PSEUDO-ANALYTIC VECTORS ON PSEUDO-KÄHLERIAN MANIFOLDS

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1. Introduction. A pseudo-Kählerian manifold is by definition a Riemannian manifold M^{2n} of class C^r $(r \geq 2)$ which has a skew-symmetric tensor field I_{AB} [1] of class C^{r-1} with non-vanishing determinant satisfying following two conditions:

$$(1) \qquad I^A{}_B I^B{}_C = -\delta^A_C, \qquad (I^{AB} I_{BC} = -\delta^A_C)$$

$$(2) \qquad I_{AB,C} = 0,$$

where

$$(3) \qquad I^A{}_B = g^{AE} I_{EB}, \qquad I^{AB} = g^{AE} g^{BF} I_{EF},$$

and a comma denotes the covariant differentiation with respect to g_{AB}. It is known that the real representation of a Kählerian manifold of complex dimension n is a pseudo-Kählerian manifold of dimension $2n$ and of class C^ω and the converse is also true. However, the problem whether a pseudo-Kählerian manifold M^{2n} of class C^r $(r \neq \omega)$ can be regarded, by introducing suitable complex coordinate systems on M^{2n}, as a real representation of a (complex) Kählerian manifold or not is, as far as we know, still an open problem. In this paper we shall generalize some theorems which concern analytic vectors on Kählerian manifolds to pseudo-Kählerian manifolds.

2. Definitions of pseudo-analyticity.

DEFINITION 1. A set of *functions* (ϕ, ψ) defined over a pseudo-Kählerian manifold M^{2n} is said to be *pseudo-analytic* if

$$(4) \qquad I^A{}_B \phi_{,A} = \psi_{,B}.$$

If (ϕ, ψ) is pseudo-analytic, then $(-\psi, \phi)$ is pseudo-analytic too.

DEFINITION 2. A *contravariant vector field* u^A defined over M^{2n} is said to be *pseudo-analytic* if

$$(5) \qquad I^A{}_B u^B{}_{,C} = u^A{}_{,B} I^B{}_C.$$

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[1] We assume that the indices run as follows:

$$\alpha, \beta, \gamma, \cdots = 1, 2, \cdots, n,$$
$$A, B, C, \cdots = 1, 2, \cdots, n, n+1, \cdots, 2n.$$

DEFINITION 3. A *covariant vector field* u_A defined over M^{2n} is said to be *pseudo-analytic* if

$$(6) \qquad I^A{}_B u_{A,C} = I^A{}_C u_{B,A} .$$

If we denote a complex coordinate system of a Kählerian manifold K^n by z^α $(=x^\alpha + iy^\alpha)$ and take (x^α, y^α) as coordinates of the real representation of K^n, then (cf. [3])

$$(7) \qquad I^\alpha{}_\beta = I^{n+\alpha}{}_{n+\beta} = 0 , \qquad I^\alpha{}_{n+\beta} = -I^{n+\alpha}{}_\beta = \delta^\alpha_\beta .$$

In this case, (4), (5) and (6) are nothing but Cauchy-Riemann equations for a complex analytic function $\phi + i\psi$, for a self-adjoint complex analytic contravariant vector $u^\alpha + iu^{n+\alpha}$ and for a self-adjoint complex analytic covariant vector $u_\alpha + iu_{n+\alpha}$. (We must take account of the fact that the real representation of a contravariant vector $u^\alpha + iu^{n+\alpha}$ is $(u^\alpha, u^{n+\alpha})$ and that of a covariant vector $u_\alpha + iu_{n+\alpha}$ is $(2u_\alpha, -2u_{n+\alpha})$). Hence, the Definitions 1, 2 and 3 are appropriate.

When $I^A{}_B$ takes the value (7), (5) means that $u^A{}_{,B}$ is a matrix which is the real representation of a unitary $(n \times n)$ matrix. Hence we may say that $u^A{}_{,B}$ is pseudo-unitary.

THEOREM 1. *If a set of functions (ϕ, ψ) is pseudo-analytic, then ϕ and ψ are both harmonic functions on our pseudo-Kählerian manifold.*

Proof. By hypothesis

$$\psi_{,B} = I^A{}_B \phi_{,A} ,$$

hence we get

$$\Delta\psi = \psi_{,BC} g^{BC} = I^{AC} \phi_{,AC} = 0 .$$

As (4) can be written also in the form

$$\phi_{,B} = -I^A{}_B \psi_{,A} ,$$

we get in the same way $\Delta\phi = 0$.

THEOREM 2. *If a contravariant vector field u^A and its associated covariant vector u_A are both pseudo-analytic, then u^A is a parallel vector field.*

Proof. If we use covariant components of u^A, then (5) can be written as

$$-I^A{}_B u_{A,C} = I^A{}_C u_{B,A} .$$

Comparing the last equation with (6), we can immediately see that our assertion is true.

THEOREM 3. *If u^A is a pseudo-analytic contravariant vector field, then $I^A{}_B u^B$ is also pseudo-analytic.*

Proof.
$$(I^A{}_B u^B)_{,C} I^C{}_D = I^A{}_B u^B{}_{,C} I^C{}_D$$
$$= I^A{}_B I^B{}_C u^C{}_{,D} = I^A{}_B (I^B{}_C u^C)_{,D} .$$

We shall remark that equation (6) can be written also as

$$XI^A{}_C = 0 \tag{8}$$

where $X = u^A(\partial/\partial x^A)$ and $XI^A{}_C$ is the Lie derivative of $I^A{}_C$ (cf. [5]). We put further $Y = v^A(\partial/\partial x^A)$ and

$$[uv]^A = u^B(v^A{}_{,B}) - v^B(u^A{}_{,B})$$

and similarly define $[Iu, v]^A$, $[u, Iv]^A$, $[Iu, Iv]^A$. Then we get the following

THEOREM 4. *Let u^A and v^A be two pseudo-analytic contravariant vector fields, then $[uv]^A$, $[Iu, v]^A$, $[u, Iv]^A$ and $[Iu, Iv]^A$ are pseudo-analytic too.*

Proof. It is sufficient to prove the pseudo-analyticity of $[uv]^A$. As

$$(XY - YX)f = [uv]^A \frac{\partial f}{\partial x^A} ,$$

it is sufficient to show that

$$(XY - YX)I^A{}_B = 0.$$

However, this follows immediately from the assumption that u^A and v^A are pseudo-analytic.

3. Curvature tensors.

THEOREM 5. (cf. [3])

$$\text{(i)} \quad R^A{}_{BCD} I^E{}_B = I^A{}_E R^E{}_{BCD} , \tag{9}$$
$$\text{(ii)} \quad R_{ABCD} = I^E{}_A I^F{}_B R_{EFCD} .$$

Proof. From (1) we get

$$0 = I^A{}_{B,CD} - I^A{}_{B,DC} = R^A{}_{ECD} I^E{}_B - R^E{}_{BCD} I^A{}_E .$$

Equation (9ii) follows immediately from (9i). The curvature tensor $R^A{}_{BCD}$ is pseudo-unitary with respect to the first two indices.

THEOREM 6.

$$(10)\quad \begin{array}{ll} \text{(i)} & R_{BC}=I^E{}_B I^F{}_C R_{EF}\,, \\ \text{(ii)} & R^A{}_B=-I^A{}_E I^F{}_B R^E{}_F\,, \\ \text{(iii)} & I^A{}_B R^B{}_C=R^A{}_B I^B{}_C\,, \\ \text{(iv)} & I^E{}_A R_{EB}=-I^E{}_B R_{EA}\,. \end{array}$$

$R^A{}_B$ is pseudo-unitary too.

THEOREM 7.

$$(11)\qquad R_{ABCD}I^{AB}=2I^E{}_C R_{ED}\ \ (=-2I^E{}_D R_{EC})\,.$$

Let us prove Theorems 6 and 7 at the same time. First

$$R_{ABCD}I^{AB}=-I^A{}_E R^E{}_{ACD}=I^A{}_E(R^E{}_{CDA}+R^E{}_{DAC})=I^E{}_C R^A{}_{EDA}+I^E{}_D R^A{}_{EA}$$

Hence we get

$$(12)\qquad R_{ABCD}I^{AB}=I^E{}_C R_{ED}-I^E{}_D R_{EC}\,.$$

Now, from (9) we see that

$$R_{BC}=I^E{}_A I^F{}_B R_{EFCD}g^{AD}=-I^{ED}I^F{}_B(R_{FCED}+R_{CEFD})\,.$$

By virtue of (12), the first term of the right hand side of equation becomes $R_{BC}-R_{EF}I^E{}_B I^F{}_C$ and the second term can be ea to be $-R_{BC}$. Hence we get

$$R_{BC}=R_{EF}I^E{}_B I^F{}_C\,.$$

(10 ii, iii, iv) can be immediately seen to be equivalent to (10 i). use (10 iii), then (12) reduces to (11).

4. Pseudo-analytic vector fields.

THEOREM 8. *Let u_A be a pseudo-analytic covariant vector fi pseudo-Kählerian manifold M^{2n}, then it satisfies the relation*

$$(13)\qquad u_{A,BC}g^{BC}-R^B{}_A u_B=0\,.$$

Especially, if M^{2n} is compact, then u_A is a harmonic vector. (cf. |

Proof. By hypothesis

$$I^A{}_E u_{A,B}=I^A{}_B u_{E,A}\,,$$

$$I^A{}_E u_{A,BC}g^{BC}=I^A{}_B u_{E,AC}g^{BC}=I^{AC}u_{E,AC}=-\tfrac{1}{2}R^F{}_{EAC}u_F I^{AC}\,.$$

The last equation can be transformed by (12) into

$$I^A{}_E u_{A,BC}g^{BC}=-I^H{}_F R_{HE}g^{FB}u_B=I^A{}_E R_A{}^B u_B\,.$$

As det $(I^A{}_E) \neq 0$, we see that (13) is true. Especially, if M^{2n} is compact and orientable, then by de Rham's theorem [2], u_A is a harmonic vector.

THEOREM 9. *Let u^A be a pseudo-analytic contravariant vector field over a pseudo-Kählerian manifold M^{2n}, then*

$$u^A{}_{,BC}g^{BC} + R^A{}_B u^B = 0 . \tag{14}$$

Especially, if $u^A{}_{,A}=0$ and M^{2n} is compact, then u^A is a Killing vector. (cf. [4], [6]).

The proof is quite similar to that of Theorem 8. Instead of de Rham's theorem we use a theorem due to one of the authors (cf. [4], [6]).

THEOREM 10. *Suppose that u^A and v_A are contravariant and covariant pseudo-analytic vector field over a compact pseudo-Kählerian manifold M^{2n}. Then $u^A v_A$ is a constant over the manifold M^{2n}.*

Proof. It is sufficient to show that $u^A v_A$ is harmonic. We put

$$\phi = u^A v_A .$$

Then we get

$$\Delta\phi = (u^A{}_{,BC}v_A + 2u^A{}_{,B}v_{A,C} + u^A v_{A,BC})g^{BC} .$$

Putting (13) and (14) into the right hand side of the last equation we get

$$\Delta\phi = 2v_{A,C}u^A{}_{,B}g^{BC} .$$

However, the right hand side can be transformed as follows:

$$\begin{aligned} 2v_{A,C}u^A{}_{,B}g^{BC} &= -2(u^A{}_{,E}I^E{}_D I^D{}_B)g^{BC}v_{A,C} = -2(I^A{}_E u^E{}_{,D}I^D{}_B)g^{BC}v_{A,C} = -2u^E{}_{,D}I^{DC}I^F{}_C v_{E,F} \\ &= -2u^E{}_{,D}(-I^F{}_C I^C{}_K g^{KD})v_{E,F} = -2u^E{}_{,D}v_{E,F}g^{FD} = -\Delta\phi . \end{aligned}$$

Hence $\Delta\phi=0$, and ϕ is a harmonic function.

THEOREM 11. *Suppose that M^{2n} is a compact pseudo-Kählerian manifold. If the Ricci tensor R_{AB} is positive definite, then there exists no pseudo-analytic covariant tensors other than the zero vector. (If R_{AB} is positive semi-definite, then the covariant derivative of any pseudo-analytic covariant vector field vanishes).* (cf. [1], [6]).

Proof. We put

$$\phi = g^{AB}u_A u_B ,$$

then we get

$$\Delta\phi=(g^{AB}u_Au_B)_{,CD}g^{CD}=2g^{AB}(u_{A,C}u_{B,D}+u_{A,CD}u_B)g^{CD} .$$

If we substitute (13) in the second term of the last equation we get

$$\Delta\phi=2g^{AB}g^{CD}u_{A,C}u_{B,D}+2R^{AB}u_Au_B .$$

Hence, by virtue of Bochner's lemma (cf. [1], [6]), we can see immediately that our assertion is true.

THEOREM 12. *Suppose that M^{2n} is a compact pseudo-Kählerian manifold. If the Ricci tensor R_{AB} is negative definite, then there exists no pseudo-analytic contravariant vector field other than the zero vector. (If R_{AB} is negative semi-definite, then the covariant derivative of any pseudo-analytic contravariant vector field vanishes).* (cf [1], [6]).

The proof is quite similar to that of Theorem 11.

THEOREM 13. *Suppose that M^{2n} is a compact pseudo-Kählerian manifold and u_B is a covariant vector field over M^{2n} such that u_B is expressible in a neighborhood of each point of M^{2n} as $\phi_{,B}+I^A{}_B\psi_{,A}$ where ϕ and ψ are harmonic functions in such neighborhood with respect to the pseudo-Kählerian metric. If the Ricci tensor R_{AB} is positive definite, then $u_B\equiv 0$, that is, the set of functions (ϕ, ψ) is pseudo-analytic. (If R_{AB} is positive semi-definite, then the covariant derivative of u_B vanishes).* (cf. [1], [6]).

Proof. We put

$$T=g^{AB}u_Au_B , \qquad u_A=\phi_{,A}+I^E{}_A\psi_{,E} ,$$

then we get

$$\Delta T=(2g^{AB}u_{A,C}u_{B,D}+2g^{AB}u_{A,CD}u_B)g^{CD} .$$

Now, the second term of the right hand side of the last equation is transformed in the following way:

$$\begin{aligned} II&=2g^{AB}(\phi_{,CAD}+I^E{}_A\psi_{,CED})(\phi_{,B}+I^F{}_B\psi_{,F})g^{CD}\\ &=2g^{AB}(-R^H{}_{CAD}\phi_{,H}-I^E{}_AR^H{}_{CED}\psi_{,H})(\phi_{,B}+I^F{}_B\psi_{,F})g^{CD}\\ &=2g^{AB}(R^H{}_A\phi_{,H}+I^E{}_AR^H{}_E\psi_{,H})(\phi_{,B}+I^F{}_B\psi_{,F})g^{CD}\\ &=2R^{HB}u_Hu_B . \end{aligned}$$

In the process of the transformation we used (10 ii) and the fact that ϕ and ψ are harmonic functions. Hence

$$\Delta T=2g^{AB}g^{CD}u_{A,C}u_{B,D}+2R^{AB}u_Au_B ,$$

so $\Delta T\geq 0$. Accordingly, by virtue of Bochner's lemma (cf. [1], [6]), we see that the theorem is true.

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