(\pi_i^{-1}(y)\cap A\big)=2r$ for $\mathcal{L}_{n-1}$ almost every y. Fix such a $y\in Q^i(c,r)$. From (3.3) and elementary properties of functions of bounded variation we obtain that there is a function $\tilde u_{i,y}\colon(c_i-r,c_i+r)\to\mathbb{R}^{n^2}$ such that $Df^{-1}(y+t\mathbf{e}_i)=\tilde u_{i,y}(t)$ for every $t\in(c_i-r,c_i+r)\cap A$ and

$$V\big(\tilde u_{i,y},(c_i-r,c_i+r)\big)\le CV\big(h_{i,y},(c_i-r,c_i+r)\big). \tag{3.4}$$

It follows that there is a function $\tilde u$ such that $\tilde u(x)=Df^{-1}(x)$ almost everywhere and this new representative has bounded variation on $Q(c,r)\cap\pi_i^{-1}(y)$ for $\mathcal{L}_{n-1}$ almost every $y\in Q^i(c,r)$. Now (3.4) and (3.1) yields

$$\int_{Q^i(c,r)} V\big(\tilde u_{i,y},(c_i-r,c_i+r)\big)\,dy<\infty \tag{3.5}$$

which verifies (2.2) for Df^{-1}. □

Proof of Theorem 1.3: First let us prove the theorem in the case $p=1$. The proof of this case is analogous to the previous proof and therefore we only sketch it and point out the differences. From Theorem 2.2 (ii) we know that $Df\circ f^{-1}\in W^{1,1}_{\mathrm{loc}}$. Fix $y\in Q^i(c,r)$ such that $h_{i,y}$ is absolutely continuous on (c_i-r,c_i+r). Given $\varepsilon>0$ find $\delta>0$ from the absolute continuity of $h_{i,y}$. Choose A_j and B_j as before and moreover assume that $\sum_{j=1}^{k} |A_j - B_j| < \delta$. Analogously to (3.3) we obtain

$$\sum_{j=1}^{k} \|Df^{-1}(A_j) - Df^{-1}(B_j)\| < C\varepsilon.$$

Reasoning analogously to the previous proof we conclude that Df^{-1} has a representative which is absolutely continuous on almost all lines parallel to coordinate axes. On those lines we have

$$V\big(\tilde{u}_{i,y}, (c_i - r, c_i + r)\big) = \int_{c_i - r}^{c_i + r} |\nabla \tilde{u}_{i,y}(t)|\, dt.$$

From Theorem 1.2 and Theorem 2.1 (ii) we already know (2.2) and thus we obtain (2.1).

Now let us return to the case $p > 1$. We already know that $Df^{-1} \in W^{1,1}_{\mathrm{loc}}$. Therefore we can use Theorem 2.2 (ii) and differentiate twice the identity $f \circ f^{-1}(y) = y$ to obtain

$$D^2 f(f^{-1}(y)) Df^{-1}(y) Df^{-1}(y) + Df(f^{-1}(y)) D^2 f^{-1}(y) = 0. \tag{3.6}$$

Here and in the sequel we identify the second derivative with an linear operator from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$. Clearly

$$\|\big(Df(f^{-1}(y))\big)^{-1}\| \leq C,\ \|Df^{-1}(y)\| \leq C \text{ and } |J_{f^{-1}}(y)| \geq C$$

at almost every point since f is bilipschitz. From (3.6) and substitution formula we now obtain

$$\begin{aligned}\int_A \|D^2 f^{-1}(y)\|^p\, dy &\leq C \int_A \|D^2 f(f^{-1}(y))\|^p |J_{f^{-1}}(y)|\, dy \\ &= C \int_{f^{-1}(A)} \|D^2 f(x)\|^p\, dx\end{aligned}$$

for every open set $A \Subset f(\Omega)$ and the claim follows. □

4. Necessity of the Lipschitz condition for f for $n \geq 4$

Example 4.1. Let $n \geq 4$. There is a homeomorphism $f\colon (-1,1)^n \to \mathbb{R}^n$ such that $Df \in W^{1,1}((-1,1)^n, \mathbb{R}^{n^2})$ and f^{-1} is Lipschitz, but $Df^{-1} \notin BV_{\mathrm{loc}}\big(f((-1,1)^n), \mathbb{R}^{n^2}\big)$.

Proof: Given $x \in \mathbb{R}^n$ we denote $\tilde{x} = [x_1, \dots, x_{n-1}] \in \mathbb{R}^{n-1}$ and $\|\tilde{x}\| = \sqrt{x_1^2 + \dots + x_{n-1}^2}$.

Let $\alpha = \frac{1}{2n}$, $\beta = \frac{3}{4}$ and set

$$f(x) = \sum_{i=1}^{n-1} \mathbf{e}_i x_i \|\tilde{x}\|^{\alpha-1} + \mathbf{e}_n \big(x_n + \|\tilde{x}\| \sin(\|\tilde{x}\|^{-\beta})\big)$$

if $\|\tilde{x}\| > 0$ and $f(x) = \mathbf{e}_n x_n$ if $\|\tilde{x}\| = 0$. Our mapping f is clearly continuous and it is easy to check that f is a one-to-one map since

$$x_i \|\tilde{x}\|^{\alpha-1} = z_i \|\tilde{z}\|^{\alpha-1} \text{ for every } i \in \{1, \dots, n-1\} \Rightarrow \\ \Rightarrow x_i = z_i \text{ for every } i \in \{1, \dots, n-1\}.$$

Therefore f is a homeomorphism.

A direct computation shows that the second partial derivatives of f_i, $i \in \{1, \dots, n-1\}$, are smaller than $C\|\tilde{x}\|^{\alpha-2}$ and therefore integrable. Moreover,

$$\frac{\partial f_n(x)}{\partial x_1} = x_1 \|\tilde{x}\|^{-1} \sin(\|\tilde{x}\|^{-\beta}) - \|\tilde{x}\| \beta \frac{x_1}{\|\tilde{x}\|^{\beta+2}} \cos(\|\tilde{x}\|^{-\beta}).$$

It is not difficult to compute that we can bound each partial derivative of this expression by $C\|\tilde{x}\|^{1-2(\beta+1)}$. Clearly $2(\beta+1)-1 < n-1$ and therefore these second partial derivatives are integrable. Analogously we can estimate other second partial derivatives of f_n. Since the second derivatives of f are smooth outside the segment $\{[0, \dots, 0, t] : t \in (-1, 1)\}$ and $|D^2 f| \in L^1((-1,1)^n)$ it is easy to see that $Df \in W^{1,1}((-1,1)^n, \mathbb{R}^{n^2})$.

The inverse of f is given by

$$f^{-1}(y) = \sum_{i=1}^{n-1} \mathbf{e}_i y_i \|\tilde{y}\|^{\frac{1}{\alpha}-1} + \mathbf{e}_n \big(y_n - \|\tilde{y}\|^{\frac{1}{\alpha}} \sin(\|\tilde{y}\|^{-\frac{\beta}{\alpha}})\big)$$

if $\|\tilde{y}\| > 0$ and $f^{-1}(y) = \mathbf{e}_n y_n$ if $\|\tilde{y}\| = 0$. The derivative of the function $\phi(t) = t^{\frac{1}{\alpha}} \sin(t^{-\frac{\beta}{\alpha}})$ is bounded on $(-1,1)$ and therefore ϕ is Lipschitz. Thus it is not difficult to see that f^{-1} is Lipschitz.

The second derivative of f^{-1} is clearly continuous outside the segment $\{[0, \dots, 0, t] : t \in \mathbb{R}\}$. Elementary computation gives us

$$\frac{\partial (f^{-1})_n(y)}{\partial y_1} = -\frac{y_1}{\alpha} \|\tilde{y}\|^{\frac{1}{\alpha}-2} \sin(\|\tilde{y}\|^{-\frac{\beta}{\alpha}}) + \|\tilde{y}\|^{\frac{1}{\alpha}} \frac{\beta}{\alpha} \frac{y_1}{\|\tilde{y}\|^{\frac{\beta}{\alpha}+2}} \cos(\|\tilde{y}\|^{-\frac{\beta}{\alpha}})$$

and therefore the second derivative $\frac{\partial^2 (f^{-1})_n(y)}{\partial y_1^2}$ contains some integrable terms and

$$-\|\tilde{y}\|^{\frac{1}{\alpha}} \frac{\beta^2}{\alpha^2} \frac{y_1^2}{\|\tilde{y}\|^{2\frac{\beta}{\alpha}+4}} \sin(\|\tilde{y}\|^{-\frac{\beta}{\alpha}}). \tag{4.1}$$

Since

$$2\left(\frac{\beta}{\alpha}+1\right)-\frac{1}{\alpha}>n-1$$

we obtain that the integral of the absolute value of (4.1) over the set

$$S=\left\{y\in f((-1,1)^n): y_1>\tfrac{1}{2}||\tilde{y}||,\ \sin(\|\tilde{y}\|^{-\frac{\beta}{\alpha}})>\tfrac{1}{2}\right\}$$

is infinite. Hence $|D^2 f^{-1}|\notin L^1_{\text{loc}}$ and it is not difficult to deduce that $Df^{-1}\notin BV_{\text{loc}}$. □

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Department of Mathematical Analysis
Charles University
Sokolovská 83
186 00 Prague 8
Czech Republic
E-mail address: hencl@karlin.mff.cuni.cz

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